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ATANGANA–BALEANU FISHER–KPP MODEL: WELL-POSEDNESS, POSITIVITY, HOMOTOPY APPROXIMATION

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Abstract. A Fisher–Kolmogorov–Petrovskii–Piskunov equation with Atangana–Baleanu–Caputo time derivative is studied on a bounded no-flux habitat. Because that operator annihilates the initial instant, the Cauchy problem is unsolvable for every founder profile; the Volterra equation obtained by inversion is adopted instead, equivalent to a differential problem with an explicit Mittag-Leffler transient. In a cosine Banach algebra, one smallness hypothesis gives existence, uniqueness, regularity, continuous dependence and Ulam–Hyers–Rassias stability for fixed Galerkin truncations. An extremum principle bounds solutions in an explicit interval containing the unit interval. The reformulated homotopy analysis method is exactly solvable at every order, with convergence certified for fixed truncations.

Keywords. Atangana–Baleanu fractional derivative; Fisher–KPP equation; mild solution; Ulam–Hyers stability; homotopy analysis method
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1. Introduction

Fractional operators built on a power-law kernel (Riemann–Liouville, Caputo) are singular at the origin [19, 29, 33]. Caputo and Fabrizio [11, 26] replaced this with a bounded exponential kernel; Atangana and Baleanu [5] replaced the exponential in turn by a Mittag-Leffler kernel, giving the Atangana–Baleanu–Caputo (ABC) derivative, simultaneously nonsingular at $s = t$ and nonlocal with an algebraically decaying tail [1, 7, 18, 6]: the memory shape one wants for populations combining a well-defined short-term response with long-tailed dispersal or dormancy statistics. A generalized non-singular kernel containing the Caputo–Fabrizio and Atangana–Baleanu derivatives as special cases is proposed in [16].

The Fisher–Kolmogorov–Petrovskii–Piskunov equation

$$u_t \;=\; D\,u_{xx} \;+\; r\,u\Big(1-\frac{u}{K}\Big), \tag{1.1}$$

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due to Fisher [14] and Kolmogorov–Petrovskii–Piskunov [20], is the canonical model of logistic growth under dispersal [28]. Work on time-fractional Fisher-type models has stayed close to numerical territory: Atangana [4] applied a locally-defined nonsingular derivative to Fisher's equation, Saad, Khader, Gómez-Aguilar and Baleanu [32] solved the ABC time-fractional Fisher equation by Chebyshev collocation, and Algahtani [2] contrasted the ABC and CF derivatives on an Allen–Cahn model. Nonlinear fractional evolution problems of this type remain actively studied, for instance for Burgers equations with mixed derivatives [39] and for a fractional p-Kirchhoff problem under a nonlocal Neumann condition [9]. A fixed-point apparatus pairing Krasnoselskii's theorem with the Banach contraction principle, reinforced by an Ulam-type stability estimate, has become a familiar toolkit for nonsingular-kernel equations [8, 24, 25, 35, 36], and serves other non-classical derivatives as well, for instance the tempered ψ-Caputo operator [3]. What is done here is to adapt that toolkit to the specific demands of the ABC Fisher–KPP problem, drawing on the author's earlier treatment of abstract Hadamard-type equations [13].

The starting point of this paper is a structural obstruction that appears to have been overlooked. The ABC derivative of any absolutely continuous function vanishes identically at $t = 0$, because its defining integral runs from 0 to 0. Consequently the ABC Cauchy problem ${}^{\mathrm{ABC}}D_t^{\alpha}u = f(u)$, $u(0) = u_0$ forces $f(u_0) = 0$, and *has no solution at all* otherwise (Proposition 3.6). For the Fisher–KPP nonlinearity on a Neumann interval the compatibility set contains no founder profile (Remark 3.7), so the differential problem is empty for the intended data. We therefore take as our model the Volterra equation obtained by applying the Atangana–Baleanu integral, and prove it equivalent to the differential problem ${}^{\mathrm{ABC}}D_t^{\alpha}u = f(u) - f(u_0)\mathcal{E}_{\alpha}(t)$ carrying an explicit *compatibility transient*. Every section of the paper treats this one object.

Four further contributions follow from that decision. First, working throughout in the cosine Banach algebra $\mathcal{A}$ rather than the supremum norm makes the Galerkin projection a norm-one map, so the Lipschitz estimates of Lemma 3.4 carry no Lebesgue-constant penalty at any truncation, although their constants still grow with N through Λ_N and no estimate below is uniform in N; combined with a ball centred at the founder profile, a single hypothesis (H1) replaces three smallness conditions and yields existence, uniqueness, regularity, continuous dependence and Ulam–Hyers–Rassias stability (Section 3.2). Second, a self-consistent argument in the defect itself, rather than an a priori ball radius, sharpens the positivity estimate and frees it from any unproved untruncated existence theory, at the cost of a projection term (Theorem 3.19); the upper bound there is unconditional, the lower bound holds under (3.10), and exact invariance of $[0, 1]$ is not established; the accompanying asymptotic analysis identifies precisely the regime in which the defect is $O(1 - \alpha)$, and shows that the comparison method degenerates at fixed horizon as $\alpha \to 1^-$ (Proposition 3.21). Third, replacing the nonlinear operator of the homotopy analysis method by the one associated with the correct model makes every deformation residual vanish at $t = 0$, so each deformation equation is solved exactly rather than through an ad hoc device, while the closed-form iterates are unchanged (Section 3.4). Fourth, a majorant argument

with an exact first term certifies convergence of the series for each fixed truncation and identifies its limit as the mild solution (Theorem 3.27, Proposition 3.28); the certificate is explicitly *not* a statement about the untruncated expansion, and a structural ceiling on the mode number is proved (Proposition 3.30).

All symbolic derivations below were cross-checked against an independent exact rational/cosine-mode implementation, and all reported numerical values were reproduced by two independently written codes [12].

2. Preliminaries

Throughout, $\alpha \in (0,1)$ is fixed, $T > 0$, and

$$\lambda := \frac{\alpha}{1-\alpha}, \qquad \theta := \frac{1-\alpha}{B(\alpha)}, \qquad \mathcal{E}_\alpha(t) := E_\alpha(-\lambda t^\alpha)\,,$$
$$\mu := \frac{\alpha}{B(\alpha)\Gamma(\alpha+1)} = \frac{1}{B(\alpha)\Gamma(\alpha)}, \qquad \Theta_\alpha(T) := \theta + \mu T^\alpha,$$

with the standard normalization $B(\alpha) = 1-\alpha+\alpha/\Gamma(\alpha)$, so $B(0) = B(1) = 1$. We write J^α for the Riemann–Liouville integral.

Definition 2.1 ([15])**.** For $\alpha > 0$, $\beta \in \mathbb{R}$, $E_{\alpha,\beta}(z) := \sum_{n\ge 0} z^n/\Gamma(\alpha n+\beta)$ and $E_\alpha(z) := E_{\alpha,1}(z)$.

For $0 < \alpha \le 1$ and $\beta \ge \alpha$ the function $x \mapsto E_{\alpha,\beta}(-x)$ is completely monotone on $[0,\infty)$ [15, 34]; in particular $\mathcal{E}_\alpha$ is positive, nonincreasing, $\mathcal{E}_\alpha(0) = 1$,

$$\mathcal{E}_\alpha'(\tau) = -\lambda\,\tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda\tau^\alpha) \le 0, \qquad \mathcal{E}_\alpha' \in L^1(0,T), \tag{2.1}$$

and $\mathcal{E}_\alpha(t) \sim (\lambda\Gamma(1-\alpha)t^\alpha)^{-1}$ as $t \to \infty$: the ABC memory is forgotten only algebraically.

Definition 2.2 ([5])**.** Let X be a Banach space and $g : [0,T] \to X$ absolutely continuous with $g' \in L^1(0,T;X)$. The ABC derivative of order $\alpha \in (0,1)$ is

$${}^{\mathrm{ABC}}D_t^\alpha g(t) = \frac{B(\alpha)}{1-\alpha}\int_0^t g'(s)\,\mathcal{E}_\alpha(t-s)\,ds = \frac{B(\alpha)}{1-\alpha}\big(g' * \mathcal{E}_\alpha\big)(t). \tag{2.2}$$

Since the integral in (2.2) is over a degenerate interval at $t = 0$,

$${}^{\mathrm{ABC}}D_t^\alpha g(0) = 0 \qquad \text{for every such } g, \tag{2.3}$$

a fact that drives Proposition 3.6. The kernel of ${}^{\mathrm{ABC}}D_t^\alpha$ consists exactly of the t-constants.

Definition 2.3 ([5, 1])**.** For $g \in C([0,T],X)$,

$${}^{\mathrm{AB}}I_t^\alpha g(t) = \theta\, g(t) + \frac{\alpha}{B(\alpha)} J^\alpha g(t), \qquad \|{}^{\mathrm{AB}}I_t^\alpha g\|_\infty \le \Theta_\alpha(T)\|g\|_\infty. \tag{2.4}$$

The following two lemmas are the only analytic input used in the sequel. We give direct convolution proofs; in particular no Laplace transform, no exponential-boundedness assumption and no density argument is required, and the hypothesis is exactly the one Definition 2.2 already imposes. This matters, because the functions to which the identities are applied below — t^α, $t^{2\alpha}$, the homotopy

iterates and the mild solution itself — are absolutely continuous but *not* C^1 at the origin.

Lemma 2.4 (Inversion). *Let $g:[0,T]\to X$ be absolutely continuous with $g'\in L^1(0,T;X)$. Then for all $t\in[0,T]$,*

$$ {}^{\mathrm{AB}}I_t^\alpha\big({}^{\mathrm{ABC}}D_t^\alpha g\big)(t)=g(t)-g(0),\qquad {}^{\mathrm{ABC}}D_t^\alpha\big({}^{\mathrm{AB}}I_t^\alpha g\big)(t)=g(t)-g(0)\,\mathcal{E}_\alpha(t). \tag{2.5}$$

Proof. Write $*$ for convolution on $[0,t]$. The whole proof rests on one identity: from $J^\alpha\mathcal{E}_\alpha=\lambda^{-1}(1-\mathcal{E}_\alpha)$,

$$\mathcal{E}_\alpha+\lambda J^\alpha\mathcal{E}_\alpha=\mathcal{E}_\alpha+(1-\mathcal{E}_\alpha)=1. \tag{2.6}$$

(For (2.6) itself: $\mathcal{L}\{J^\alpha\mathcal{E}_\alpha\}=s^{-\alpha}\,s^{\alpha-1}/(s^\alpha+\lambda)=\lambda^{-1}\big(s^{-1}-s^{\alpha-1}/(s^\alpha+\lambda)\big)$, and $\mathcal{E}_\alpha$ is the inverse transform of $s^{\alpha-1}/(s^\alpha+\lambda)$; both sides are continuous, so the identity holds pointwise.)

