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STOCHASTIC $p(t,x)$- LAPLACE EQUATIONS WITH NONLINEAR PERTURBATIONS

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Abstract. We establish a well-posedness result of solution of a stochastic evolution equation with variable exponent under homogeneous Dirichlet boundary conditions, with multiplicative noise. We consider a singular or degenerated parabolic problem of Laplace type, disturbed by an additional nonlinear term. Since the function spaces are of Orlicz type and fall outside the Bochner setting, we adapt monotonicity arguments of this framework to obtain the existence, uniqueness, and stability results of solutions

1. Introduction

The aim of this paper is to extend previous results in [4] to the case of variable exponent $p(t,x)$. The newness of this work is the introduction of a function β into the stochastic PDE framework and the use of time-variable exponent $p(t,x)$. Indeed, we consider the following problem.

$$\begin{cases} du(w,t,x) - \Delta_{p(.,.)}u(w,t,x)\,dt = h(.,u(w,t,x))\,dW(w,t,x) + \beta(u(w,t,x))\,dt \\ \qquad\qquad \text{in } \Omega \times (0,T) \times D \\ \qquad\qquad u(w,t,x) = 0 \text{ on } \Omega \times (0,T) \times \partial D \\ \qquad\qquad u(w,0,.) = u_0(w,.) \text{ in } \Omega \times D, \end{cases} \tag{1.1}$$

where β is L_β- Lipschitz continuous function and h models the stochastic forcing, with $T > 0$, $D \subset \mathbb{R}^d$ a bounded Lipschitz domain, $\mathcal{Q}_T := (0,T) \times D$, W a standard $\mathcal{F}_t$-adapted unidimensional Brownian motion with continuous sample paths, constructed on the classical Wiener space $(\Omega, \mathcal{F}, P)$ over $[0,T]$. Problems of this type have been widely studied by authors when the exponent depends only on the position in space or on the solution (see for example [14, 20]). In the present paper, the exponent p depends on the space and time variables (t,x). Consequently, if u is a solution, then $u(t,.) \in W_0^{1,p(t,.)}(D)$, whereas $u(s,.) \in W_0^{1,p(s,.)}(D)$. Since

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these Sobolev spaces are generally not comparable, the standard proof of the Itô formula based on time discretization cannot be applied (see [4]). Furthermore, the variable-exponent Lebesgue space

$$L^{p(.,.)}(\mathcal{Q}_T) := \left\{ u : \mathcal{Q}_T \to \mathbb{R} \text{ measurable } : \int_{\mathcal{Q}_T} |u(t,x)|^{p(t,x)}\, dx\, dt < \infty \right\}$$

is not stable under partial convolution with respect to the spatial variable x. As a consequence, the second standard approach for establishing the Itô formula also fails. Nevertheless, each term in the identity

$$E\|u(t)\|^2_{L^2(D)} - E\|u(s)\|^2_{L^2(D)} + 2E\int_s^t \int_D |\nabla u|^{p(\sigma,x)}\, dx\, d\sigma = E\int_s^t \int_D |h(u)|^2\, dx\, d\sigma + E\int_s^t \int_D u\beta(u)\, dx\, d\sigma$$

is well defined. This naturally leads to the conjecture that u admits an almost surely continuous modification with values in $L^2(D)$, and that the above identity holds for every $0 \le s < t \le T$. Due to the lack of a Bochner space structure (see [4]), we follow the approach developed in [4, 13, 21], which consists of constructing suitable functions spaces as analogues of the classical one. The paper is organized as follows. We start by recalling the definitions and the main properties of variable-exponent Lebesgue and Sobolev spaces (see [4, 12]). The following section is devoted to the presentation of the functional framework. The third section states a preliminary result and provides its proof. The fourth section is devoted to the establishment of two intermediate results, together with their proofs. Finally, the last section is dedicated to the statement of the main result and its proof. Throughout the remainder of this paper, in order to simplify the notation, we shall simply write h in place of $h(w,t,x)$.

2. **Preliminary results**

We make the following assumptions on the data (see [3] and [4]).

(H_1) The map $h : \Omega \times (0,T) \times D \times \mathbb{R} \to \mathbb{R}, \quad (w,t,x,\lambda) \mapsto h(w,t,x,\lambda)$ is a Carathéodory function. Moreover, the mapping $\lambda \mapsto h(w,t,x,\lambda)$ is L_h-Lipschitz continuous for $P_T \otimes \mathcal{L}^d$-almost surely (w,t), where P_T denotes the predictable σ-algebra and $\mathcal{L}^d$ the d-dimensional Lebesgue measure.

(H_2) $\beta : \mathbb{R} \to \mathbb{R}$ is continuous and L_β-Lipschitz, with $\beta(0) = 0$.

(H_3) $p : \mathcal{Q}_T \to (1,\infty)$ satisfies the following assumptions.

$$\begin{cases} (H_{3,1}) \quad 1 < p^- := \operatorname*{ess\,inf}_{(t,x)} p(t,x) \le p^+ := \operatorname*{ess\,sup}_{(t,x)} p(t,x) < \infty, \\ (H_{3,2}) \quad (t,x) \mapsto p(t,x) \text{ is log-Hölder continuous, i.e., there exists } C \ge 0 \text{ such that} \\ \qquad \forall (t,x),(s,y) \in \mathcal{Q}_T, \quad |p(t,x) - p(s,y)| \le \dfrac{C}{\ln\left(e + \frac{1}{|(t,x)-(s,y)|}\right)}. \end{cases}$$

The following notations and definitions are due to Bauzet *et al* (see [4]) and Vallet and Zimmermann (see [21]).

Definition 2.1 (Modular function)**.**

$$\rho_{p(.,.)}(f) = \int_0^T \int_D |f(w,t,x)|^{p(t,x)}\, dx\, dt.$$

Definition 2.2 (Luxemburg norm)**.**

$$\|f\|_{p(.,.)} = \inf\left\{\lambda > 0 : \rho_{p(.,.)}(f/\lambda) \le 1\right\},$$

which defines a norm on $L^{p(.,.)}(\Omega \times (0,T) \times D)$, if $p^+ < \infty$.

Definition 2.3 (Variable exponent Lebesgue space)**.**

$$L^{p(.,.)}(\Omega\times(0,T)\times D) = \left\{ f : \Omega \times Q_T \to \mathbb{R} \text{ measurable } : E\int_0^T \int_D |f(w,t,x)|^{p(t,x)}\, dx\, dt < \infty \right\}.$$

Definition 2.4 (Variable exponent Sobolev space)**.**

$$W^{1,p(.,.)}(\Omega\times(0,T)\times D) = \left\{ f \in L^{p(.,.)}(\Omega \times (0,T) \times D) : |\nabla f| \in L^{p(.,.)}(\Omega \times (0,T) \times D)\right\}.$$

Definition 2.5 (Sobolev space with zero Dirichlet boundary condition)**.**

$$W_0^{1,p(.,.)}(\Omega\times(0,T)\times D) = \left\{ f \in W^{1,p(.,.)}(\Omega \times (0,T) \times D) : f = 0 \text{ on } \Omega \times (0,T) \times \partial D\right\}.$$

Definition 2.6 (Norms on Sobolev spaces)**.**

$$\|f\|_{W^{1,p(.,.)}(\Omega\times(0,T)\times D)} = \|f\|_{p(.,.)} + \|\nabla f\|_{p(.,.)},$$
$$\|f\|_{W_0^{1,p(.,.)}(\Omega\times(0,T)\times D)} = \|\nabla f\|_{p(.,.)}.$$

Definition 2.7 (Bochner-type spaces)**.** For $s = \min(2, p^-)$ and $r = \min(2, (p')^-)$,

$$L^s\big(0,T;\, L^2(D)\cap W_0^{1,p^-}(D)\big) = \begin{cases} u : (0,T) \to L^2(D) \cap W_0^{1,p^-}(D) \text{ strongly measurable } / \\ \int_0^T \left(\|u\|_{L^2(D)} + \|u\|_{W_0^{1,p^-}(D)}\right)^s dt < \infty \end{cases},$$

$$L^r\big(0,T;\, L^2(D)+W^{-1,(p')^-}(D)\big) = \begin{cases} f : (0,T) \to L^2(D) + W^{-1,(p')^-}(D) \text{ strongly measurable } / \\ \int_0^T \|f\|^r_{L^2(D)+W^{-1,(p')^-}(D)} dt < \infty, \end{cases}$$

where

$$\|f\|_{L^2(D)+W^{-1,(p')^-}(D)} := \inf_{\substack{f=f_1+f_2 \\ f_1\in L^2(D),\, f_2\in W^{-1,(p')^-}(D)}} \left(\|f_1\|_{L^2(D)} + \|f_2\|_{W^{-1,(p')^-}(D)}\right).$$