Using $\theta B(\alpha)/(1-\alpha)=1$, $\frac{\alpha}{B(\alpha)}\frac{B(\alpha)}{1-\alpha}=\lambda$, and associativity of convolution,

$$ {}^{\mathrm{AB}}I_t^\alpha\big[{}^{\mathrm{ABC}}D_t^\alpha g\big]=(g'*\mathcal{E}_\alpha)+\lambda J^\alpha(g'*\mathcal{E}_\alpha)=g'*\big[\mathcal{E}_\alpha+\lambda J^\alpha\mathcal{E}_\alpha\big]=g'*1=g(t)-g(0),$$

which is the first identity; the convolutions converge because $g'\in L^1$ and $\mathcal{E}_\alpha$ is bounded. For the second, put $h:={}^{\mathrm{AB}}I_t^\alpha g$. Since g is absolutely continuous, $J^\alpha g=g(0)t^\alpha/\Gamma(\alpha+1)+J^{\alpha+1}g'$ is absolutely continuous with $(J^\alpha g)'=g(0)t^{\alpha-1}/\Gamma(\alpha)+J^\alpha g'\in L^1$, hence so is h, and

$$ {}^{\mathrm{ABC}}D_t^\alpha h=(g'*\mathcal{E}_\alpha)+\lambda\Big[g(0)\,J^\alpha\mathcal{E}_\alpha+J^\alpha(g'*\mathcal{E}_\alpha)\Big]=\big[g(t)-g(0)\big]+g(0)\big(1-\mathcal{E}_\alpha(t)\big),$$

where $(t^{\alpha-1}/\Gamma(\alpha))*\mathcal{E}_\alpha=J^\alpha\mathcal{E}_\alpha=\lambda^{-1}(1-\mathcal{E}_\alpha)$ was used on the first bracketed term and the first identity on the pair $(g'*\mathcal{E}_\alpha)+\lambda J^\alpha(g'*\mathcal{E}_\alpha)$. □

Lemma 2.5 (The transient integrates to a constant). *For every t-independent $c\in X$ and all $t\in[0,T]$, ${}^{\mathrm{AB}}I_t^\alpha[c\,\mathcal{E}_\alpha](t)=\theta\,c$.*

Proof. ${}^{\mathrm{AB}}I_t^\alpha[c\mathcal{E}_\alpha]=\theta c\,\mathcal{E}_\alpha+\frac{\alpha}{B(\alpha)}c\,J^\alpha\mathcal{E}_\alpha=\theta c\,\mathcal{E}_\alpha+\frac{\alpha}{B(\alpha)\lambda}c\,(1-\mathcal{E}_\alpha)=\theta c\,\mathcal{E}_\alpha+\theta c\,(1-\mathcal{E}_\alpha)=\theta c$, since $\alpha/(B(\alpha)\lambda)=\theta$. □

Lemma 2.5 is elementary but pivotal: it is why the compatibility transient can be removed by a single inversion, and why the Ulam–Hyers estimate of Theorem 3.15 carries the constant $\Theta_\alpha(T)$ rather than $2\theta+\mu T^\alpha$.

Lemma 2.6 (Power rules). *For $j\ge0$, with $\nu:=\frac{\alpha\Gamma(\alpha+1)}{B(\alpha)\Gamma(2\alpha+1)}$ and $\xi:=\frac{\alpha\Gamma(2\alpha+1)}{B(\alpha)\Gamma(3\alpha+1)}$,*

$$\frac{\alpha}{B(\alpha)}J^\alpha\big[t^{j\alpha}\big]=\frac{\alpha\,\Gamma(j\alpha+1)}{B(\alpha)\Gamma((j+1)\alpha+1)}\,t^{(j+1)\alpha}\quad\big(=\mu t^\alpha,\ \nu t^{2\alpha},\ \xi t^{3\alpha}\ \text{for } j=0,1,2\big), \tag{2.7}$$

and

$$ {}^{\mathrm{ABC}}D_t^\alpha\big[\mu t^\alpha\big]=1-\mathcal{E}_\alpha(t),\qquad {}^{\mathrm{ABC}}D_t^\alpha\big[t^{2\alpha}\big]=\frac{1}{\nu}\Big(t^\alpha-\frac{\theta}{\mu}\big(1-\mathcal{E}_\alpha(t)\big)\Big).$$

Proof. (2.7) is the classical formula $J^\alpha t^\beta=\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}t^{\beta+\alpha}$. The maps $t\mapsto t^\alpha, t^{2\alpha}$ are absolutely continuous with derivative in $L^1(0,T)$, so Lemma 2.4 applies to them. From ${}^{\mathrm{AB}}I_t^\alpha[\mathbf{1}]=\theta+\mu t^\alpha$ and the second identity of (2.5) with $g\equiv1$ we get

${}^{\mathrm{ABC}}D_t^\alpha[\mu t^\alpha] = 1-\mathcal{E}_\alpha$, constants lying in $\ker {}^{\mathrm{ABC}}D_t^\alpha$. Likewise ${}^{\mathrm{AB}}I_t^\alpha[t^\alpha] = \theta t^\alpha + \nu t^{2\alpha}$ and $g(t) = t^\alpha$ has $g(0) = 0$, so $\theta\,{}^{\mathrm{ABC}}D_t^\alpha[t^\alpha] + \nu\,{}^{\mathrm{ABC}}D_t^\alpha[t^{2\alpha}] = t^\alpha$; substituting ${}^{\mathrm{ABC}}D_t^\alpha[t^\alpha] = \mu^{-1}(1-\mathcal{E}_\alpha)$ gives the second relation. □

Two classical tools are used below as standard results, not as contributions of this paper, and are recorded here in the form required. Krasnoselskii's fixed point theorem [21]: if Y is a nonempty closed bounded convex subset of a Banach space Z and $\mathcal{A}, \mathcal{B} : Y \to Z$ satisfy $\mathcal{A}u+\mathcal{B}v \in Y$ for all $u, v \in Y$, $\mathcal{A}$ is a contraction on Y, and $\mathcal{B}$ is continuous with $\mathcal{B}(Y)$ relatively compact, then $\mathcal{A}+\mathcal{B}$ has a fixed point in Y. The generalized Gronwall inequality [38]: if $\alpha > 0$, $b \geq 0$, $a : [0,T] \to [0,\infty)$ is nondecreasing and $w \in C([0,T],[0,\infty))$ satisfies $w(t) \leq a(t)+b\int_0^t (t-s)^{\alpha-1} w(s)\,ds$ on $[0,T]$, then $w(t) \leq a(t)\,E_\alpha\big(b\,\Gamma(\alpha)\,t^\alpha\big)$.

3. Main Result & Proof

3.1. The model, the solution concept, and the Galerkin setting.

3.1.1. *Formulation.* A single population of density $u(x,t)$ inhabits $\Omega = (0,L)$; $D > 0$ is the dispersal coefficient, $r > 0$ the intrinsic growth rate, $K > 0$ the carrying capacity. After rescaling $u \mapsto u/K$ we consider

$$ {}^{\mathrm{ABC}}D_t^\alpha u(x,t) = f(u)(x,t), \qquad f(u) := D\,u_{xx} + r\,u(1-u), \tag{3.1}$$

on $\Omega \times (0,T]$, with

$$ u_x(0,t) = u_x(L,t) = 0, \qquad u(x,0) = u_0(x), \tag{3.2}$$

and, unless stated otherwise, the founder profile

$$ u_0(x) = \frac{\sigma}{2}\Big(1+\cos\frac{\pi x}{L}\Big), \qquad 0 < \sigma \leq 1. \tag{3.3}$$

The no-flux condition models a closed habitat, so $\int_\Omega u_{xx}\,dx = 0$; the order α is a memory index, and $\alpha \to 1^-$ formally recovers (1.1) since $B(1) = 1$ and $\theta \to 0$.

3.1.2. *The cosine algebra.*

Definition 3.1. For $N \in \mathbb{N}$ let $X_N := \operatorname{span}\{c_k\}_{k=0}^N$, $c_k(x) := \cos(k\pi x/L)$, normed by

$$ \|v\|_{\mathcal{A}} := \sum_{k=0}^N |v_k|, \qquad v = \sum_{k=0}^N v_k c_k, \tag{3.4}$$

and let P_N denote truncation. Put $\Lambda_N := D(N\pi/L)^2$ and equip $C([0,T],X_N)$ with $\|v\|_{\mathcal{A},\infty} := \sup_{t\leq T}\|v(t)\|_{\mathcal{A}}$.

Lemma 3.2. (a) $\|v\|_{C(\overline{\Omega})} \leq \|v\|_{\mathcal{A}}$. (b) $\|vw\|_{\mathcal{A}} \leq \|v\|_{\mathcal{A}}\|w\|_{\mathcal{A}}$ *for every* N. (c) $\|P_N v\|_{\mathcal{A}} \leq \|v\|_{\mathcal{A}}$. (d) $\|\partial_{xx} v\|_{\mathcal{A}} \leq (N\pi/L)^2\|v\|_{\mathcal{A}}$. (e) *For* $N = 1$, $\|v\|_{\mathcal{A}} = \|v\|_{C(\overline{\Omega})}$.

Proof. (a), (c), (d) are immediate from (3.4). (b) follows from $c_i c_m = \frac{1}{2}(c_{|i-m|} + c_{i+m})$: each coefficient product distributes its modulus over two modes with total weight one. (e) holds because $\cos(\pi x/L)$ sweeps $[-1,1]$ on $[0,L]$, so $\sup_x |a_0 + a_1\cos(\pi x/L)| = |a_0| + |a_1|$. □

Parts (b) and (c) are what the supremum norm does not provide: there P_N has the Lebesgue constant as its norm. The single norm (3.4) is used in every estimate of this paper, including the convergence analysis of Section 3.4.

Definition 3.3. $f_N(u) := P_N\big(Du_{xx} + ru(1-u)\big) : X_N \to X_N$.

The restriction to a finite-dimensional X_N is a necessity, not a convenience: $u \mapsto Du_{xx}$ is unbounded, so no Lipschitz bound can hold on the untruncated space. *All results of Sections 3.2 and 3.4 are theorems about f_N for an arbitrary but fixed N*; no limit $N \to \infty$ is claimed anywhere.

Lemma 3.4 (Lipschitz and bound constants)**.** *For $u, v \in X_N$ with $\|u\|_{\mathcal{A}}, \|v\|_{\mathcal{A}} \le R$,*

$$\|f_N(u) - f_N(v)\|_{\mathcal{A}} \le L_f(R)\|u - v\|_{\mathcal{A}}, \qquad \|f_N(u)\|_{\mathcal{A}} \le \Phi(R),$$
$$L_f(R) := \Lambda_N + r + 2rR, \qquad \Phi(R) := R\big(\Lambda_N + r + rR\big).$$

Proof. By Lemma 3.2(c) it suffices to bound before projecting. By (d), $\|D(u-v)_{xx}\|_{\mathcal{A}} \le \Lambda_N\|u-v\|_{\mathcal{A}}$; the linear term contributes $r\|u-v\|_{\mathcal{A}}$; and by (b), $\|u^2 - v^2\|_{\mathcal{A}} = \|(u+v)(u-v)\|_{\mathcal{A}} \le 2R\|u-v\|_{\mathcal{A}}$. Since $f_N(0) = 0$, $\|f_N(u)\|_{\mathcal{A}} \le L_f(\|u\|_{\mathcal{A}})\|u\|_{\mathcal{A}} \le \Phi(R)$. □

3.1.3. *Compatibility and the solution concept.* Define the *partial inverse*

$$\widetilde{\mathcal{I}}_t^{\alpha}[g](t) := \theta\big(g(t) - g(0)\big) + \frac{\alpha}{B(\alpha)} J^{\alpha} g(t) = {}^{\mathrm{AB}}I_t^{\alpha} g(t) - \theta g(0), \tag{3.5}$$

so that $\widetilde{\mathcal{I}}_t^{\alpha}[g](0) = 0$ always, and by Lemma 2.4,

$${}^{\mathrm{ABC}}D_t^{\alpha}\big(\widetilde{\mathcal{I}}_t^{\alpha}[g]\big)(t) = g(t) - g(0)\,\mathcal{E}_{\alpha}(t). \tag{3.6}$$

Definition 3.5 (The model)**.** Given $u_0 \in X_N$, the problem is the Volterra equation

$$u(t) = u_0 + \widetilde{\mathcal{I}}_t^{\alpha}\big[f_N(u)\big](t) = u_0 + \theta\big[f_N(u(t)) - f_N(u_0)\big] + \frac{\alpha}{B(\alpha)} J^{\alpha} f_N(u(\cdot))(t), \tag{M}$$

equivalently, in pointwise differential form,

$${}^{\mathrm{ABC}}D_t^{\alpha} u(t) = f_N(u(t)) - f_N(u_0)\,\mathcal{E}_{\alpha}(t), \qquad u(0) = u_0. \tag{M′}$$

The term $-f_N(u_0)\mathcal{E}_{\alpha}(t)$ is the *compatibility transient*. A solution of (M) is called a *mild solution*.

Proposition 3.6 (Trichotomy)**.** *Let $u \in C([0,T], X_N)$ be absolutely continuous in t with $\partial_t u \in L^1$. Then*

(i) u solves (M) if and only if u solves (M′);
(ii) if u solves (3.1) then $f_N(u_0) = 0$ and u solves (M);
(iii) if $f_N(u_0) = 0$, then (3.1), (M) and (M′) have exactly the same solutions;
(iv) if $f_N(u_0) \neq 0$, problem (3.1) has no solution, whereas (M) does (Theorem 3.9).

Proof. (i) Apply ${}^{\mathrm{AB}}I_t^\alpha$ to (M′). By Lemma 2.4, ${}^{\mathrm{AB}}I_t^\alpha[{}^{\mathrm{ABC}}D_t^\alpha u] = u - u_0$; by Lemma 2.5, ${}^{\mathrm{AB}}I_t^\alpha[f_N(u_0)\mathcal{E}_\alpha] = \theta f_N(u_0)$. Hence $u - u_0 = \theta f_N(u) + \frac{\alpha}{B(\alpha)} J^\alpha f_N(u) - \theta f_N(u_0) = \widetilde{\mathcal{I}}_t^\alpha[f_N(u)]$, which is (M). Conversely apply ${}^{\mathrm{ABC}}D_t^\alpha$ to (M) and use (3.6) with $g = f_N(u)$, $g(0) = f_N(u_0)$. (ii) and (iv): by (2.3), evaluating (3.1) at $t = 0$ gives $0 = f_N(u_0)$. (iii) is then immediate from (i) and (ii). □

Remark 3.7. The compatibility set is empty for the intended data. Writing (3.3) and reducing $\cos^2$, the constant Fourier mode of $f(u_0)$ is

$$\big[f(u_0)\big]_0 = \frac{r\sigma}{2}\Big(1 - \frac{3\sigma}{4}\Big) \;\geq\; \frac{r\sigma}{8} \;>\; 0 \qquad \text{for every } \sigma \in (0,1],$$

so $f(u_0) \not\equiv 0$ for every founder profile. Restricting the theory to compatible data would leave only the equilibria $u_0 \equiv 0$ and $u_0 \equiv 1$. Unlike the exponentially decaying transient of the Caputo–Fabrizio operator, the ABC transient $f_N(u_0)\mathcal{E}_\alpha(t)$ decays only algebraically, so it cannot be dismissed as a short initial layer; it is tracked explicitly in Sections 3.3, 4 and 5.

3.2. **Well-posedness.** Fix N and $u_0 \in X_N$, and for $\varrho > 0$ put $R := \|u_0\|_{\mathcal{A}} + \varrho$ and $B_\varrho(u_0) := \{v \in C([0,T], X_N) : \|v - u_0\|_{\mathcal{A},\infty} \leq \varrho\}$, a closed convex set on which $\|v\|_{\mathcal{A},\infty} \leq R$.

Assumption 3.8. *There exists $\varrho > 0$ such that, with $\rho(R) := \Theta_\alpha(T) L_f(R)$ and $\rho_1(R) := \theta L_f(R)$,*

$$\rho(R)\,\varrho + \mu T^\alpha \|f_N(u_0)\|_{\mathcal{A}} \;\leq\; \varrho. \tag{H1}$$

Since $\mu T^\alpha \|f_N(u_0)\|_{\mathcal{A}} > 0$ whenever $f_N(u_0) \neq 0$, (H1) forces $\rho(R) < 1$ and hence $\rho_1(R) < 1$; one hypothesis therefore replaces the three smallness conditions usually imposed separately.

Theorem 3.9 (Existence and uniqueness)**.** *Under Assumption 3.8, problem* (M) *has exactly one solution $u \in B_\varrho(u_0)$, and $\|u\|_{\mathcal{A},\infty} \leq R$.*

Proof. Let $\mathcal{T}v := u_0 + \widetilde{\mathcal{I}}_t^\alpha[f_N(v)]$, and note $g := f_N(v)$ has $g(0) = f_N(u_0)$.

Invariance. For $v \in B_\varrho(u_0)$, by (3.5), (2.4) and Lemma 3.4,

$$\|\mathcal{T}v - u_0\|_{\mathcal{A}} \leq \theta \|f_N(v(t)) - f_N(u_0)\|_{\mathcal{A}} + \mu T^\alpha \sup_t \|f_N(v)\|_{\mathcal{A}}$$

$$\leq \theta L_f(R)\varrho + \mu T^\alpha \big(\|f_N(u_0)\|_{\mathcal{A}} + L_f(R)\varrho\big),$$

which is the left-hand side of (H1), hence $\leq \varrho$. Thus $\mathcal{T}(B_\varrho(u_0)) \subset B_\varrho(u_0)$.

Contraction. For $v, w \in B_\varrho(u_0)$ the constant terms $f_N(u_0)$ cancel, so

$$\|\mathcal{T}v - \mathcal{T}w\|_{\mathcal{A}} \leq \theta L_f(R)\|v - w\|_{\mathcal{A},\infty} + \mu T^\alpha L_f(R)\|v - w\|_{\mathcal{A},\infty} = \rho(R)\|v - w\|_{\mathcal{A},\infty},$$

and $\rho(R) < 1$. Banach's theorem gives a unique fixed point, which by Proposition 3.6 is the unique mild solution in the ball. □

The result is local in both T and ϱ, and uniqueness is asserted only in $B_\varrho(u_0)$; no global existence is claimed anywhere in this paper.

Remark 3.10 (The Krasnoselskii route). The splitting $\mathcal{T} = \mathcal{A} + \mathcal{B}$ with $(\mathcal{A}u)(t) = u_0 - \theta f_N(u_0) + \theta f_N(u(t))$ and $(\mathcal{B}v)(t) = \frac{\alpha}{B(\alpha)} J^\alpha f_N(v)(t)$ yields *existence*, by Krasnoselskii's theorem [21], under the weaker contraction requirement $\rho_1(R) < 1$ in place of $\rho(R) < 1$, at the cost of the cruder invariance condition $\theta L_f(R)\varrho + \mu T^\alpha \Phi(R) \le \varrho$. Its compactness ingredient is the uniform α-Hölder estimate: for $0 \le t_1 < t_2 \le T$ and $v \in B_\varrho(u_0)$,

$$\|(\mathcal{B}v)(t_2) - (\mathcal{B}v)(t_1)\|_{\mathcal{A}} \le 2\mu\Phi(R)\,(t_2 - t_1)^\alpha,$$

since $\int_0^{t_1}[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}]ds = \alpha^{-1}[(t_2 - t_1)^\alpha + t_1^\alpha - t_2^\alpha]$ and $\mu = 1/(B(\alpha)\Gamma(\alpha))$; this is the sharpest modulus a weakly singular kernel allows, and Arzelà–Ascoli applies because X_N is finite-dimensional.

Lemma 3.11 (Temporal regularity)**.** *Let Assumption 3.8 hold and set*

$$\omega := \rho_1(R) + \frac{\Gamma(\alpha+1)L_f(R)T^\alpha}{B(\alpha)\Gamma(2\alpha)}.$$

If $\omega < 1$, the solution of (M) is absolutely continuous on $[0,T]$, differentiable on $(0,T]$, and

$$\|\partial_t u(t)\|_{\mathcal{A}} \le C_{\mathrm{reg}}\, t^{\alpha-1}, \qquad C_{\mathrm{reg}} := \frac{\mu\|f_N(u_0)\|_{\mathcal{A}}}{1-\omega}.$$

In particular $\partial_t u \in L^1(0,T;X_N)$, and when $f_N(u_0) \neq 0$ the solution is not *C^1 at $t = 0$.*

Proof. Let $Y := \{w \in C((0,T],X_N) : |\!|\!|w|\!|\!| := \sup_{0<t\le T} t^{1-\alpha}\|w(t)\|_{\mathcal{A}} < \infty\}$, a Banach space; let $u^{(0)} := u_0$ and $u^{(n+1)} := \mathcal{T}u^{(n)}$ be the iterates, which converge uniformly to u by Theorem 3.9.

We show by induction that each $u^{(n)}$ is absolutely continuous with $(u^{(n)})' \in Y$. This holds for $n = 0$. If it holds for n, then $g := f_N(u^{(n)})$ is absolutely continuous with $g' = f_N'(u^{(n)})(u^{(n)})'$ and $\|g'\|_{\mathcal{A}} \le L_f(R)\|(u^{(n)})'\|_{\mathcal{A}}$; moreover $g(0) = f_N(u_0)$. Since g is absolutely continuous, $(J^\alpha g)' = g(0)t^{\alpha-1}/\Gamma(\alpha) + J^\alpha g'$, so by (3.5),

$$(u^{(n+1)})' = \theta g' + \frac{\alpha}{B(\alpha)}\Big[\frac{f_N(u_0)}{\Gamma(\alpha)}t^{\alpha-1} + J^\alpha g'\Big].$$

The first term on the right is t-independent in the weighted norm:

$$|\!|\!|\tfrac{\alpha}{B(\alpha)\Gamma(\alpha)} f_N(u_0)\, t^{\alpha-1}|\!|\!| = \tfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\|f_N(u_0)\|_{\mathcal{A}} = \alpha\mu\|f_N(u_0)\|_{\mathcal{A}} \le \mu\|f_N(u_0)\|_{\mathcal{A}}.$$

For the remaining two, $\|\theta g'\|_{\mathcal{A}} \le \rho_1(R)\|(u^{(n)})'\|_{\mathcal{A}}$, while by the Beta integral $\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds = \Gamma(\alpha)^2 t^{2\alpha-1}/\Gamma(2\alpha)$,

$$t^{1-\alpha}\Big\|\tfrac{\alpha}{B(\alpha)}J^\alpha g'(t)\Big\|_{\mathcal{A}} \le \frac{\Gamma(\alpha+1)L_f(R)T^\alpha}{B(\alpha)\Gamma(2\alpha)}\,|\!|\!|(u^{(n)})'|\!|\!|.$$

Hence $|\!|\!|(u^{(n+1)})'|\!|\!| \le \omega\,|\!|\!|(u^{(n)})'|\!|\!| + \mu\|f_N(u_0)\|_{\mathcal{A}}$, and by induction $|\!|\!|(u^{(n)})'|\!|\!| \le \mu\|f_N(u_0)\|_{\mathcal{A}}/(1-\omega) = C_{\mathrm{reg}}$ for all n.

The derivatives are thus dominated in $L^1(0,T)$ by the fixed integrable majorant $C_{\mathrm{reg}}t^{\alpha-1}$, hence uniformly integrable; by the Dunford–Pettis theorem a subsequence converges weakly in L^1 to some w, and passing to the limit in $u^{(n)}(t) = u_0 + \int_0^t (u^{(n)})'$ gives $u(t) = u_0 + \int_0^t w$. Therefore u is absolutely

continuous with $\partial_t u = w$ a.e., and $\|w(t)\|_{\mathcal{A}} \le C_{\mathrm{reg}} t^{\alpha-1}$ a.e. because the set $\{v : \|v(t)\|_{\mathcal{A}} \le C_{\mathrm{reg}} t^{\alpha-1}\}$ is convex and strongly closed, hence weakly closed. The final assertion follows since $t^{\alpha-1} \to \infty$ and, by (3.5), $\partial_t u(t) \sim \alpha\mu f_N(u_0) t^{\alpha-1}$ as $t \to 0^+$. □

Lemma 3.11 supplies exactly the regularity required by the extremum principle of Section 3.3, and shows that a C^1 hypothesis there would be unattainable in the incompatible case.