Definition 2.8. (Stochastic adapted process space, see [8])

$$S^2_W(0,T,H^k_0(D)) = \left\{ u : \Omega \times [0,T] \to H^k_0(D) : \begin{array}{l} u \text{ is } (\mathcal{F}_t)\text{-adapted}, \\ u \text{ is continuous in } t \text{ a.s.}, \\ E\left[\sup\limits_{t\in[0,T]} \|u(t)\|^2_{H^k_0(D)}\right] < \infty \end{array} \right\}.$$

Lemma 2.9. *(see [12]) If $p^- > 1$ and $p^+ < \infty$, then we have the following properties.*

$\|f\|_{p(.,.)} > 1 \Rightarrow \|f\|^{p^-}_{p(.,.)} \le \rho_{p(.,.)}(f) \le \|f\|^{p^+}_{p(.,.)}$;

$\|f\|_{p(.,.)} < 1 \Rightarrow \|f\|^{p^+}_{p(.,.)} \le \rho_{p(.,.)}(f) \le \|f\|^{p^-}_{p(.,.)}$;

$\|f\|_{p(.,.)} < 1 (= 1; > 1) \Leftrightarrow \rho_{p(.,.)}(f) < 1 (= 1; > 1)$;
$\lim_{n\longrightarrow\infty} \|f_n\|_{p(.,.)} = 0\ (\infty) \Leftrightarrow \lim_{n\longrightarrow\infty} \rho_{p(.,.)}(f_n) = 0\ (\infty)$.
$\rho_{p(.)}(f/\|f\|_{p(.,.)}) = 1$.

Proposition 2.10. *(see [9], Theorem 1.10) If $p^- > 1$, $L^{p(.)}(D)$ is a uniformly convex (reflexive) space and its dual is $L^{p'(.)}(D)$ with*

$$\frac{1}{p(.)} + \frac{1}{p'(.)} = 1.$$

If $1 < p^- < p(.) \leq p^+ < \infty$ then $(W^{1,p(.)}, \|.\|_{W^{1,p(.)}})$ and $(W_0^{1,p(.)}, \|.\|_{W_0^{1,p(.)}})$ are reflexive Banach spaces.

Lemma 2.11. *(Hölder-type inequality)(see [7])*

$$\int_{\mathcal{Q}_T} |fg|\, dxdt \leq (\frac{1}{p^-} + \frac{1}{p^+}) \|f\|_{L^{p(.,.)}(\mathcal{Q}_T)} \|g\|_{L^{p'(.,.)}(\mathcal{Q}_T)}.$$

One defines analogue of the classical Bochner space framework (see [4]).

$$L^{p(.,.)}(Q_T) := \left\{ u : Q_T \to \mathbb{R} \text{ measurable} \;\middle|\; \int_{Q_T} |u(t,x)|^{p(t,x)}\, dx\, dt < \infty \right\}$$

and

$$V_t := \left\{ u \in L^2(D) \cap W_0^{1,1}(D) \;\middle|\; \nabla u \in (L^{p(t,.)}(D, \mathbb{R}^d)) \right\},$$

equipped with the norm $\|u\|_{V_t} = \|u\|_{L^2(D)} + \|\nabla u\|_{L^{p(t,.)}(D,\mathbb{R}^d)}$.

Lemma 2.12. *(see [4]) For any $t \in [0,T]$, V_t is a separable and reflexive Banach space. Furthermore, V_t is topologically equivalent to $\left(L^2(D) \cap W_0^{1,p(t,.)}(D)\right)$.*

Lemma 2.13. *(see [4], [13]) The space*

$X_T := \left\{ u \in L^2(\mathcal{Q}_T) : \nabla u \in L^{p(.)}(\mathcal{Q}_T; \mathbb{R}^d),\ u(t,.) \in V_t \text{ for almost every } t \in (0,T) \right\}$

equipped with the norm

$$\|u\|_{X_T} := \|u\|_{L^2(\mathcal{Q}_T)} + \|\nabla u\|_{L^{p(.)}(\mathcal{Q}_T;\mathbb{R}^d)}$$

is a separable and reflexive Banach space. It is continuously embedded in the Bochner space $L^s(0,T; L^2(D) \cap W^{1,p^-}(D))$, where $s := \min(2, p^-)$. Under the log-Hölder continuity assumption on the exponent $p(.,.)$, the space $D(\mathcal{Q}_T)$ is dense in X_T. Its dual space can be identified with

$$X_T' = \left\{ T = g - \operatorname{div} G \ \in D'(\mathcal{Q}_T) \;\middle|\; g \in L^2(\mathcal{Q}_T),\ G \in L^{p'(.,.)}(\mathcal{Q}_T; \mathbb{R}^d) \right\}.$$

Remark 2.14. (see [13] , Remark 5.6) Given X_T a subset of $L^s(0,T; L^2(D) \cap W^{1,p^-}(D))$ where $s := \min(2, p^-)$. The duality properties of the product space imply that $X_T' \hookrightarrow L^r(0,T; L^2(D) + W^{-1,(p')^-}(D))$, where $r := \min(2, (p')^-)$.

Definition 2.15. (see [4]) We define the space $N_w^2(0,T; L^2(D))$ as the space of predictable processes with values in $L^2(D)$, which is identified with $L^2((0,T) \times$

$\Omega; L^2(D))$ equipped with the product measure $dt \otimes dP$ on the predictable σ-algebra P_T and with the $L^2(\mathcal{Q}_T \times \Omega)$ norm.

$$N_w^2(0,T,L^2(D)) := \left\{ S \text{ predictable } : E\left[\int_0^T \int_D |S(\omega,t,x)|^2 \, dx \, dt\right] < \infty \right\}. \tag{2.1}$$

Definition 2.16. (see [4]) One defines the space

$\mathcal{W}_T := \{u \in X_T \mid u_t \in X_T^{'}\}$, endowed with the norm $\|u\|_{\mathcal{W}_T} := \|u\|_{X_T} + \|u_t\|_{X_T^{'}}$,

where u_t is the distributional derivative of u with respect to t.

Proposition 2.17. *(see [4]) The space $\mathcal{W}_T$ is a Banach space. Moreover, the set $C^\infty([0,T];\mathcal{D}(D))$ is dense in $\mathcal{W}_T$.*

Lemma 2.18. *(see [4]) The space $\mathcal{W}_T$ is continuously embedded in $C([0,T];L^2(\Omega))$. That is, there exists a constant $C > 0$ such that*

$$\|u\|_{C([0,T];L^2(\Omega))} \leq C\|u\|_{\mathcal{W}_T}, \text{ for all } u \in \mathcal{W}_T.$$

Theorem 2.19. *(see [4],[12] [13]) (Integration by parts formula) For all $u, v \in \mathcal{W}_T$ and for all $s, t \in [0,T]$, the following integration by parts formula holds.*

$$\langle u_t, \mathbf{1}_{[s,t]} v\rangle_{X_T} = (u(t), v(t))_{L^2(D)} - (u(s), v(s))_{L^2(D)} - \langle v_t, \mathbf{1}_{[s,t]} u\rangle_{X_T}, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle_{X_T}$ denotes the duality product between $X_T^{'}$ and X_T and $(.,.)_{L^2(D)}$ the inner product in $L^2(D)$.