Theorem 3.12 (Continuous dependence)**.** *Let $u, \hat{u}$ solve (M) for data $u_0, \hat{u}_0$, both satisfying Assumption 3.8 with the same R, and let $\kappa := 1 - \rho_1(R) > 0$. Then*

$$\|u - \hat{u}\|_{\mathcal{A},\infty} \le \frac{1 + \rho_1(R)}{\kappa} E_\alpha\!\left(\frac{\alpha L_f(R)}{\kappa B(\alpha)} T^\alpha\right) \|u_0 - \hat{u}_0\|_{\mathcal{A}}.$$

Proof. Subtracting the two copies of (M) and setting $\psi(t) := \|u(t) - \hat{u}(t)\|_{\mathcal{A}}$,

$$\psi(t) \le \big(1 + \rho_1(R)\big)\|u_0 - \hat{u}_0\|_{\mathcal{A}} + \theta L_f(R)\psi(t) + \frac{\alpha L_f(R)}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\psi(s)\, ds.$$

Moving $\theta L_f(R)\psi(t)$ to the left and dividing by κ puts this in the form of the generalized Gronwall inequality with the constant (hence nondecreasing) majorant $a \equiv \kappa^{-1}(1 + \rho_1)\|u_0 - \hat{u}_0\|_{\mathcal{A}}$ and $b = \alpha L_f/(\kappa B\Gamma(\alpha))$; the factor $\Gamma(\alpha)$ cancels in $b\Gamma(\alpha)t^\alpha$. □

3.2.1. *Ulam–Hyers stability.*

Definition 3.13. Problem (M) is *Ulam–Hyers stable* if there is $C_f > 0$ such that for every $\varepsilon > 0$ and every $v \in C([0,T], X_N)$ with

$$\|v(t) - v(0) - \widetilde{\mathcal{I}}_t^\alpha[f_N(v)](t)\|_{\mathcal{A}} \le \varepsilon \qquad (t \in [0,T]), \tag{3.7}$$

there is a solution u of (M) with $u(0) = v(0)$ and $\|v - u\|_{\mathcal{A},\infty} \le C_f \varepsilon$.

Proposition 3.14 (The two residuals agree exactly)**.** *Let v be absolutely continuous with $\partial_t v \in L^1$ and put $\varphi := {}^{\mathrm{ABC}}D_t^\alpha v - f_N(v) + f_N(v(0))\mathcal{E}_\alpha$, the residual of (M′). Then*

$$v(t) - v(0) - \widetilde{\mathcal{I}}_t^\alpha[f_N(v)](t) = {}^{\mathrm{AB}}I_t^\alpha[\varphi](t),$$

so $\|\varphi\|_{\mathcal{A}} \le \varepsilon_D$ implies (3.7) with $\varepsilon = \Theta_\alpha(T)\varepsilon_D$.

Proof. Apply ${}^{\mathrm{AB}}I_t^\alpha$ termwise: Lemma 2.4 gives ${}^{\mathrm{AB}}I_t^\alpha[{}^{\mathrm{ABC}}D_t^\alpha v] = v - v(0)$, Lemma 2.5 gives ${}^{\mathrm{AB}}I_t^\alpha[f_N(v(0))\mathcal{E}_\alpha] = \theta f_N(v(0))$, and ${}^{\mathrm{AB}}I_t^\alpha[f_N(v)] = \theta f_N(v) + \frac{\alpha}{B(\alpha)} J^\alpha f_N(v)$. Collecting, the two θ-terms assemble $\widetilde{\mathcal{I}}_t^\alpha[f_N(v)]$ by (3.5). The bound is (2.4). □

Theorem 3.15 (Ulam–Hyers)**.** *Let Assumption 3.8 hold with $u_0 := v(0)$ and $v \in B_\varrho(u_0)$. Then (M) is Ulam–Hyers stable with*

$$C_f = \frac{1}{\kappa} E_\alpha\!\left(\frac{\alpha L_f(R)}{\kappa B(\alpha)} T^\alpha\right), \tag{3.8}$$

and, measured against the residual of (M′), with $C_f^D = \Theta_\alpha(T)\, C_f$.

Proof. Let u solve (M) with $u(0) = v(0)$ (Theorem 3.9) and $\psi := \|v - u\|_{\mathcal{A}}$. Subtracting (M) from (3.7) and noting that the constants $f_N(v(0))$ and $f_N(u_0)$ cancel because the data agree,

$$\psi(t) \le \varepsilon + \theta L_f(R)\psi(t) + \frac{\alpha L_f(R)}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\psi(s)\, ds.$$

Dividing by κ after transposing and applying the generalized Gronwall inequality with $a \equiv \varepsilon/\kappa$ gives (3.8); the differential form follows from Proposition 3.14. □

Corollary 3.16 (Ulam–Hyers–Rassias)**.** *If $\Theta \in C([0,T],\mathbb{R}_{\ge 0})$ is non-decreasing and ε in (3.7) is replaced by $\Theta(t)$, then $\|v(t) - u(t)\|_{\mathcal{A}} \le \kappa^{-1}\Theta(t)\, E_\alpha\big(\alpha L_f(R) T^\alpha/(\kappa B(\alpha))\big)$ on $[0,T]$.*

The vocabulary used here follows the classical Hyers–Ulam–Rassias line of work [17, 30, 37] and its extension to fractional differential equations by Rus [31], and, specifically for nonsingular-kernel operators, by Başcı–Öğrekçi–Mısır [8] and Liu–Fečkan–O'Regan–Wang [24, 25]; an estimate of Mittag-Leffler–Hyers–Ulam type for quaternion-valued equations is obtained in [27].

Remark 3.17. If instead v satisfies the residual bound $\|{}^{\mathrm{ABC}}D_t^\alpha v - f_N(v)\| \le \varepsilon$ against (3.1), then evaluating at $t = 0$ and using (2.3) gives $\|f_N(v(0))\|_{\mathcal{A}} \le \varepsilon$: an ε-approximate solution of (3.1) is automatically ε-compatible. Thus (3.1) and (M′) are *approximately* interchangeable even though, by Proposition 3.6, they are not exactly so.

3.2.2. *The admissible parameter region.* We instantiate the hypotheses for the two-mode truncation $N = 1$, $X_1 = \mathrm{span}\{1, \cos(\pi x/L)\}$. Writing $u = a_0 + a_1 c_1$ and using $\cos^2 = \frac{1}{2}(1 + \cos 2\cdot)$, (M) is the planar system with

$$f_1(a) = \Big(r\big(a_0 - a_0^2 - \tfrac{1}{2}a_1^2\big),\ -D\pi^2 a_1 + r a_1(1 - 2a_0)\Big),$$

and by Lemma 3.2(e) the norm (3.4) coincides with the supremum norm of the represented function. Take $D = 0.05$, $r = 1$, $L = 1$, $\sigma = 0.2$, so $\|u_0\|_{\mathcal{A}} = \sigma = 0.2$, $\Lambda_1 = D\pi^2 = 0.493480$, $\|P_1 f(u_0)\|_{\mathcal{A}} = 0.115652$ and $\|f(u_0)\|_{C(\overline{\Omega})} = 0.110652$.

Condition (H1) is satisfiable precisely for T below a threshold $T_{\max}(\alpha)$, which quantifies the local character of the theory:

α	0.6	0.7	0.8	0.9	0.95
$T_{\max}(\alpha)$	0.0002	0.0292	0.1142	0.2289	0.2897

In particular the primary setting of Sections 4 and 5, $(\alpha, T) = (0.9, 0.15)$, lies inside the certified region, with admissible radii $\varrho \in [0.0485, 0.7516]$. The orders $\alpha = 0.6, 0.7$ are certified only on very short horizons; this is structural, since $\Theta_\alpha(T) \ge \theta$ however small T is (Proposition 3.30). Table 1 collects the resulting constants at the smallest admissible ϱ.

All hypotheses hold at both points: $\omega < 1$, so Lemma 3.11 applies; and the Gronwall constant improves on the elementary contraction bound $(1+\rho_1)/(1-\rho)$ by a factor 1.31 at the first point and 4.74 at the second, where the elementary bound is nearly vacuous because ρ approaches 1.

TABLE 1. Constants of Section 3.2 at two admissible parameter points, $N = 1$.

	$\alpha = 0.95$, $T = 0.06$	$\alpha = 0.9$, $T = 0.15$
ϱ (minimal); $R = \sigma + \varrho$	0.0104; 0.2104	0.0485; 0.2485
$B(\alpha)$; θ; μ	0.971031; 0.051492; 0.998430	0.942201; 0.106134; 0.993184
$\Theta_\alpha(T)$	0.120446	0.286234
$L_f(R)$; $\Phi(R)$	1.914280; 0.358496	1.990480; 0.432882
$\rho_1(R)$; $\rho(R)$; κ	0.098570; 0.230567; 0.901430	0.211259; 0.569743; 0.788741
ω; C_{reg}	0.237283; 0.151394	0.606841; 0.292156
C_{contdep} (elem. bound)	1.411634 (1.847758)	2.444146 (11.596864)
C_f; C_f^D	1.284974; 0.154770	2.017857; 0.577579

3.3. **Positivity and boundedness.**

Lemma 3.18 (Extremum principle for ${}^{\mathrm{ABC}}D_t^\alpha$, after [10]). *Let $g : [0, t_0] \to \mathbb{R}$ be absolutely continuous with $g' \in L^1(0, t_0)$. If $g(t_0) = \min_{[0,t_0]} g$ then*

$$ {}^{\mathrm{ABC}}D_t^\alpha g(t_0) \le \frac{B(\alpha)}{1-\alpha}\, \mathcal{E}_\alpha(t_0)\, \big[g(t_0) - g(0)\big] \le 0, \tag{3.9}$$

and the inequalities are reversed at a maximum.

Proof. Put $h(\tau) := \mathcal{E}_\alpha(\tau)$ and $w(s) := (t_0 - s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t_0 - s)^\alpha)$, so that $\frac{d}{ds} h(t_0 - s) = \lambda w(s)$ by (2.1). By complete monotonicity of $E_{\alpha,\alpha}(-\cdot)$ we have $w \ge 0$, and $w \in L^1(0, t_0)$ with $\int_0^{t_0} w = \lambda^{-1}(1 - \mathcal{E}_\alpha(t_0))$, obtained by integrating (2.1) over $(0, t_0)$. Integration by parts is therefore legitimate for g absolutely continuous and bounded:

$$\int_0^{t_0} g'(s) h(t_0 - s)\, ds = g(t_0) - g(0)\mathcal{E}_\alpha(t_0) - \lambda \int_0^{t_0} g(s) w(s)\, ds.$$

At a minimum $g(s) \ge g(t_0)$, so $\lambda \int gw \ge g(t_0)(1 - \mathcal{E}_\alpha(t_0))$, whence ${}^{\mathrm{ABC}}D_t^\alpha g(t_0) \le \frac{B(\alpha)}{1-\alpha}[g(t_0) - g(0)\mathcal{E}_\alpha(t_0) - g(t_0)(1 - \mathcal{E}_\alpha(t_0))]$, which is (3.9); and $g(t_0) \le g(0)$ gives the sign. The maximum case is identical with the inequalities reversed. □

Extending $u(\cdot, t)$ evenly about $x = 0$ and $x = L$ — equivalently, expanding in the cosine basis, as throughout — produces a function that is smooth across the reflection points precisely because $u_x = 0$ there. Hence at any spatial extremum of $u(\cdot, t)$, interior or at the boundary, the second-derivative test $u_{xx} \ge 0$ (minimum) or $u_{xx} \le 0$ (maximum) holds.