Proposition 2.20. *(see [4]) The dual space of $V_t(D)$ can be identified with the space*

$$V_t^{'}(D) := \left\{ T \in \mathcal{D}'(D) \;\middle|\; T = g - \operatorname{div} G,\ g \in L^2(D),\ G \in L^{p'(t,.)}(D;\mathbb{R}^d) \right\},$$

where $p'(t,.)$ denotes the conjugate exponent of $p(t,.)$. The duality product $\langle T, \phi\rangle_{X_T}$ for $T \in V_t^{'}(D)$ and $\phi \in X_T$ is given explicitly by

$$\begin{aligned}
\langle T, \phi\rangle_{X_T} &= \int_{\mathcal{Q}_T} g\phi \, dx \, dt + \int_{\mathcal{Q}_T} G \cdot \nabla\phi \, dx \, dt \\
&= \int_0^T (g(t), \phi(t))_{L^2(D)} \, dt + \int_0^T \langle G(t), \nabla\phi(t)\rangle_{L^{p'(t,.)}(D)} \, dt \\
&= \int_0^T (g(t), \phi(t))_{V_t} \, dt.
\end{aligned}$$

Lemma 2.21. *(see [4]) Let D be a measurable domain, $T > 0$, $\alpha > 0$, and $\mathcal{Q}_T := (0,T) \times D$. For every $u \in L^2(\mathcal{Q}_T)$, define the weighted norm by*

$$\|u\|_{L^2_\alpha(0,T;L^2(D))} := \left(\int_0^T \|u(t,\cdot)\|^2_{L^2(D)} \, e^{-\alpha t} \, dt \right)^{\frac{1}{2}}.$$

Then, this norm is equivalent to the standard norm of $L^2(0,T;L^2(D))$. More precisely, we have the estimate

$$e^{-\alpha T}\|u\|_{L^2(0,T;L^2(D))} \;\leq\; \|u\|_{L^2_\alpha(0,T;L^2(D))} \;\leq\; \|u\|_{L^2(0,T;L^2(D))},$$

for all $u \in L^2(\mathcal{Q}_T)$, with constants $e^{-\alpha T}$ and 1. In particular, the two norms are topologically equivalent.

Corollary 2.22. *Under the same assumptions, the space $L^2_\alpha(0,T;L^2(D))$ coincides, as a normed vector space, with $L^2(0,T;L^2(D))$ and their norms are topologically equivalent.*

Lemma 2.23. *(see [4]) Let $q \geq \max(2, p^+)$ and $\varepsilon > 0$. Define*

$$A(t,u) := -\varepsilon \Delta_q u - \Delta_{p(t,x)} u, \qquad A : (0,T) \times W_0^{1,q}(D) \to W^{-1,q'}(D).$$

Then $A(t,.)$ satisfies the standard assumptions of the Pardoux monotonicity framework (monotonicity, hemicontinuity, coercivity, boundedness, and measurability in t).

3. The additive case for a nice h

Let us first consider the following problem, in which h is independent of S and $(S,h) \in L^2(\Omega \times \mathcal{Q}_T) \times S^2_w(0,T;H^k_0(D))$.

$$(P,h,\beta) \begin{cases} \partial_t \left[u - \displaystyle\int_0^t h(w,t,x) dW(w,\sigma,x) \right] - \Delta_{p(.)} u = \beta(S) \text{ in } \Omega \times (0,T) \times D \\ u(w,t,x) = 0 \text{ on } \Omega \times (0,T) \times \partial D \\ u(w,0,.) = u_0(w,x) \in \Omega \times D. \end{cases} \tag{3.1}$$

Definition 3.1. A strong solution to problem (3.1) is a predictable process u such that
$u \in L^2\left(\Omega; C\left([0,T]; L^2(D)\right)\right) \cap X_T \cap N^2_w(0,T;L^2(D))$, satisfying $u(w,0,.) = u_0(w,x) \in L^2(\Omega \times D)$ with $\nabla u \in \left(L^{p(.,.)}(\Omega, \mathcal{Q}_T)\right)^d$, and for all $t \in [0,T]$, $x \in D$ and almost surely w in Ω,

$$\begin{aligned} u(w,t,x) =& u(w,s,x) + \int_s^t \operatorname{div}\left(|\nabla u(w,\sigma,x)|^{p(\sigma,x)-2} \nabla u(w,\sigma,x) \right) d\sigma \\ &+ \int_s^t \beta(S(w,\sigma,x)) d\sigma + \int_s^t h(w,\sigma,x) dW(w,\sigma,x). \end{aligned} \tag{3.2}$$

Lemma 3.2. *Assume that (H_2) and (H_3) hold true. Then, for every $h \in S^2_W(0,T;H^k_0(D))$, there exists a unique solution u to Problem (3.1), in the sense of Definition 3.1 . Moreover, if $h_1, h_2 \in S^2_W(0,T;H^k_0(D))$ and u_1, u_2 are the corresponding solutions, then for every $t \in (0,T)$,*

$$\begin{aligned} &E\|u_1 - u_2\|^2_{C([0,T],L^2(D))} + E \int_0^t \int_D \left(|\nabla u_1|^{p(\sigma,x)-2} \nabla u_1 - |\nabla u_2|^{p(\sigma,x)-2} \nabla u_2 \right) \cdot \nabla(u_1 - u_2)\, dx\, d\sigma \\ &\leq C E \|h_1 - h_2\|^2_{N^2_W(0,T;L^2(D))}. \end{aligned} \tag{3.3}$$

In the following subsections we establish the proof of Lemma 3.2

3.1. **Proof of Lemma 3.2.**

Remark 3.3. The direct application of Itô formula to a solution of (3.1) is not straightforward, since the exponent $p(t,x)$ depends on time, thus, the natural Sobolev space $W_0^{1,p(t,.)}(D)$ varies with t. Therefore, we are not in the standard framework of Bochner spaces with a constant exponent. To overcome this difficulty, we first apply the Itô formula to the regularized solution u^ε, which lies in a fixed space $W_0^{1,q}(D)$ and we pass to the limit as $\varepsilon \to 0$. However, this limiting procedure only preserves an energy inequality, the equality being lost due to the lack of strong convergence of the nonlinear gradient terms.

We take in this section $h \in S_W^2(0,T,H_0^k(D))$ with a sufficiently large value of k i.e $h \in N_W^2(0,T,H_0^1(D))$ such that, for any $i = 1, \cdots, d, \quad \partial_{x_i} h \in L^\infty(\Omega \times \mathcal{Q}_T)$. Moreover, for such h, for any $r > 1$, we know that there exists a constant $C > 0$ such that

$$E\left(\left|\nabla \int_0^t h(\sigma,x,w)dW(\sigma)\right|^r\right) \leq C \left\|\nabla h\right\|^2_{L^\infty(\Omega\times(0,T)\times W_0^{1,r}(D))}, \tag{3.4}$$

for almost all t,x and therefore $\int_0^\cdot h dW \in L^r(\Omega \times (0,T) \times W_0^{1,r}(D))$ for any $r > 1$ (see [4]). One has by Lipschitz property of β, and as $\beta(0) = 0$ that $|\beta(S)| \leq L_\beta |S|$, and thanks to (2.1),

$$E\left(\int_0^t \int_D |\beta(S(w,\sigma,x))|^2 d(\sigma,x)\right) \leq L_\beta^2 E \int_0^t \int_D |S(w,\sigma,x)|^2 d(\sigma,x) < \infty. \tag{3.5}$$

For such h, (see [17]) for any $q \geq \max(2, p^+)$, there exists a unique solution u^ε to

$$\begin{cases} du - \varepsilon \Delta_q u \, dt - \Delta_{p(.)} u \, dt = h \, dW + \beta(S) dt, \text{ in } \Omega \times (0,T) \times D \\ u(w,t,x) = 0 \text{ on } \Omega \times (0,T) \times \partial D \\ u(w,0,.) = u_0(w,x) \in \Omega \times D, \end{cases} \tag{3.6}$$

belonging to $L^q(\Omega, L^q(0,T,W_0^{1,q}(D)) \cap N_W^2(0,T,L^2(D)))$, with $u^\varepsilon(w,0,x) = u_0(w,x)$ and satisfying

$$\begin{aligned} &u^\varepsilon(w,t,x) = u^\varepsilon(w,s,x) \\ &+ \int_s^t \operatorname{div}\left(\varepsilon |\nabla u^\varepsilon(w,\sigma,x)|^{q-2} \nabla u^\varepsilon(w,\sigma,x) + |\nabla u^\varepsilon(w,\sigma,x)|^{p(\sigma,x)-2} \nabla u^\varepsilon(w,\sigma,x)\right) d\sigma \\ &+ \int_s^t \beta(S(w,\sigma,x)) d\sigma + \int_s^t h(w,\sigma,x) dW(w,\sigma,x). \end{aligned} \tag{3.7}$$

Remark 3.4. If u^ε is a solution to Problem (3.6), then the limit u of the sequence $(u^\varepsilon)_{\varepsilon>0}$ is a solution to the limit problem

$$\begin{cases} du - \Delta_{p(.)} u \, dt = h \, dW + \beta(S) dt, \text{ in } \Omega \times (0,T) \times D \\ u(w,t,x) = 0 \text{ on } \Omega \times (0,T) \times \partial D \\ u(w,0,.) = u_0(w,x) \in \Omega \times D, \end{cases} \tag{3.8}$$

and belonging to $L^q(\Omega, L^q(0,T,W_0^{1,q}(D)) \cap N_W^2(0,T,L^2(D)))$ with $u(w,0,x) = u_0(w,x)$.

For convenience, we systematically set

$$v(t) := u(t) - \int_0^t h\, dW. \tag{3.9}$$

We note that u^ε is a solution of the above problem (3.6) if and only if v^ε is a solution of the following problem.

$$\partial_t v - \varepsilon\Delta_q\Big[v + \int_0^t h dW\Big] - \Delta_{p(\cdot)}\Big[v + \int_0^t h dW\Big] = \beta(S). \tag{3.10}$$

Thus,

$$\partial_t v = \varepsilon\Delta_q\Big[v + \int_0^t h dW\Big] + \Delta_{p(\cdot)}\Big[v + \int_0^t h dW\Big] + \beta(S). \tag{3.11}$$

Since $v^\varepsilon + \displaystyle\int_0^t h dW \in L^q\big(\Omega, L^q(0,T,W_0^{1,q}(D))\big)$, the operators Δ_q and $\Delta_{p(.,.)}$ take values in $W^{-1,q'}(D)$ (see [17], Remark 1.2). Hence,

$$\partial_t v^\varepsilon \in L^{q'}(\Omega, L^{q'}(0,T,W^{-1,q'}(D))). \tag{3.12}$$

Indeed, as $(p(t,x)-1)q' \le (q-1)q' = q$, $1 < p^- \le p(t,x) \le p^+ \le q$ and D is bounded then $W^{-1,p'(t,.)}(D) \hookrightarrow W^{-1,q'}(D)$. Since

$|\nabla v^\varepsilon|^{(p(t,x)-1)q'} \le 1 + |\nabla v^\varepsilon|^q$ then $\big\||\nabla v^\varepsilon|^{p(t,x)-2}\nabla v^\varepsilon\big\|_{(L^{q'}(D))^d} \le C\left(1 + \|\nabla v^\varepsilon\|^q_{L^q(D)}\right)$.

Therefore,

$$\|\Delta_{p(t,.)} v^\varepsilon\|^{q'}_{W^{-1,q'}} \le C\big(1 + \|v^\varepsilon\|^q_{W_0^{1,q}}\big).$$

Measurability with respect to t stems from the Carathéodory property of the mapping $(t,x,v) \mapsto |v|^{p(t,x)-2}\, v$ and Pettis theorem, using the separability of $W_0^{1,q}(D)$.

Moreover, from (3.4), $\displaystyle\int_0^t h dW \in L^q\big(\Omega, L^q(0,T,W_0^{1,q}(D))\big)$, which implies that $v^\varepsilon \in L^q\big(\Omega, L^q(0,T,W_0^{1,q}(D))\big)$. By the Lions-Magenes lemma (see [16]) with pivot space $L^2(D)$, we deduce that $v^\varepsilon \in L^2\big(\Omega, C([0,T],L^2(D))\big)$ and $u^\varepsilon \in L^2\big(\Omega, C([0,T],L^2(D))\big)$ as well as their weak limits.

3.2. **A priori Estimates.**

Remark 3.5. The formal expression $\dfrac{d\|u^\varepsilon\|^2_{L^2(D)}}{dt}$, involving the heuristic derivative $\dfrac{dW}{dt}$, is strictly formal and lacks classical mathematical rigor, as the Wiener process $W(t)$ is almost surely nowhere differentiable with respect to t. To rigorously establish the energy equality, one applies the infinite-dimensional Itô formula (see [17], Theorem 4.1) to the squared norm functional $\phi(u) = \|u\|^2_{L^2(D)}$ defined on the Hilbert space $H = L^2(D)$. The first Fréchet derivative of ϕ at $u \in H$ is given by the linear functional $\phi'(u)\cdot v = 2\int_D uv\, dx$, and its second Fréchet derivative

$\phi''(u)\cdot(v,w) = 2\int_D vw\,dx$ is a continuous bilinear form that is independent of u. Applying Itô's formula in integral form to the semimartingale $u^\varepsilon(t)$, the drift term yields $2\int_D u^\varepsilon\left[\Delta_{p(.)}u^\varepsilon + \varepsilon\Delta_q u^\varepsilon + \beta(S)\right]dx$, the stochastic term is expressed as the Itô integral $2\int_0^t\int_D u^\varepsilon h\,dx\,dW(s)$, and the second-order trace term (the Itô correction arising from $\frac{1}{2}\phi''(u)$ evaluated on the diffusion operator) naturally accounts for the integral $\int_0^t\int_D h^2\,dx\,ds$.

Using the Itô multiplication rules

$$dt\,dt = 0, \qquad dt\,dW = 0, \qquad dW\,dW = dt, \tag{3.13}$$

Itô's formula yields

$$d\|u^\varepsilon\|^2_{L^2(D)} = 2\int_D u^\varepsilon\,du^\varepsilon\,dx + \int_D (du^\varepsilon)^2\,dx. \tag{3.14}$$

Therefore

$$(du^\varepsilon)^2 = \left[\left(\Delta_{p(.)}u^\varepsilon + \varepsilon\Delta_q u^\varepsilon + \beta(S)\right)dt + hdW\right].\left[\left(\Delta_{p(.)}u^\varepsilon + \varepsilon\Delta_q u^\varepsilon + \beta(S)\right)dt + hdW\right] = h^2dt \tag{3.15}$$

and (3.14) becomes

$$d\,\|u^\varepsilon\|^2_{L^2(D)} = 2\int_D\left[u^\varepsilon\left(\Delta_{p(.)}u^\varepsilon + \varepsilon\Delta_q u^\varepsilon + \beta(S)\right)\right]dtdx + 2\int_D u^\varepsilon h dW dx + \int_D h^2 dt dx, \tag{3.16}$$

which is equivalent to say

$$\frac{d\,\|u^\varepsilon\|^2_{L^2(D)}}{dt} = 2\int_D\left[u^\varepsilon\left(\Delta_{p(.)}u^\varepsilon + \varepsilon\Delta_q u^\varepsilon + \beta(S)\right)\right]dx + 2\int_D u^\varepsilon h\frac{dW}{dt}dx + \int_D h^2 dx.$$

We know that

$$\int_D u^\varepsilon\Delta_{p(.)}u^\varepsilon dx = -\int_D |\nabla u^\varepsilon|^{p(t,x)}\,dx$$

and

$$\int_D \varepsilon u^\varepsilon\Delta_q u^\varepsilon dx = -\int_D \varepsilon\,|\nabla u^\varepsilon|^q\,dx,$$

since

$$u^\varepsilon(w,t,x) = 0 \text{ on } \Omega\times(0,T)\times\partial D.$$

Thus,

$$\frac{d\,\|u^\varepsilon\|^2_{L^2(D)}}{dt} = -2\int_D\left[\varepsilon\,|\nabla u^\varepsilon|^q + |\nabla u^\varepsilon|^{p(t,x)}\right]dx + 2\int_D u^\varepsilon\beta(S)dx + 2\int_D u^\varepsilon h\frac{dW}{dt}dx + \int_D h^2 dx. \tag{3.17}$$

Integrating (3.17) between 0 and t, gives

$$\begin{aligned}\|u^\varepsilon(t)\|^2_{L^2(D)} - \|u_0\|^2_{L^2(D)} + 2\int_0^t\int_D\left[\varepsilon\,|\nabla u^\varepsilon|^q + |\nabla u^\varepsilon|^{p(\sigma,x)}\right]dxd\sigma \\ = 2\int_0^t\int_D u^\varepsilon\beta(S)dxd\sigma + 2\int_0^t\int_D u^\varepsilon h dx dW(\sigma) + \int_0^t\int_D h^2 dx d\sigma.\end{aligned} \tag{3.18}$$

Applying the Burkholder-Davis-Gundy inequality as in [4], one deduces that

$$E\left[\sup_t\left|\int_0^t\int_D hu^\varepsilon dxdW\right|\right]\leq\frac{1}{4}E\left[\sup_t\|u^\varepsilon(t)\|^2_{L^2(D)}\right]+C\|h\|^2_{L^2(\Omega\times\mathcal{Q}_T)},\quad\text{where } C>0. \tag{3.19}$$