Theorem 3.19 (Positivity and boundedness, Galerkin system). *Let $N = 1$ and let u be the solution of (M) given by Theorem 3.9, with $0 \le u_0 \le 1$ and $\omega < 1$. Put*

$$F := \|P_1 f(u_0)\|_{C(\overline{\Omega})}, \qquad A_1 := \tfrac{\sigma}{2} + \varrho, \qquad \widetilde{F} := F + \frac{r A_1^2}{2\,\mathcal{E}_\alpha(T)},$$

$$a := \frac{r}{\mathcal{E}_\alpha(T)}, \qquad \gamma_0 := \frac{B(\alpha)}{1-\alpha} - \frac{r}{\mathcal{E}_\alpha(T)}.$$

Then $u(x,t) \le 1+\theta\widetilde{F}$ *on* $\overline{\Omega}\times[0,T]$*; and if*

$$\gamma_0 > 0, \qquad \gamma_0^2 > 4a\widetilde{F}, \qquad R < \delta_+ := \frac{\gamma_0 + \sqrt{\gamma_0^2 - 4a\widetilde{F}}}{2a}, \tag{3.10}$$

then

$$u(x,t) \;\ge\; -\delta_-, \qquad \delta_- := \frac{2\widetilde{F}}{\gamma_0 + \sqrt{\gamma_0^2 - 4a\widetilde{F}}}\,. \tag{3.11}$$

Proof. By Lemma 3.11 the map $t \mapsto u(x,t)$ is absolutely continuous with integrable derivative, so Lemma 3.18 applies to it. For $u = a_0 + a_1 c_1 \in X_1$ only the quadratic term leaves X_1, and $(I-P_1)f(u) = -r(a_1^2/2)c_2$; moreover $|a_1(t)| \le |a_1(0)| + \|u-u_0\|_{\mathcal{A},\infty} \le A_1$. Hence (M′) reads, pointwise,

$$^{\mathrm{ABC}}D_t^{\alpha} u = Du_{xx} + ru(1-u) + r\tfrac{a_1^2}{2}c_2 - P_1 f(u_0)\,\mathcal{E}_\alpha(t). \tag{3.12}$$

Lower bound. Let $m(t) := \min_x u(x,t)$, let t_0 minimise m over $[0,T]$, let x^* minimise $u(\cdot,t_0)$, and set $g(t) := u(x^*,t)$, so $g(t_0) = \min_{[0,t_0]} g$. If $m(t_0) \ge 0$ there is nothing to prove; otherwise put $\delta := -m(t_0) \in (0,R]$. Using Lemma 3.18, (3.12), $u_{xx}(x^*,t_0) \ge 0$ and $|c_2| \le 1$, and evaluating the logistic term *exactly* as $rm(1-m) = -r\delta(1+\delta)$,

$$\tfrac{B(\alpha)}{1-\alpha}\mathcal{E}_\alpha(t_0)\big[-\delta - u_0(x^*)\big] \;\ge\; {}^{\mathrm{ABC}}D_t^{\alpha} g(t_0) \;\ge\; -r\delta(1+\delta) - \tfrac{rA_1^2}{2} - F\,\mathcal{E}_\alpha(t_0).$$

With $u_0(x^*) \ge 0$, dividing by $\mathcal{E}_\alpha(t_0) > 0$ and using $\mathcal{E}_\alpha(t_0) \ge \mathcal{E}_\alpha(T)$ in the two negative terms,

$$\delta\Big[\frac{B(\alpha)}{1-\alpha} - \frac{r(1+\delta)}{\mathcal{E}_\alpha(T)}\Big] \;\le\; F + \frac{rA_1^2}{2\mathcal{E}_\alpha(T)} = \widetilde{F}, \qquad \text{i.e.} \qquad a\delta^2 - \gamma_0\delta + \widetilde{F} \;\ge\; 0.$$

Under (3.10) this quadratic has two positive roots $\delta_- < \delta_+$, so $\delta \notin (\delta_-,\delta_+)$; since $\delta \le R < \delta_+$, the upper branch is excluded and $\delta \le \delta_-$. As $m(t_0)$ is the global minimum, (3.11) holds everywhere.

Upper bound. Let t_1 maximise $M(t) := \max_x u(x,t)$ over $[0,T]$ and x^{**} maximise $u(\cdot,t_1)$. By Lemma 3.18(max), (3.12) and $u_{xx}(x^{**},t_1) \le 0$,

$$\tfrac{B(\alpha)}{1-\alpha}\mathcal{E}_\alpha(t_1)\big[M(t_1) - u_0(x^{**})\big] \le rM(t_1)\big(1-M(t_1)\big) + \tfrac{rA_1^2}{2} - P_1 f(u_0)(x^{**})\mathcal{E}_\alpha(t_1).$$

If $M(t_1) \ge 1$ the logistic term is ≤ 0, so the right-hand side is at most $\widetilde{F}\mathcal{E}_\alpha(t_1)$ (using $\mathcal{E}_\alpha(t_1) \ge \mathcal{E}_\alpha(T)$ on the A_1 term); cancelling $\mathcal{E}_\alpha(t_1) > 0$ and using $u_0 \le 1$ gives $M(t_1) \le 1 + \theta\widetilde{F}$. If $M(t_1) < 1$ the bound is trivial. □

Theorem 3.19 is unconditional in the sense that it concerns exactly the object whose existence Theorem 3.9 establishes; the price is the projection term $rA_1^2/(2\mathcal{E}_\alpha(T))$ inside $\widetilde{F}$. For comparison we record the untruncated statement, which is sharper but conditional on existence, not proved here.

Theorem 3.20 (Untruncated version)**.** *Let* u *solve* (M) *with* f *in place of* f_1*, with* $u(\cdot,t) \in C^2(\overline{\Omega})$ *satisfying* (3.2)*,* $u(x,\cdot)$ *absolutely continuous with* $\partial_t u \in L^1$*,* $\sup|u| \le R$ *and* $0 \le u_0 \le 1$*. Then the conclusions of Theorem 3.19 hold with* $\widetilde{F}$

TABLE 2. Left: defects of Theorem 3.19, giving the invariant interval $[-\delta_-, 1+\theta\widetilde{F}]$ (†: untruncated values of Theorem 3.20). Right: the ratio $\delta_-/(1-\alpha)$ along $T = \frac{1}{2}(1-\alpha)^{1/\alpha}$, converging to $\|f(u_0)\|_{C(\overline{\Omega})} = 0.110652$ as Proposition 3.21(iii) predicts.

α	γ_0	δ_-	$1+\theta\widetilde{F}$	α	$\delta_-/(1-\alpha)$
0.95 ($T=0.06$)	15.8091	0.008725	1.007088	0.9	0.1424
0.90 ($T=0.15$)	4.8362	0.035569	1.017641	0.99	0.1132
0.90 ($T=0.06$)†	7.3817	0.015053	1.011744	0.999	0.1109
0.80 ($T=0.06$)†	2.8980	0.038989	1.024946	0.9999	0.1107

replaced by $F = \|f(u_0)\|_{C(\overline{\Omega})}$. *In particular, if* $f(u_0) = 0$ *then* $\delta_- = 0$ *and* $[0,1]$ *is exactly invariant.*

Proof. Identical, with (3.12) replaced by (M′) and the term $rA_1^2/2$ absent. □

Proposition 3.21 (Asymptotics of the defects). *(i) The upper defect satisfies* $\theta F = (1-\alpha)F/B(\alpha) = O(1-\alpha)$ *as* $\alpha \to 1^-$, *unconditionally.*

(ii) For fixed $T > 0$, $\mathcal{E}_\alpha(T) \to 0$ *and* $\gamma_0 \approx \big[(1-\alpha) - rT^\alpha\big]/(1-\alpha)^2 \to -\infty$ *as* $\alpha \to 1^-$: *hypothesis* (3.10) *fails in a left neighbourhood of* $\alpha = 1$, *and the lower estimate is not uniform in* α *there.*

(iii) Along the scaling $T = \tau(1-\alpha)^{1/\alpha}$ *with* $\tau > 0$ *fixed,* $\lambda T^\alpha = \alpha\tau^\alpha$ *remains bounded,* $\mathcal{E}_\alpha(T)$ *stays bounded away from* 0, $\gamma_0 \sim (1-\alpha)^{-1}$, *and* $\delta_- = \theta F\,(1+o(1)) = O(1-\alpha)$.

Proof. (i) is immediate since $B(\alpha) \to 1$. (ii): $\lambda = \alpha/(1-\alpha) \to \infty$, so $\lambda T^\alpha \to \infty$ and $\mathcal{E}_\alpha(T) = E_\alpha(-\lambda T^\alpha) \to 0$; using $E_\alpha(-z) \sim 1/(z\Gamma(1-\alpha))$ and $\Gamma(1-\alpha) \sim (1-\alpha)^{-1}$ gives $\mathcal{E}_\alpha(T) \approx (1-\alpha)^2/(\alpha T^\alpha)$, hence $r/\mathcal{E}_\alpha(T)$ diverges one power of $(1-\alpha)$ faster than $B(\alpha)/(1-\alpha)$. (iii): $\lambda T^\alpha = \frac{\alpha}{1-\alpha}\tau^\alpha(1-\alpha) = \alpha\tau^\alpha \to \tau$, so $\mathcal{E}_\alpha(T) \to e^{-\tau} > 0$ and $a \to re^\tau$ is bounded, while $B(\alpha)/(1-\alpha) \to \infty$; thus $\gamma_0 \sim (1-\alpha)^{-1}$, $\gamma_0^2 > 4a\widetilde{F}$ eventually, and $\delta_- \sim F/\gamma_0 \sim F(1-\alpha) = \theta F(1+o(1))$. □

The mechanism behind (ii) deserves comment: the restoring factor in (3.9) is $\frac{B(\alpha)}{1-\alpha}\mathcal{E}_\alpha(T) \approx (1-\alpha)/T^\alpha \to 0$, so the ABC comparison principle degenerates in the classical limit at fixed horizon. The failure is one of *method*, not of the model: at $\alpha = 1$ the classical maximum principle applies directly and $[0,1]$ is invariant. Table 2 reports both defects; the rightmost columns confirm Proposition 3.21(iii) numerically along $T = \frac{1}{2}(1-\alpha)^{1/\alpha}$.

What is established is therefore an *invariant interval containing* $[0,1]$, conditional on (3.10), which collapses to $[0,1]$ exactly on the compatibility set. At $(\alpha, T) = (0.9, 0.15)$ it is $[-0.035569, 1.017641]$, and $R = 0.2485 < \delta_+ = 1.0190$, so (3.10) holds. This is not exact positivity, and $[0,1]$ itself is not shown to be invariant. Numerically the certified excursion is not seen. For the Galerkin system of Theorem 3.19 itself — the planar system of Section 3.2.2, integrated by the product-integration scheme of Section 4, for which $\min_x u = a_0 - |a_1|$ — the computed approximations are nonnegative at every tested time node

at $(\alpha, T) = (0.9, 0.15)$: the minimum over the tested positive time nodes is 1.21×10^{-5}, 6.49×10^{-6} and 3.48×10^{-6} for $N_t = 2000, 4000$ and 8000 respectively, decreasing towards the initial value $u_0(L) = 0$ under refinement, while $\max u = 0.223159$; the certified bound $-\delta_- = -0.035569$ is never approached. For the full model, the reference solver of Section 4, on $(N_x, N_t) = (40, 8000)$, approaches the carrying capacity from below without overshoot at horizons up to $t = 400$: the largest value of the field itself is $\max_{x,t} u = 0.999977$ at $\alpha = 0.99$, unchanged to the reported precision on $(80, 16000)$. This is a pointwise maximum, not the total population $N(400) = 0.99998$ of Section 5, which the nearly homogeneous late profile makes numerically close to it. These computations suggest that the slack may lie in the comparison argument rather than in the model, for the parameters and discretisations tested; they do not establish invariance of $[0, 1]$.

3.4. **Homotopy analysis method.**

3.4.1. *Deformation equations.* The homotopy analysis method [22, 23] must be built on the operator whose zero set is the model.

Definition 3.22. $\mathcal{N}'[\phi] := {}^{\mathrm{ABC}}D_t^\alpha \phi - D\phi_{xx} - r\phi + r\phi^2 + f(u_0)\mathcal{E}_\alpha(t)$, so that $\mathcal{N}'[w] = 0$ if and only if w solves (M′), equivalently (M). With $q \in [0, 1]$ the embedding parameter and $\hbar \neq 0$ the convergence-control parameter, the zeroth-order deformation equation is

$$(1-q)\,{}^{\mathrm{ABC}}D_t^\alpha\big[\phi(x,t;q) - u_0(x)\big] = q\,\hbar\,\mathcal{N}'[\phi](x,t;q), \qquad \phi(x,0;q) = u_0(x). \tag{3.13}$$

At $q = 0$ the right-hand side vanishes and $\ker {}^{\mathrm{ABC}}D_t^\alpha$ consists of the t-constants, so $\phi(\cdot,\cdot;0) = u_0$; at $q = 1$, $\mathcal{N}'[\phi(\cdot,\cdot;1)] = 0$, so $\phi(\cdot,\cdot;1)$ is the mild solution. Writing $\phi = u_0 + \sum_{m\geq 1} u_m q^m$ and $\Phi_M := \sum_{m=0}^{M} u_m$, differentiating (3.13) $m-1$ times in q, dividing by $(m-1)!$ and setting $q = 0$ gives

$${}^{\mathrm{ABC}}D_t^\alpha\big[u_m - \chi_m u_{m-1}\big] = \hbar\, R'_m, \qquad u_m(x,0) = 0, \tag{3.14}$$

with $\chi_m = 0$ for $m \leq 1$, $\chi_m = 1$ for $m > 1$, and, since $f(u_0)\mathcal{E}_\alpha$ is q-independent,

$$R'_1 = -f(u_0)\big(1 - \mathcal{E}_\alpha(t)\big),$$
$$R'_m = {}^{\mathrm{ABC}}D_t^\alpha u_{m-1} - D(u_{m-1})_{xx} - r u_{m-1} + r A_{m-1} \quad (m \geq 2),$$

where $A_{m-1} := \sum_{i=0}^{m-1} u_i u_{m-1-i}$.