Therefore, from (3.19) and (3.18) we arrive at

$$\begin{aligned}&E\left[\sup_t\|u^\varepsilon(t)\|^2_{L^2(D)}+4\int_0^T\int_D\left[\varepsilon|\nabla u^\varepsilon|^q+|\nabla u^\varepsilon|^{p(\sigma,x)}\right]dxd\sigma\right]\\&\leq C'\|h\|^2_{L^2(\Omega\times\mathcal{Q}_T)}+2\|u_0\|^2_{L^2(D)}+4E\sup_{0\leq t\leq T}\int_0^t\int_D u^\varepsilon\beta(S)dxd\sigma,\end{aligned} \tag{3.20}$$

where $C'=4C+1$. Using Young's inequality, for any $\delta>0$ and for all $(\sigma,x)\in\mathcal{Q}_t$, one has

$$2u^\varepsilon(\sigma,x)\beta(S)\leq\delta|u^\varepsilon(\sigma,x)|^2+\frac{1}{\delta}|\beta(S)|^2. \tag{3.21}$$

Integrating this pointwise estimate over D and then over the time interval $(0,T)$ yields

$$4\sup_{t\in[0,T]}\int_{\mathcal{Q}_t}u^\varepsilon\beta(S)\,dxd\sigma\leq 2\delta\int_0^T\|u^\varepsilon(\sigma)\|^2_{L^2(D)}\,d\sigma+\frac{2}{\delta}\int_0^T\|\beta(S)\|^2_{L^2(D)}\,d\sigma. \tag{3.22}$$

We introduce the following quantity $0<\mathbf{B}:=C'\|h\|^2_{L^2(\Omega\times\mathcal{Q}_T)}+2\|u_0\|^2_{L^2(D)}+\frac{2}{\delta}E\int_0^T\|\beta(S)\|^2_{L^2(D)}\,ds<\infty$, to deduce by using (3.20),

$$\begin{aligned}&E\left[\sup_{0\leq\sigma\leq t}\|u^\varepsilon(\sigma)\|^2_{L^2(D)}+4\int_0^T\int_D\left(\varepsilon|\nabla u^\varepsilon(\sigma,x)|^q+|\nabla u^\varepsilon(\sigma,x)|^{p(\sigma,x)}\right)dx\,d\sigma\right]\\&\leq\mathbf{B}+2\delta\int_0^T E\left[\sup_{0\leq r\leq\sigma}\|u^\varepsilon(r)\|^2_{L^2(D)}+4\int_0^\sigma\int_D\left(\varepsilon|\nabla u^\varepsilon(r,x)|^q+|\nabla u^\varepsilon(r,x)|^{p(r,x)}\right)dx\,dr\right]d\sigma.\end{aligned} \tag{3.23}$$

By applying the integral form of Gronwall's Lemma (see [1], Lemma 5.1) to (3.23), one has

$$E\left[\sup_{0\leq\sigma\leq T}\|u^\varepsilon(\sigma)\|^2_{L^2(D)}+4\int_0^T\int_D\left(\varepsilon|\nabla u^\varepsilon(\sigma,x)|^q+|\nabla u^\varepsilon(\sigma,x)|^{p(\sigma,x)}\right)dx\,d\sigma\right]\leq\mathbf{B}e^{2\delta T}. \tag{3.24}$$

Using (3.24), one deduces that

$$E\left[\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^2_{L^2(D)}\right]\leq K,\text{where}\quad K:=\mathbf{B}e^{2\delta T}+1>0. \tag{3.25}$$

Therefore $(u^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(0,T,L^2(\Omega\times D))$, $(\nabla u^\varepsilon)_{\varepsilon>0}$ is bounded in $\left(L^{p(.,.)}(\Omega\times\mathcal{Q}_T)\right)^d$ and also $\varepsilon E\int_{\mathcal{Q}_T}|\nabla u^\varepsilon|^q\,dxds<\infty$.

Since the stochastic integral is continuous almost surely (see [18]), there exists a set $\widetilde{\Omega} \subset \Omega$ with $P(\widetilde{\Omega}) = 1$ such that, for every $\omega \in \widetilde{\Omega}$, the trajectory

$$t \mapsto \int_0^t h\, dW$$

is continuous on $[0,T]$. Therefore, for every $\omega \in \widetilde{\Omega}$, the process

$$v^\varepsilon(t) := u^\varepsilon(t) - \int_0^t h\, dW$$

is well-defined for all $t \in [0,T]$ and satisfies, for a.e. $t \in (0,T)$, the following deterministic equation:

$$\partial_t v^\varepsilon - \varepsilon \Delta_q \Big[v^\varepsilon + \int_0^t h dW \Big] \; - \; \Delta_{p(.)} \Big[v^\varepsilon + \int_0^t h dW \Big] \; = \beta(S). \tag{3.26}$$

Multiplying (3.26) by v^ε and integrating over D, one obtains

$$\int_D v^\varepsilon \partial_t v^\varepsilon dx - \varepsilon \int_D v^\varepsilon \Delta_q \Big[v^\varepsilon + \int_0^t h dW \Big] dx - \int_D v^\varepsilon \Delta_{p(.)} \Big[v^\varepsilon + \int_0^t h dW \Big] dx \; = \int_D v^\varepsilon \beta(S) dx. \tag{3.27}$$

By using integration by parts formula and according to the boundary condition ($v^\varepsilon = 0$ on ∂D), one gets

$$\int_D v^\varepsilon \partial_t v^\varepsilon dx = \frac{1}{2} \frac{d}{dt} \left(\|v^\varepsilon\|^2_{L^2(D)} \right)$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|^2_{L^2(D)} = & -\varepsilon \int_D \left| \nabla \left(v^\varepsilon + \int_0^t h dW \right) \right|^{q-2} \nabla \left(v^\varepsilon + \int_0^t h dW \right) \cdot \nabla v^\varepsilon \, dx \\ & - \int_D \left| \nabla \left(v^\varepsilon + \int_0^t h dW \right) \right|^{p(t,x)-2} \nabla \left(v^\varepsilon + \int_0^t h dW \right) \cdot \nabla v^\varepsilon \, dx \\ & + \int_D v^\varepsilon \beta(S) \, dx. \end{aligned}$$

Using relation

$$\nabla v^\varepsilon = \nabla \left(v^\varepsilon + \int_0^t h dW \right) - \nabla \left(\int_0^t h dW \right)$$

in the above equality, one obtains

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|^2_{L^2(D)} + \varepsilon \int_D \left| \nabla \left(v^\varepsilon + \int_0^t h dW \right) \right|^q dx + \int_D \left| \nabla \left(v^\varepsilon + \int_0^t h dW \right) \right|^{p(t,x)} dx \\ = \varepsilon \int_D \left| \nabla \left(v^\varepsilon + \int_0^t h dW \right) \right|^{q-2} \nabla \left(v^\varepsilon + \int_0^t h dW \right) \cdot \nabla \left(\int_0^t h dW \right) dx \\ + \int_D \left| \nabla \left(v^\varepsilon + \int_0^t h dW \right) \right|^{p(t,x)-2} \nabla \left(v^\varepsilon + \int_0^t h dW \right) \cdot \nabla \left(\int_0^t h dW \right) dx \\ + \int_D v^\varepsilon \beta(S) \, dx. \end{aligned}$$

Applying Young's inequality, we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v^\varepsilon\|^2_{L^2(D)} &+ \varepsilon\int_D\left|\nabla\left(v^\varepsilon+\int_0^t hdW\right)\right|^q dx+\int_D\left|\nabla\left(v^\varepsilon+\int_0^t hdW\right)\right|^{p(t,x)}dx\\ &\le \frac{\varepsilon}{q'}\int_D\left|\nabla\left(v^\varepsilon+\int_0^t hdW\right)\right|^q dx+\frac{\varepsilon}{q}\int_D\left|\nabla\left(\int_0^t hdW\right)\right|^q dx\\ &+\int_D\frac{1}{p'(t,x)}\left|\nabla\left(v^\varepsilon+\int_0^t hdW\right)\right|^{p(t,x)}dx\\ &+\int_D\frac{1}{p(t,x)}\left|\nabla\left(\int_0^t hdW\right)\right|^{p(t,x)}dx+\int_D v^\varepsilon\beta(S)\,dx.\end{aligned}$$