Proposition 3.23 (Exact solvability)**.** *$R'_m(x,0) = 0$ for every $m \geq 1$, and the unique solution of (3.14) is*

$$u_m = \chi_m u_{m-1} + \hbar\,\widetilde{\mathcal{I}}_t^\alpha[R'_m]. \tag{3.15}$$

Consequently $u_1 = \hbar\mu t^\alpha \Psi_0$ with $\Psi_0 := -f(u_0)$, and for $m \geq 2$

$$u_m = (1+\hbar)u_{m-1} - \hbar\,\widetilde{\mathcal{I}}_t^\alpha\big[D u_{m-1,xx} + r u_{m-1} - r A_{m-1}\big]. \tag{3.16}$$

Proof. That $R'_m(x,0)=0$ is immediate in both cases: $R'_1(x,0)=-f(u_0)(1-\mathcal{E}_\alpha(0))=0$ as $\mathcal{E}_\alpha(0)=1$, while for $m\geq 2$ each term of R'_m vanishes at $t=0$, since $u_{m-1}(x,0)=0$, every term of A_{m-1} carries a factor vanishing there, and ${}^{\mathrm{ABC}}D_t^\alpha u_{m-1}(x,0)=0$ by (2.3). Given g with $g(x,0)=0$, (3.6) gives ${}^{\mathrm{ABC}}D_t^\alpha\widetilde{\mathcal{I}}_t^\alpha[g]=g$ and $\widetilde{\mathcal{I}}_t^\alpha[g](x,0)=0$; uniqueness holds because $\ker {}^{\mathrm{ABC}}D_t^\alpha\cap\{w(x,0)=0\}=\{0\}$. For u_1: $\widetilde{\mathcal{I}}_t^\alpha[1-\mathcal{E}_\alpha]=\theta(1-\mathcal{E}_\alpha)+\mu t^\alpha-\theta(1-\mathcal{E}_\alpha)=\mu t^\alpha$ by (2.7) and Lemma 2.5, so $u_1=-\hbar\mu t^\alpha f(u_0)$. For $m\geq 2$, substituting $R'_m={}^{\mathrm{ABC}}D_t^\alpha u_{m-1}-F_{m-1}$ with $F_{m-1}:=Du_{m-1,xx}+ru_{m-1}-rA_{m-1}$ into (3.15) and using $\widetilde{\mathcal{I}}_t^\alpha[{}^{\mathrm{ABC}}D_t^\alpha u_{m-1}]=u_{m-1}$ (Lemma 2.4, $u_{m-1}(x,0)=0$, ${}^{\mathrm{ABC}}D_t^\alpha u_{m-1}(x,0)=0$) gives (3.16). □

With the operator $\mathcal{N}$ of (3.1) in place of $\mathcal{N}'$ one would have $R_1(x,0)=-f(u_0)\neq 0$, and (3.14) would be unsolvable at $m=1$: this is precisely the obstruction that Definition 3.22 removes, at no cost in the iterates.

3.4.2. *Algebraic closure and closed forms.*

Proposition 3.24. *$\mathcal{V}:=\operatorname{span}_{\mathbb{R}}\{t^{j\alpha}\cos(k\pi x/L):j,k\in\mathbb{Z}_{\geq 0}\}$ is closed under ∂_{xx}, pointwise multiplication and $\widetilde{\mathcal{I}}_t^\alpha$. Hence every u_m and every Φ_M is a finite element of $\mathcal{V}$, and no Mittag-Leffler transient appears in any iterate.*

Proof. Closure under ∂_{xx} and multiplication follows from $\partial_{xx}c_k=-(k\pi/L)^2c_k$, $c_ic_m=\frac{1}{2}(c_{|i-m|}+c_{i+m})$ and $t^{j_1\alpha}t^{j_2\alpha}=t^{(j_1+j_2)\alpha}$. For $\widetilde{\mathcal{I}}_t^\alpha$, (2.7) gives $J^\alpha[t^{j\alpha}c_k]\in\mathcal{V}$, and the pointwise part preserves $\mathcal{V}$. Induction on m via (3.16) completes the proof; the transient never survives because $\widetilde{\mathcal{I}}_t^\alpha$ is applied only to fields whose value at $t=0$ has been subtracted or vanishes. □

Writing $\mathcal{K}[w]:=-\big(Dw''+r(1-2u_0)w\big)$ for the (negated) linearisation of f about u_0, and $\Psi_0:=-f(u_0)$, $\Psi_1:=\mathcal{K}[\Psi_0]$, $\Psi_2:=\mathcal{K}[\Psi_1]$:

Proposition 3.25 (Closed forms)**.** *With ν,ξ as in Lemma 2.6,*

$$u_1=\hbar\,\mu\,t^\alpha\,\Psi_0, \tag{3.17}$$

$$u_2=\hbar(1+\hbar)\,\mu\,t^\alpha\Psi_0+\hbar^2\mu\theta\,t^\alpha\Psi_1+\hbar^2\mu\nu\,t^{2\alpha}\Psi_1, \tag{3.18}$$

$$\begin{aligned}u_3=\;&\mu\hbar(1+\hbar)^2t^\alpha\Psi_0+2\mu\hbar^2(1+\hbar)\big[\theta t^\alpha+\nu t^{2\alpha}\big]\Psi_1\\&+\mu\hbar^3\big[\theta^2t^\alpha+2\theta\nu t^{2\alpha}+\nu\xi t^{3\alpha}\big]\Psi_2+r\mu^2\hbar^3\big[\theta t^{2\alpha}+\xi t^{3\alpha}\big]\Psi_0^2.\end{aligned} \tag{3.19}$$

Proof. (3.17) is Proposition 3.23. For $m=2$, $A_1=2u_0u_1$, so $Du_{1,xx}+r(1-2u_0)u_1=-\hbar\mu t^\alpha\mathcal{K}[\Psi_0]=-\hbar\mu t^\alpha\Psi_1$; applying $\widetilde{\mathcal{I}}_t^\alpha$ to $t^\alpha\Psi_1$ produces $\theta t^\alpha\Psi_1$ from the pointwise part and $\nu t^{2\alpha}\Psi_1$ from (2.7) with $j=1$, giving (3.18). For $m=3$, $A_2=2u_0u_2+u_1^2$, so $Du_{2,xx}+ru_2-rA_2=-\mathcal{K}[u_2]-ru_1^2$ with $u_1^2=\hbar^2\mu^2t^{2\alpha}\Psi_0^2$; applying $\mathcal{K}$ and $\widetilde{\mathcal{I}}_t^\alpha$ term by term, using (2.7) with $j=0,1,2$, yields (3.19). □

Each u_m satisfies $u_m(x,0)=0$ identically, as (3.14) requires. The action of $\mathcal{K}$ on cosine coefficients, with $u_0=\frac{\sigma}{2}+\frac{\sigma}{2}c_1$, is

$$\mathcal{K}[w]_0=-r(1-\sigma)w_0+\tfrac{r\sigma}{2}w_1,\qquad \mathcal{K}[w]_1=\big[D\pi^2-r(1-\sigma)\big]w_1+r\sigma w_0+\tfrac{r\sigma}{2}w_2, \tag{3.20}$$

$$\mathcal{K}[w]_k=\Big[D\Big(\tfrac{k\pi}{L}\Big)^2-r(1-\sigma)\Big]w_k+\tfrac{r\sigma}{2}\big(w_{k-1}+w_{k+1}\big),\qquad k\geq 2, \tag{3.21}$$

the case $k = 1$ differing from (3.21) because $c_1 \cdot 1 = c_1$ carries full weight into mode 1; see Appendix A.

3.4.3. *Convergence.* Fix a truncation to the first $N+1$ cosine modes and work in $C([0,T], X_N)$ with the norm (3.4), which is a Banach algebra by Lemma 3.2(b).

Lemma 3.26. *If $g(x,0) = 0$ then $\|\widetilde{\mathcal{I}}_t^{\alpha}[g]\|_{\mathcal{A},\infty} \le \Theta_\alpha(T)\|g\|_{\mathcal{A},\infty}$, and $\|\widetilde{\mathcal{I}}_t^{\alpha}[f(u_0)]\|_{\mathcal{A},\infty} = \mu T^\alpha \|P_N f(u_0)\|_{\mathcal{A}}$.*

Proof. The first is (3.5) with $g(0) = 0$ together with (2.4); the second is the identity $\widetilde{\mathcal{I}}_t^{\alpha}[c] = \mu t^\alpha c$ for t-independent c. □

Theorem 3.27 (Majorant convergence, fixed N)**.** *Put $\beta_0 := \|u_0\|_{\mathcal{A}}$, $\beta_1 := |\hbar|\mu T^\alpha \|P_N f(u_0)\|_{\mathcal{A}}$, $a := |\hbar|\Theta_\alpha(T) r$ and $b := |1+\hbar| + |\hbar|\Theta_\alpha(T)(\Lambda_N + r)$. If*

$$b + 2a\beta_0 + 2\sqrt{a\beta_1} \;<\; 1, \tag{3.22}$$

then $\sum_{m\ge 0}\|u_m\|_{\mathcal{A},\infty} < \infty$: the series converges absolutely in (3.4), uniformly on $[0,T]$.

Proof. By (3.16), Lemma 3.2(b),(d) and Lemma 3.26, for $m \ge 2$

$$\|u_m\|_{\mathcal{A},\infty} \le b\,\|u_{m-1}\|_{\mathcal{A},\infty} + a\sum_{i=0}^{m-1}\|u_i\|_{\mathcal{A},\infty}\|u_{m-1-i}\|_{\mathcal{A},\infty},$$

while $\|u_1\|_{\mathcal{A},\infty} = \beta_1$ *exactly*, by (3.17). Define β_m by the same recurrence for $m \ge 2$; since $a, b \ge 0$, induction gives $\|u_m\|_{\mathcal{A},\infty} \le \beta_m$. The generating function $\beta(z) := \sum_m \beta_m z^m$ satisfies $az\beta^2 + (bz-1)\beta + \big[\beta_0 + \beta_1 z - bz\beta_0 - az\beta_0^2\big] = 0$, whose discriminant is

$$\Delta(z) = 1 - 2pz + \big(p^2 - 4a\beta_1\big)z^2, \qquad p := b + 2a\beta_0,$$

so $\Delta(1) = (1-p)^2 - 4a\beta_1 > 0$ is exactly (3.22). Moreover $\Delta'(z) \le \Delta'(1) = 2p(p-1) - 8a\beta_1 < 0$ on $[0,1]$, so Δ is decreasing and positive throughout $[0,1]$. The branch regular at $z = 0$ is therefore analytic on $[0,1]$, and $\beta(1) = \sum_m \beta_m < \infty$; comparison concludes. □

Taking instead the cruder bound $\beta_1 = b\beta_0 + a\beta_0^2$ reduces (3.22) to $(1-b)^2 > 4a\beta_0$, so Theorem 3.27 contains that form as a special case.