By collecting similar terms, using the conjugate exponent identities

$$1-\frac{1}{q'}=\frac{1}{q}\quad\text{and}\quad 1-\frac{1}{p'(t,x)}=\frac{1}{p(t,x)},$$

and applying the Cauchy-Schwarz inequality followed by Young's inequality, one gets

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v^\varepsilon\|^2_{L^2(D)}+\varepsilon\int_D\frac{1}{q}\left|\nabla\left(v^\varepsilon+\int_0^t hdW\right)\right|^q dx+\int_D\frac{1}{p(t,x)}\left|\nabla\left(v^\varepsilon+\int_0^t hdW\right)\right|^{p(t,x)}dx\\ &\le\varepsilon\int_D\frac{1}{q}\left|\nabla\int_0^t hdW\right|^q dx+\int_D\frac{1}{p(t,x)}\left|\nabla\int_0^t hdW\right|^{p(t,x)}dx+\int_D v^\varepsilon\beta(S)\,dx\\ &\le\varepsilon\int_D\frac{1}{q}\left|\nabla\int_0^t hdW\right|^q dx+\int_D\frac{1}{p(t,x)}\left|\nabla\int_0^t hdW\right|^{p(t,x)}dx+\frac{1}{2}\|\beta(S)\|^2_{L^2(D)}+\frac{1}{2}\|v^\varepsilon(t)\|^2_{L^2(D)}\\ &\le R+\frac{1}{2}\|v^\varepsilon(t)\|^2_{L^2(D)},\end{aligned}\tag{3.28}$$

where

$$\begin{aligned}R(t):=&\varepsilon\int_D\frac{1}{q}\left|\nabla\int_0^t hdW\right|^q dx+\int_D\frac{1}{p^-}\left|\nabla\int_0^t hdW\right|^{p(t,x)}dx+\frac{1}{2}\|\beta(S)\|^2_{L^2(D)}\\ &\le\varepsilon\int_D\frac{1}{q}\left|\nabla\int_0^t hdW\right|^q dx+\int_D\frac{1}{p^-}\left|\nabla\int_0^t hdW\right|^{p^+}dx+C(|D|)+\frac{1}{2}\|\beta(S)\|^2_{L^2(D)},\end{aligned}\tag{3.29}$$

with $C(|D|)>0$ not depending on w.
Dropping the nonnegative term on the left-hand side of (3.28), yields

$$\frac{1}{2}\frac{d}{dt}\|v^\varepsilon(t)\|^2_{L^2(D)}\le R+\frac{1}{2}\|v^\varepsilon(t)\|^2_{L^2(D)}.\tag{3.30}$$

We define

$$G(t):=\|v^\varepsilon(t)\|^2_{L^2(D)}.$$

Then,

$$G'(t)\le 2R+G(t).$$

Multiplying both sides by the positive factor e^{-t} gives

$$\frac{d}{dt}\left(e^{-t}G(t)\right) \leq 2Re^{-t}. \tag{3.31}$$

By integrating (3.31) over $[0,t]$, we obtain

$$G(t)e^{-t} - G(0) \leq 2\int_0^t R(s)e^{-s}ds,$$

and therefore

$$G(t) \leq G(0)e^{t} + 2e^{t}\int_0^t R(s)e^{-s}ds.$$

Since $t \leq T$, $e^{-s} \leq 1$ and $R(s) > 0$, it follows that, for every $t \in [0,T]$,

$$\|v^\varepsilon(t)\|^2_{L^2(D)} \leq \|u_0\|^2_{L^2(D)}e^T + 2e^T\int_0^T R(s)ds. \tag{3.32}$$

Using (3.32) in (3.28) and as $e^T \geq 1$, one obtains

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v^\varepsilon\|^2_{L^2(D)} + \varepsilon\int_D \frac{1}{q}\left|\nabla\Big[v^\varepsilon + \int_0^t hdW\Big]\right|^q dx + \int_D \frac{1}{p(t,x)}\left|\nabla\Big[v^\varepsilon + \int_0^t hdW\Big]\right|^{p(t,x)} dx \\
&\leq \varepsilon\int_D \frac{1}{q}\left|\nabla\int_0^t hdW\right|^q dx + \int_D \frac{1}{p(t,x)}\left|\nabla\int_0^t hdW\right|^{p(t,x)} dx + \frac{1}{2}\|\beta(S)\|^2_{L^2(D)} + \frac{1}{2}\|v^\varepsilon\|^2_{L^2(D)} \\
&\leq R + \|u_0\|^2_{L^2(D)}e^T + e^T\int_0^T R(s)ds \\
&\leq \left(\|u_0\|^2_{L^2(D)} + R(t) + \int_0^T R(s)ds\right)e^T \in L^1(0,T).
\end{aligned} \tag{3.33}$$

Therefore $(v^\varepsilon)_{\varepsilon>0}$ is bounded in $C([0,T],L^2(D))$ and in X_T, and

$$\varepsilon\int_{\mathcal{Q}_T}\left|\nabla\Big[v^\varepsilon + \int_0^t hdW\Big]\right|^q dxdt < C_1 := (T+1)\left(\|u_0\|^2_{L^2(D)} + \int_0^T R(t)dt\right)e^T > 0.$$

By extracting a subsequence, still denoted by v^ε (see [10], p.24), one deduces that there exists $v \in X_T \cap L^\infty(0,T;L^2(D))$, such that v^ε converges to v weakly in X_T and weak-* in $L^\infty(0,T;L^2(D))$.
Let $\mathbf{A}$ denote any limit point of the weak convergence in $L^{p'(.)}(\mathcal{Q}_T)$ of

$$\left|\nabla(v^\varepsilon + \int_0^t h\,dW)\right|^{p(t,x)-2}\nabla(v^\varepsilon + \int_0^t h\,dW),$$

and let χ be a weak limit in $L^2(D)$ of $v^\varepsilon(T)$.
Therefore, by integrating equation (3.28) over $[0,T]$ and passing to the limit, thanks to $(H_{3,1})$ and the weak lower semicontinuity of the modular ensured by

Theorem 3.2.9 in [12], we obtain

$$\begin{aligned}
&\int_0^T \frac{1}{2}\frac{d}{dt}\left\|v^\varepsilon\right\|^2_{L^2(D)} dt + \varepsilon\int_0^T\int_D \left|\nabla\Big[v^\varepsilon + \int_0^t hdW\Big]\right|^q dxdt + \int_0^T\int_D \left|\nabla\Big[v^\varepsilon + \int_0^t hdW\Big]\right|^{p(t,x)} dxdt \\
&\quad \le \varepsilon\int_0^T\int_D \frac{1}{q}\left|\nabla\int_0^t hdW\right|^q dxdt + \int_D \frac{1}{p(t,x)}\left|\nabla\int_0^t hdW\right|^{p(t,x)} dx + \int_0^T\int_D v^\varepsilon\beta(S)dxdt \\
&\quad \le \varepsilon\int_0^T\int_D \frac{1}{q}\left|\nabla\int_0^t hdW\right|^q dxdt + \int_0^T\int_D \frac{1}{p(t,x)}\left|\nabla\int_0^t hdW\right|^{p(t,x)} dxdt + \int_0^T\int_D v^\varepsilon\beta(S)dxdt.
\end{aligned} \tag{3.34}$$

Thus, as $\varepsilon \to 0$, one has

$$\begin{aligned}
&\frac{1}{2}\left\|\chi\right\|^2_{L^2(D)} + \int_0^T\int_D \frac{1}{p(t,x)}\left|\nabla\Big[v + \int_0^t hdW\Big]\right|^{p(t,x)} dxdt \\
&\qquad \le \frac{1}{2}\left\|u_0\right\|^2_{L^2(D)} + \int_0^T\int_D \frac{1}{p(t,x)}\left|\nabla\int_0^t hdW\right|^{p(t,x)} dxdt \\
&\qquad + \int_0^T\int_D v\beta(S)dxdt.
\end{aligned} \tag{3.35}$$

For any $\phi \in \mathcal{D}([0,T]\times D)$ and by definition of v^ε (see (3.10)), we get by (3.26),

$$\begin{aligned}
&\int_D v^\varepsilon(T,x)\phi(T,x)dx - \int_{\mathcal{Q}_T} v^\varepsilon(t,x)\partial_t\phi(t,x)dxdt \\
&+ \varepsilon\int_{\mathcal{Q}_T}\left|\nabla\left(v^\varepsilon + \int_0^t hdW\right)\right|^{q-2}\left(v^\varepsilon + \int_0^t hdW\right).\nabla\phi dxdt \\
&+ \int_{\mathcal{Q}_T}\left|\left(v^\varepsilon + \int_0^t hdW\right)\right|^{p(t,x)-2}\left(v^\varepsilon + \int_0^t hdW\right).\nabla\phi dxdt \\
&= \int_D u_0(x)\phi(0,x)dx + \int_{\mathcal{Q}_T}\beta(S)\phi dxdt.
\end{aligned} \tag{3.36}$$

Thus, as $\varepsilon \to 0$, one has

$$\begin{aligned}
&\int_D \chi\phi(T,x)dx - \int_{\mathcal{Q}_T} v(t,x)\partial_t\phi(t,x)dxdt + \int_{\mathcal{Q}_T}\mathbf{A}.\nabla\phi dxdt \\
&\quad = \int_D u_0(x)\phi(0,x)dx + \int_{\mathcal{Q}_T}\beta(S)\phi dxdt.
\end{aligned} \tag{3.37}$$

Hence,

$$\partial_t v - \operatorname{div}\mathbf{A} = \beta(S) \tag{3.38}$$

in the sense of distributions and $\partial_t v \in X_T'$.
Since $C^\infty([0,T],\mathcal{D}(D))$ is dense in $\mathcal{W}_T$, then for all $\phi \in \mathcal{W}_T$, it follows

$$\int_D \chi(x)\phi(T,x)dx - \langle\partial_t\phi, v\rangle + \int_{\mathcal{Q}_T}\mathbf{A}.\nabla\phi dxdt = \int_D u_0(x)\phi(0,x)dx + \int_{\mathcal{Q}_T}\beta(S)\phi dxdt. \tag{3.39}$$

Since $\partial_t v \in X_T'$ and $v \in X_T$, $v \in \mathcal{W}_T$ and thanks to (2.2),

$$\langle \partial_t \phi, \mathbf{1}_{[0,T]} v \rangle_{X_T} = \langle \phi(T,x), v(T,x) \rangle_{L^2(D)} - \langle \phi(0,x), v(0,x) \rangle_{L^2(D)} - \langle v_t, \mathbf{1}_{[0,T]} \phi \rangle_{X_T}. \tag{3.40}$$

From (3.40) we have

$$\begin{aligned} &\int_D [\chi(x) - v(T,x)]\, \phi(T,x) dx + \langle \partial_t v, \phi \rangle + \int_{\mathcal{Q}_T} \mathbf{A}.\nabla \phi dx dt \\ &\quad = \int_D [u_0(x) - v(0,x)]\, \phi(0,x) dx + \int_{\mathcal{Q}_T} \beta(S) \phi dx dt, \end{aligned} \tag{3.41}$$

As $\langle \partial_t v - \operatorname{div} \mathbf{A}, \phi \rangle = \int_{\mathcal{Q}_T} \beta(S) \phi dx dt$, then

$$\int_D [\chi(x) - v(T,x)]\, \phi(T,x) dx = \int_D [u_0(x) - v(0,x)]\, \phi(0,x) dx$$

and one gets that $v(T) = \chi$ and $v(0) = u_0$. Observe that in our reasoning the choice of T is arbitrary. Consequently, we can conclude that $v^\varepsilon(t)$ converges weakly to $v(t)$ in $L^2(D)$.
We also recall that, since $v \in \mathcal{W}_T$, then

$$v \in C\big([0,T]; L^2(D)\big).$$

Moreover, testing (3.38) with v and integrating over $\mathcal{Q}_t$ yield the following energy identity, valid for every fixed $t \in [0,T]$.

$$\frac{1}{2} \|v(t)\|^2_{L^2(D)} + \int_{\mathcal{Q}_t} \mathbf{A}.\nabla v dx ds = \frac{1}{2} \|u_0\|^2_{L^2(D)} + \int_{\mathcal{Q}_t} v.\beta(S) dx ds. \tag{3.42}$$

Moreover, since

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|^2_{L^2(D)} + \varepsilon \int_D \left| \nabla \Big[v^\varepsilon + \int_0^t h dW \Big] \right|^q dx + \int_D \left| \nabla \Big[v^\varepsilon + \int_0^t h dW \Big] \right|^{p(t,x)} dx \\ &\quad = \varepsilon \int_D \left| \nabla \Big[v^\varepsilon + \int_0^t h dW \Big] \right|^{q-2} \nabla \left(v^\varepsilon + \int_0^t h dW \right) . \nabla \int_0^t h dW dx \\ &\quad + \int_D \left| \nabla \Big[v^\varepsilon + \int_0^t h dW \Big] \right|^{p(t,x)-2} \nabla \left(v^\varepsilon + \int_0^t h dW \right) . \nabla \int_0^t h dW dx + \int_D v^\varepsilon \beta(S) dx, \end{aligned}$$

the following holds.

$$\begin{aligned} &\frac{1}{2} \|v^\varepsilon\|^2_{L^2(D)} + \int_{\mathcal{Q}_t} \left| \nabla \Big[v^\varepsilon + \int_0^s h dW \Big] \right|^{p(s,x)} dx ds \\ &\leq \varepsilon \int_{\mathcal{Q}_t} \left| \nabla \Big[v^\varepsilon + \int_0^s h dW \Big] \right|^{q-2} \nabla \left(v^\varepsilon + \int_0^s h dW \right) . \nabla \int_0^s h dW dx ds + \frac{1}{2} \|u_0\|^2_{L^2(D)} \\ &+ \int_{\mathcal{Q}_t} \left| \nabla \Big[v^\varepsilon + \int_0^s h dW \Big] \right|^{p(s,x)-2} \nabla \left(v^\varepsilon + \int_0^s h dW \right) . \nabla \int_0^s h dW dx ds + \int_{\mathcal{Q}_t} v^\varepsilon \beta(S) dx ds, \end{aligned} \tag{3.43}$$

and at the limit as $\varepsilon \to 0$, one gets

$$\begin{aligned}&\frac{1}{2}\left\|v\right\|_{L^2(D)}^2+\limsup_{\varepsilon}\int_{\mathcal{Q}_t}\left|\nabla\left[v^{\varepsilon}+\int_0^s hdW\right]\right|^{p(s,x)}dxds\\&\leq\int_{\mathcal{Q}_t}A.\nabla\int_0^s hdWdxds+\int_{\mathcal{Q}_t}v\beta(S)dxds+\frac{1}{2}\left\|u_0\right\|_{L^2(D)}^2.\end{aligned}$$

Using (3.42),

$$\limsup_{\varepsilon}\int_{\mathcal{Q}_t}\left|\nabla\left[v^{\varepsilon}+\int_0^s hdW\right]\right|^{p(s,x)}dxds\leq\int_{\mathcal{Q}_t}\mathbf{A}.\nabla\left(v+\int_0^s hdW\right)dxds, \tag{3.44}$$

which is equivalent to say

$$\begin{aligned}\limsup_{\varepsilon}\int_{\mathcal{Q}_t}\left|\nabla\left[v^{\varepsilon}+\int_0^s hdW\right]\right|^{p(s,x)-2}\nabla\left[v^{\varepsilon}+\int_0^s hdW\right].\nabla\left[v^{\varepsilon}+\int_0^s hdW\right]dxds\\\leq\int_{\mathcal{Q}_t}\mathbf{A}.\nabla\left(v+\int_0^s hdW\right)dxds.\end{aligned} \tag{3.45}$$