Proposition 3.28 (Identification of the limit)**.** *If $\sum_{m\ge 0} u_m$ converges in $C([0,T], X_N)$, its sum u satisfies $u - u_0 = \widetilde{\mathcal{I}}_t^{\alpha}[f_N(u)]$, i.e. u solves (M).*

Proof. Since $\sum_{j\ge 0} A_j = (\sum_i u_i)^2 = u^2$, we get $\sum_{j\ge 1} A_j = u^2 - u_0^2$ and hence $\sum_{m\ge 2} F_{m-1} = f_N(u) - f_N(u_0)$, where F_{m-1} is as in the proof of Proposition 3.23. Summing (3.16) over $m \ge 2$ and adding $u_1 = -\hbar\widetilde{\mathcal{I}}_t^{\alpha}[f_N(u_0)]$,

$$u - u_0 = -\hbar\widetilde{\mathcal{I}}_t^{\alpha}[f_N(u_0)] + (1+\hbar)(u-u_0) - \hbar\widetilde{\mathcal{I}}_t^{\alpha}\big[f_N(u) - f_N(u_0)\big],$$

and cancelling $-\hbar \ne 0$ gives the claim. □

Remark 3.29 (Certified intervals). For the data of Section 4 ($D = 0.05$, $r = 1$, $\sigma = 0.2$, $\alpha = 0.9$, $T = 0.15$, $\Theta_\alpha(T) = 0.286234$), condition (3.22) gives the admissible window $\hbar \in [-1.1789, 0)$ at $N = 1$, which contains the residual-optimal value $\hbar^* = -1.057$ of Section 4; at $N = 2, 3, 4$ the window is empty at this horizon. The largest horizon certifying $\hbar = -1$ is $T_N = 0.2620,\ 0.1223,\ 0.0393,\ 0.0003$ for $N = 1, 2, 3, 4$.

Proposition 3.30 (Structural mode-number ceiling). *$\Theta_\alpha(T) \geq \theta = (1-\alpha)/B(\alpha) > 0$ for every $T > 0$. Consequently $b \geq |1+\hbar| + |\hbar|\theta(\Lambda_N + r)$ cannot be made less than 1 by shrinking T once $\Lambda_N = D(N\pi/L)^2$ is large enough, and for each α there is a finite $N_{\max}(\alpha)$ beyond which no horizon certifies (3.22) for a given $\hbar$.*

Proof. Immediate from $\Theta_\alpha(T) = \theta + \mu T^\alpha$ and the monotonicity of b in Λ_N. Letting $T \to 0$ in (3.22) with $\hbar = -1$ sends $\beta_1 \to 0$, leaving the explicit criterion $\theta(\Lambda_N + r) < 1 - 2\theta r \|u_0\|_{\mathcal{A}}$, which is satisfied only for finitely many N. □

With $\hbar = -1$ one finds $N_{\max}(\alpha) = 1, 2, 4, 6, 14$ at $\alpha = 0.7, 0.8, 0.9, 0.95, 0.99$: the ceiling grows only as $\alpha \to 1^-$, where $\theta \to 0$ and the classical theory is recovered. This is a genuine structural feature of the ABC memory term, absent both from classical parabolic theory and from Caputo-type models, for which the analogue of $\Theta_\alpha(T)$ is $O(T^\alpha)$.

Remark 3.31 (What is and is not certified). Theorem 3.27 is a statement about the series for the N-mode problem, for fixed N; since $\Lambda_N \to \infty$, no limit $N \to \infty$ follows, and none is claimed. The iterates of the untruncated recursion occupy modes $0, \dots, m+1$ at order m, so Φ_3 lies in X_4, for which the certified horizon at $\hbar = -1$ is only $T_4 = 0.0003$. The computations of Section 4 are therefore validated *a posteriori*, by the residual (4.1) and by the independent reference solution, and not by Theorem 3.27.

4. Numerical results

Throughout, $D = 0.05$, $r = 1$, $L = 1$, $\sigma = 0.2$; the primary order is $\alpha = 0.9$ with $T = 0.15$, with an α-sweep over $\{0.6, 0.7, 0.8, 0.9, 0.95\}$ where indicated. All fields are evaluated from the closed forms of Proposition 3.25 without time-stepping.

4.1. Selection of $\hbar$ and the four residual notions.

We distinguish, once and for all: the residual of the model (M),

$$E_M^{\mathrm{int}}(\hbar) := \int_0^L \int_0^T \Big[\Phi_M - u_0 - \widetilde{\mathcal{I}}_t^{\alpha}\big[f(\Phi_M)\big]\Big]^2 dt\, dx, \tag{4.1}$$

a polynomial in $\hbar$ of degree $4M$ evaluated exactly by Proposition 3.24 and mode orthogonality; the residual of (M′), obtained by replacing the bracket with ${}^{\mathrm{ABC}}D_t^\alpha \Phi_M - f(\Phi_M) + f(u_0)\mathcal{E}_\alpha$, which also tends to 0; the discrepancy against an

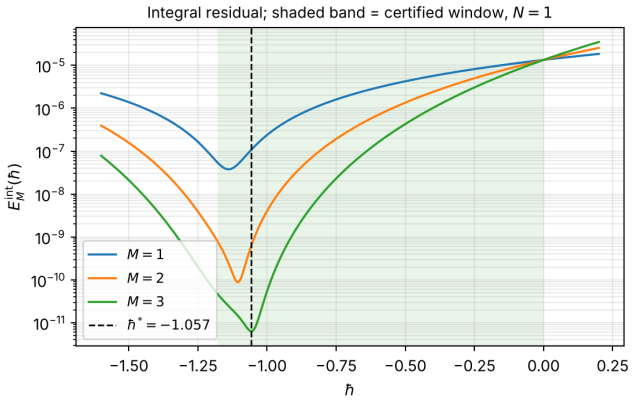


FIGURE 1. Integral residual $E_M^{\mathrm{int}}(\hbar)$ for $M = 1, 2, 3$ at $\alpha = 0.9$, $T = 0.15$. The shaded band is the window certified by Theorem 3.27 at $N = 1$; the dashed line marks $\hbar^* = -1.057$.

independently computed reference; and the *incompatibility measure*

$$E_\infty(\alpha) := \Big(\int_0^L f(u_0)^2 dx\Big)\Big(\int_0^T \mathcal{E}_\alpha(t)^2 dt\Big) = \lim_{M\to\infty} E_M^{\mathrm{pde}},$$

$$E_M^{\mathrm{pde}} := \int_0^L \int_0^T \left[{}^{\mathrm{ABC}}D_t^\alpha \Phi_M - f(\Phi_M)\right]^2 dt\, dx,$$

which measures the distance between the model and (3.1) and is *not* an error of the approximation.

Minimizing (4.1) over $\hbar \in [-1.6, 0.2]$ at $\alpha = 0.9$, $T = 0.15$ gives $\hbar_1^* = -1.13928$, $\hbar_2^* = -1.10549$, $\hbar_3^* = -1.05667$, with $E_M^{\mathrm{int}}(\hbar_M^*) = 3.78 \times 10^{-8}$, 8.80×10^{-11}, 6.32×10^{-12}. We adopt $\hbar = -1.057$ throughout, giving $E_M^{\mathrm{int}}(-1.057) = 1.078 \times 10^{-7}$, 6.24×10^{-10}, 6.32×10^{-12} — a reduction of over four orders of magnitude from $M = 1$ to $M = 3$ (Figure 1), and inside the certified window of Remark 3.29. By contrast $E_\infty(0.9) = 3.14 \times 10^{-4}$, and the residual of Φ_3 against (3.1) is 3.1422×10^{-4}: a ratio of about 5×10^7 to the integral residual, confirming that (M), not (3.1), is the problem being solved.

4.2. **Successive orders.** Table 3 reports Φ_1, Φ_2, Φ_3 at $x = 0$ for the extremes of the sweep. Convergence is monotone and accelerates with α: at $\alpha = 0.9$ the increments are at the 10^{-5} level by $t = 0.10$, whereas at $\alpha = 0.6$ they remain at 10^{-3}–10^{-2}, consistent with the shrinking admissible range of Remark 3.29.

4.3. **Comparison with an independent reference solution.** The reference solver discretises (M) — the same problem the HAM series approximates, by Proposition 3.28 — by piecewise-linear product integration of the memory term in the manner of Diethelm–Ford–Freed, a second-order central-difference Neumann Laplacian, and, at each level, an exact tridiagonal solve of the stiff linear part with Picard iteration on the reaction. The linear part must be treated implicitly: the pointwise weight θ multiplies Du_{xx} in (M), so a plain fixed-point iteration diverges

TABLE 3. Successive HAM approximations at $x = 0$, $\hbar = -1.057$.

α	t	$\Phi_1(0,t)$	$\Phi_2(0,t)$	$\Phi_3(0,t)$
0.6	0.02	0.209355	0.211860	0.212439
	0.05	0.216211	0.220954	0.222091
	0.10	0.224571	0.232503	0.234491
0.9	0.02	0.203436	0.203485	0.203484
	0.05	0.207837	0.208045	0.208039
	0.10	0.214624	0.215286	0.215268

TABLE 4. Third-order HAM approximation versus the product-integration reference at $x = 0$, $\alpha = 0.9$, $\hbar = -1.057$.

t	$\Phi_3(0,t)$	$u_{\mathrm{ref}}(0,t)$	abs. error	rel. error
0.01	0.201859	0.201860	5.8×10^{-7}	0.0003%
0.05	0.208039	0.208043	4.0×10^{-6}	0.0019%
0.10	0.215268	0.215279	1.1×10^{-5}	0.0049%
0.15	0.222360	0.222380	1.9×10^{-5}	0.0087%

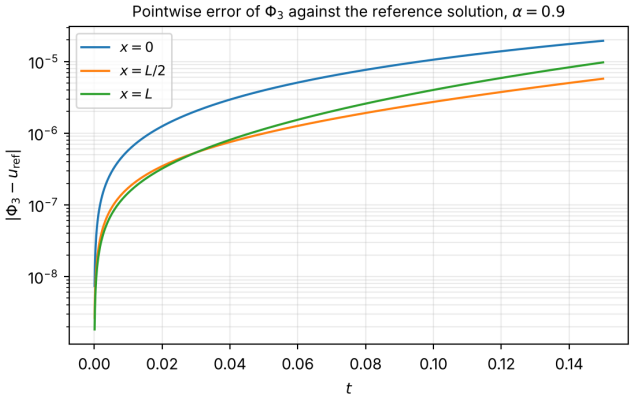


FIGURE 2. Pointwise error $|\Phi_3 - u_{\mathrm{ref}}|$ at $x = 0, L/2, L$, $\alpha = 0.9$, $\hbar = -1.057$.

for realistic mesh ratios — a numerical manifestation of the floor $\Theta_\alpha(T) \geq \theta$ of Proposition 3.30. Grid refinement at $(x,t) = (0, 0.15)$, $\alpha = 0.9$:

(N_x, N_t)	$(100, 750)$	$(200, 1500)$	$(300, 3000)$	$(400, 6000)$
$u(0, 0.15)$	0.2223802	0.2223797	0.2223796	0.2223796

so the $(200, 1500)$ reference is accurate to $\sim 10^{-6}$, an order of magnitude finer than the discrepancies of Table 4.

The relative error stays below $10^{-2}\%$ throughout $[0, 0.15]$. Figure 2 shows the pointwise error at three stations; its t^α-like growth is the signature of the truncated tail $\sum_{m \geq 4} u_m$, whose leading term carries t^α.

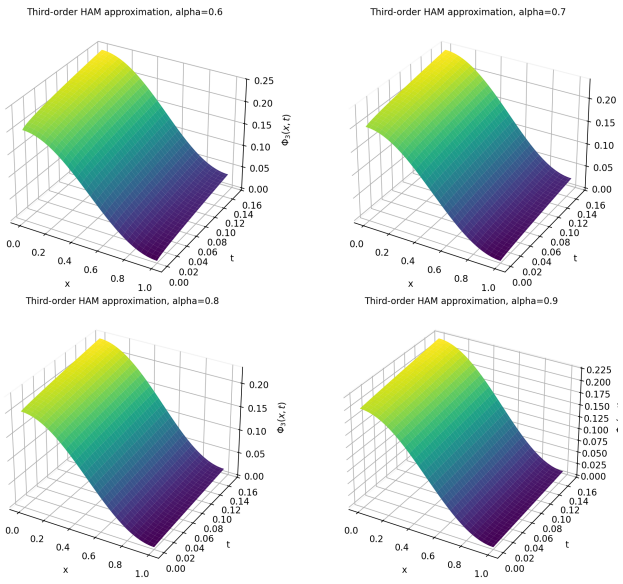


FIGURE 3. Third-order approximation $\Phi_3(x,t)$ for $\alpha = 0.6, 0.7$ (top) and $\alpha = 0.8, 0.9$ (bottom), $\hbar = -1.057$.

Figure 3 displays Φ_3 over $[0,L]\times[0,T]$: the founder profile spreads and grows monotonically at every α, and smaller α visibly slows the approach to spatial homogeneity.

5. Biological interpretation

Let $N(t) := \int_0^L u(x,t)\,dx$, so that $N(0) = \sigma L/2$. Integrating (M′) over Ω and using $\int_0^L u_{xx}\,dx = u_x(L,t) - u_x(0,t) = 0$,

$$ {}^{\mathrm{ABC}}D_t^{\alpha} N(t) = r\int_0^L u(1-u)\,dx \;-\; \mathcal{E}_\alpha(t)\int_0^L f(u_0)\,dx, \tag{5.1}$$

$$\int_0^L f(u_0)\,dx = \frac{r\sigma L}{2}\Big(1 - \frac{3\sigma}{4}\Big),$$

which for the data above equals 0.085 — the same size as the retained term at $t = 0$. The compatibility transient is therefore not negligible in the mass balance, and (5.1) is the equation from which all statements below are read.

5.1. **Approach to the carrying capacity.** Extending the reference solver of Section 4 to long horizons gives

α	0.6	0.7	0.8	0.9	0.99
$N(20)$	0.87766	0.93794	0.97290	0.99128	0.99942
$N(400)$	0.98374	—	0.99812	—	0.99998

every trajectory approaching $N = 1$ at a rate that slows as α decreases (Figure 4). Since $\mathcal{E}_\alpha(t) \sim (\lambda\Gamma(1-\alpha)t^{\alpha})^{-1}$ decays algebraically and, by (5.1), is the only memory-dependent forcing surviving once u is near equilibrium, one expects an

TABLE 5. Numerical evidence for (5.2): ratios $(1 - N(t))/(1 - N(2t))$ against 2^α, and the effective exponent $\log_2$ of the ratio, which decreases monotonically towards α. Successive refinement changes $N(400)$ by less than 3×10^{-9}, so the residual discrepancies are not quadrature error.

α	interval	observed ratio	2^α	rel. discrepancy	effective exponent
0.6	$100 \to 200$	1.570	1.516	3.6%	0.6504
	$200 \to 400$	1.550	1.516	2.3%	0.6323
	$400 \to 800$	1.538	1.516	1.5%	0.6207
0.8	$100 \to 200$	1.793	1.741	3.0%	0.8421
	$200 \to 400$	1.769	1.741	1.6%	0.8227
	$400 \to 800$	1.756	1.741	0.9%	0.8124

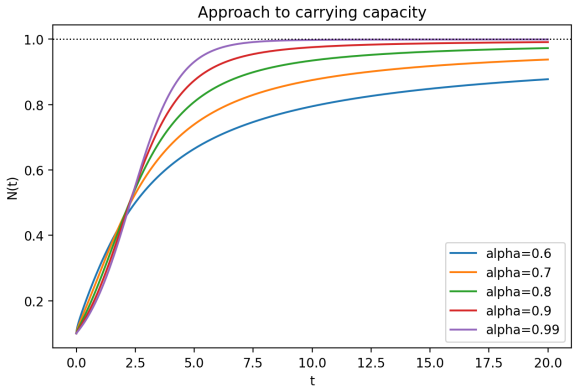


FIGURE 4. Approach of the total population $N(t)$ to the carrying capacity for five values of α.

algebraic approach,

$$1 - N(t) \;\sim\; C(\alpha)\, t^{-\alpha} \qquad (t \to \infty). \tag{5.2}$$

Remark 5.1 (Status of (5.2)). Equation (5.2) is a *numerical observation*, not a theorem. Table 5 supports it across three successive doublings at two values of α, with the effective exponent decreasing monotonically towards α and the solver verified to be accurate at the long horizon; but no proof is offered here. A proof would require linearising (5.1) about $N = 1$ and invoking the Mittag-Leffler tail, together with an error estimate; we note also that algebraic $t^{-\alpha}$ relaxation is generic for Mittag-Leffler memory and is not by itself evidence that the compatibility transient is its cause. The contrast with the exponential relaxation of the classical and Caputo–Fabrizio cases is a statement about the kernels and is unaffected.

5.2. **Sensitivity analysis.** The normalized forward sensitivities $S_p := \frac{p}{N}\frac{\partial N}{\partial p}$, evaluated at $t = 0.10$ by a centred difference on the exact field (Φ_3, $\hbar = -1.057$,

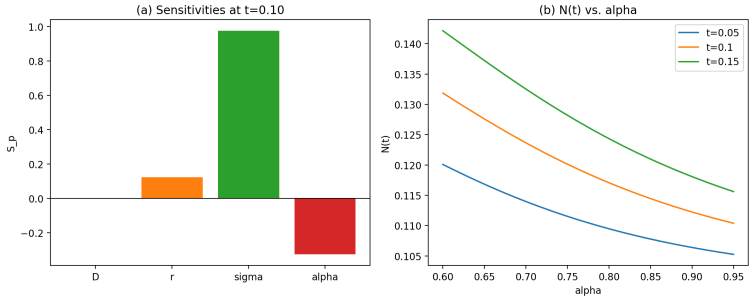


FIGURE 5. (a) Normalized sensitivities at $t = 0.10$. (b) $N(t)$ versus α at $t = 0.05, 0.10, 0.15$.

TABLE 6. Total population $N(t) = \int_0^1 u\,dx$ across the α-sweep, $\hbar = -1.057$.

α	$N(0.05)$	$N(0.10)$	$N(0.15)$
0.60	0.120087	0.131863	0.142143
0.70	0.113972	0.123651	0.132525
0.80	0.109476	0.117071	0.124344
0.90	0.106391	0.112243	0.118078
0.95	0.105256	0.110385	0.115603

$N(0.10) = 0.112243$), are

$$S_D = +0.00126, \qquad S_r = +0.12401, \qquad S_\sigma = +0.97607, \qquad S_\alpha = -0.32488.$$

Dispersal has no direct effect: $S_D \approx 0$ reflects $\int_0^L u_{xx}dx \equiv 0$ in (5.1), so dispersal makes no *direct* contribution to the spatial integral. It does not follow that N is insensitive to D, because D reshapes the profile and the profile enters $\int u(1-u)dx$ nonlinearly; the small residual value is exactly that second-order coupling. Founder size dominates instead, $S_\sigma \approx 0.98$: with $N(0) = \sigma L/2$ and short-time growth still close to the unsaturated regime, $N(t)$ is nearly proportional to σ over $[0, 0.15]$. The growth rate matters less, $S_r \approx 0.12$ being moderate because larger r accelerates recruitment but is partly saturated by $t = 0.10$. The memory order acts with the opposite sign: $S_\alpha \approx -0.32$ is negative because at small-to-moderate t a smaller α produces a larger apparent growth rate, the ABC integral over-weighting the large initial reaction rate relative to its later, saturated values. The effect reverses at long times, where $N(20)$ and $N(400)$ increase with α; the sign of S_α is therefore time-dependent, and α reshapes the *timing* of growth.

Figure 5 displays both panels and Table 6 tabulates $N(t)$ across the sweep: at every $t \le 0.15$, N decreases in α, consistent with $S_\alpha < 0$; the crossover occurs only in the saturation regime.

6. CONCLUSION

The ABC Fisher–KPP Cauchy problem on a bounded no-flux habitat has been shown to be unsolvable for every founder profile, because the ABC derivative

annihilates the initial instant; the Volterra equation (M), equivalently the differential problem (M′) with its compatibility transient, is therefore adopted as the model, and is the single object treated throughout. Within the cosine Banach algebra, one smallness hypothesis yields existence, uniqueness, temporal regularity, continuous dependence and Ulam–Hyers–Rassias stability for each fixed Galerkin truncation, with explicit constants and an explicit admissible region in (α, T). An extremum principle for ${}^{\mathrm{ABC}}D_t^{\alpha}$, combined with a self-consistent estimate of the defect, confines the solution to an explicit interval containing $[0,1]$ that collapses to $[0,1]$ exactly on the compatibility set; the accompanying asymptotics identify the scaling along which the defect is $O(1-\alpha)$ and show that the comparison method, not the model, degenerates at fixed horizon as $\alpha \to 1^-$. Reformulating the homotopy operator makes every deformation equation exactly solvable at no cost in the iterates, and a majorant argument certifies convergence for each fixed truncation while identifying the limit as the mild solution.

Four limitations should be stated plainly. The well-posedness theory is local in (T, ϱ) and no global result is claimed. The well-posedness, regularity, stability and homotopy-convergence results are established for each fixed Galerkin truncation; no passage to the limit $N \to \infty$ is proved, and Proposition 3.30 shows that the present estimates cannot deliver one, since the floor $\Theta_\alpha(T) \geq \theta$ caps the certified mode number at $N_{\max}(\alpha)$ whatever the horizon. The untruncated positivity statement of Theorem 3.20 is conditional on existence and does not supply such a passage. The lower bound is conditional on (3.10), which fails near $\alpha = 1$ at fixed T; the upper bound is not. And the relaxation law (5.2) is numerical evidence, not a theorem. Natural next steps are a semigroup argument removing the finite-N restriction, an extension to heterogeneous coefficients in two dimensions, and a proof of (5.2) from the corrected balance (5.1).

Appendix A. The linearised operator and harmonic tables

Let $w = \sum_k w_k c_k$ be t-independent. Since $1 - 2u_0 = (1-\sigma) - \sigma c_1$, $\mathcal{K}[w] = -Dw'' - r(1-\sigma)w + r\sigma c_1 w$. Expanding $c_1 c_k = \frac{1}{2}(c_{k-1} + c_{k+1})$ for $k \geq 1$, with the degenerate case $c_1 \cdot c_0 = c_1$ at *full* weight because the two product-to-sum branches coincide, the coefficient of c_k in $c_1 w$ is $\frac{1}{2}w_1$ for $k = 0$; $w_0 + \frac{1}{2}w_2$ for $k = 1$; and $\frac{1}{2}(w_{k-1} + w_{k+1})$ for $k \geq 2$. Substituting into $\mathcal{K}[w]_k = D(k\pi/L)^2 w_k - r(1-\sigma)w_k + r\sigma[c_1 w]_k$ gives (3.20)–(3.21); using the symmetric pattern at $k = 1$ instead omits the term $\frac{r\sigma}{2}w_0$. With $D = 0.05$, $r = 1$, $\sigma = 0.2$, $L = 1$ the cosine coefficients of Ψ_0, Ψ_1, Ψ_2 and Ψ_0^2 (all independent of α) are as follows (Table 7); each application of $\mathcal{K}$ generates one further harmonic, which is the mechanism behind Proposition 3.24.

TABLE 7. Cosine-mode coefficients used in (3.17)–(3.19).

mode k	Ψ_0	Ψ_1	Ψ_2	Ψ_0^2
0	-0.085000	0.064935	-0.052658	0.007707
1	-0.030652	-0.007105	0.015445	0.005058
2	0.005000	0.002804	0.002632	-0.000380
3	—	0.000500	0.002101	-0.000153
4	—	—	0.000050	0.000013

Data Availability. No datasets were generated or analysed during the current study. The Python scripts deriving the closed-form coefficients, constructing the reference solution, evaluating the residuals and producing Figures 1–5, with independent verification scripts and their output log, are openly available at Zenodo [12].

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