Since $\left|\nabla\left[v^{\varepsilon}+\int_0^s hdW\right]\right|^{p(s,x)-2}\nabla\left[v^{\varepsilon}+\int_0^s hdW\right]\rightharpoonup\mathbf{A}$ in $L^{p'}(\mathcal{Q}_T)$ and $\nabla v^{\varepsilon}\rightharpoonup\nabla v$ in $L^{p(.)}(\mathcal{Q}_T)^d$, then, by the definition of weak convergence,

$$\int_{\mathcal{Q}_t}\mathbf{A}.\left(\nabla(v^{\varepsilon}-v)\right)dxdt\to 0 \tag{3.46}$$

and

$$\begin{aligned}&\int_{\mathcal{Q}_t}\left[\left|\nabla\left[v^{\varepsilon}+\int_0^s hdW\right]\right|^{p(s,x)-2}\nabla\left[v^{\varepsilon}+\int_0^s hdW\right]\right].\nabla\left(v+\int_0^s hdW\right)dxds\\&\quad\to\int_{\mathcal{Q}_t}\mathbf{A}.\nabla\left(v+\int_0^s hdW\right)dxds,\end{aligned} \tag{3.47}$$

as $\varepsilon\to 0$, since

$$\nabla\left(v^{\varepsilon}-v\right)=\nabla\left(v^{\varepsilon}+\int_0^t hdW\right)-\nabla\left(v+\int_0^t hdW\right),$$

one concludes that, thanks to (3.45), (3.46) and (3.47),

$$\begin{aligned}
&\limsup_{\varepsilon}\int_{\mathcal{Q}_t}\left[\left|\nabla\left[v^\varepsilon+\int_0^s hdW\right]\right|^{p(s,x)-2}\nabla\left[v^\varepsilon+\int_0^s hdW\right]-\mathbf{A}\right].\nabla\left(v^\varepsilon-v\right)dxds\\
&=\limsup_{\varepsilon}\int_{\mathcal{Q}_t}\left[\left|\nabla\left[v^\varepsilon+\int_0^s hdW\right]\right|^{p(s,x)-2}\nabla\left[v^\varepsilon+\int_0^s hdW\right]\right]\\
&.\nabla\left(\left(v^\varepsilon+\int_0^s hdW\right)-\left(v+\int_0^s hdW\right)\right)dxds\\
&=\limsup_{\varepsilon}\int_{\mathcal{Q}_t}\left|\nabla\left[v^\varepsilon+\int_0^s hdW\right]\right|^{p(s,x)}dxds-\int_{\mathcal{Q}_t}\mathbf{A}.\nabla\left(v+\int_0^s hdW\right)dxds\leq 0.
\end{aligned}\tag{3.48}$$

Consider the functional $J:X_T\to\mathbb{R}$ defined by

$$J(w)=\int_{\mathcal{Q}_T}\frac{1}{p(t,x)}|\nabla w(t,x)|^{p(t,x)}\,dx\,dt.\tag{3.49}$$

This functional is proper, convex, and continuous (see [4]). Furthermore, it is Gâteaux differentiable throughout its domain. Consequently, the Gâteaux derivative DJ constitutes a single-valued maximal monotone operator mapping X_T to its topological dual X_T' (see [2]). It follows directly from (3.49) that DJ coincides with the negative $p(.)$-Laplacian operator, i.e., $DJ=-\Delta_{p(.)}$. Indeed,

$$\begin{aligned}
\langle DJ(w),\phi\rangle&=\lim_{\epsilon\to 0}\frac{J(w+\epsilon\phi)-J(w)}{\epsilon}\\
&=\int_{\mathcal{Q}_T}\frac{1}{p(t,x)}\lim_{\epsilon\to 0}\frac{|\nabla w+\epsilon\phi|^{p(t,x)}-|\nabla w|^{p(t,x)}}{\epsilon}dxdt\\
&=\int_{\mathcal{Q}_T}|\nabla w(t,x)|^{p(t,x)-2}\nabla w(t,x)\cdot\nabla\phi(t,x)\,dx\,dt.
\end{aligned}\tag{3.50}$$

For any pair of functions $w,z\in X_T$, the duality product is given explicitly by

$$\langle DJ(w),z\rangle_{X_T',X_T}=\int_{\mathcal{Q}_T}|\nabla w(t,x)|^{p(t,x)-2}\nabla w(t,x)\cdot\nabla z(t,x)\,dx\,dt.\tag{3.51}$$

Since

$$u^\varepsilon=v^\varepsilon+\int_0^s hdW\in X_T\text{ and }v^\varepsilon-v\in X_T.$$

One deduces from (3.51) that

$$\begin{aligned}
&\langle DJ(v^\varepsilon+\int_0^s hdW),v^\varepsilon-v\rangle_{X_T',X_T}\\
&=\int_{\mathcal{Q}_T}\left|\nabla v^\varepsilon+\int_0^s hdW\right|^{p(t,x)-2}\nabla\left(v^\varepsilon+\int_0^s hdW\right)\cdot\nabla\left(v^\varepsilon-v\right)\,dx\,dt
\end{aligned}\tag{3.52}$$

and

$$-\int_{\mathcal{Q}_T}\mathbf{A}\cdot\nabla\left(v^\varepsilon-v\right)dxdt=\int_{\mathcal{Q}_T}\operatorname{div}\mathbf{A}\cdot\left(v^\varepsilon-v\right)dxdt.\tag{3.53}$$

In terms of the functional DJ, the preceding estimates (3.48), (3.52) and (3.53) yields

$$\limsup_{\varepsilon} \left\langle DJ\Big(v^{\varepsilon} + \int_0^s h dW\Big) + \operatorname{div} \mathbf{A}, v^{\varepsilon} - v \right\rangle_{X_T} \leq 0.$$

By the properties of maximal monotone operators in reflexive Banach spaces (see [2]), it follows that

$$DJ\Big(v + \int_0^t h\, dW\Big) = -\operatorname{div} \mathbf{A},$$

and consequently

$$\Big\langle DJ\Big(v^{\varepsilon} + \int_0^s h\, dw\Big),\ v^{\varepsilon} + \int_0^s h\, dW \Big\rangle_{X_T} \to \Big\langle DJ\Big(v + \int_0^s h\, dW\Big),\ v + \int_0^s h\, dW \Big\rangle_{X_T}.$$

Hence, recalling that weak convergence together with the convergence of the modular in $L^{p(\cdot,\cdot)}(\mathcal{Q}_T)$ and (H_3) implies strong convergence (see [11], Appendix Corollary A.3), we obtain, on the one hand,

$$\nabla\left(v^{\varepsilon} + \int_0^t h\, dW\right) \to \nabla\left(v + \int_0^t h\, dW\right) \text{ in } \left(L^{p(.,.)}(\mathcal{Q}_T)\right)^d$$

and on the other hand,

$$\nabla v^{\varepsilon} \to \nabla v \text{ in } \left(L^{p(.,.)}(\mathcal{Q}_T)\right)^d.$$

Therefore, we can identify

$$\mathbf{A} = \left|\nabla\left(v + \int_0^t h\, dW\right)\right|^{p(t,x)-2} \nabla\left(v + \int_0^t h\, dW\right) \tag{3.54}$$

and then, replacing $\mathbf{A}$ by its value in (3.42), one obtains the following equality.

$$\begin{aligned} &\frac{1}{2}\|v(t)\|^2_{L^2(D)} + \int_{\mathcal{Q}_t} \left|\nabla(v + \int_0^t h\, dW)\right|^{p(s,x)-2} \nabla(v + \int_0^s h\, dW).\nabla v dx dt \\ &\quad = \frac{1}{2}\|u_0\|^2_{L^2(D)} + \int_{\mathcal{Q}_t} v.\beta(S) dx ds. \end{aligned} \tag{3.55}$$

By passing to the lim sup in (3.43).

$$\begin{aligned} &\frac{1}{2}\limsup_{\varepsilon}\|v^{\varepsilon}\|^2_{L^2(D)} + \int_{\mathcal{Q}_t}\left|\nabla\Big[v + \int_0^s h dW\Big]\right|^{p(s,x)} dx ds \\ &\leq \int_{\mathcal{Q}_t}\left|\nabla\Big[v + \int_0^s h dW\Big]\right|^{p(s,x)-2} \nabla\left(v + \int_0^s h dW\right).\nabla \int_0^s h dW dx ds + \int_{\mathcal{Q}_t} v\beta(S) dx ds \\ &+ \frac{1}{2}\|u_0\|^2_{L^2(D)}. \end{aligned} \tag{3.56}$$

Let's replace (3.55) in (3.56) to get

$$\begin{aligned}
&\frac{1}{2}\limsup_{\varepsilon}\|v^{\varepsilon}\|^2_{L^2(D)}+\int_{\mathcal{Q}_t}\left|\nabla\Big[v+\int_0^s hdW\Big]\right|^{p(s,x)}dxds\\
&\leq\int_{\mathcal{Q}_t}\left|\nabla\Big[v+\int_0^s hdW\Big]\right|^{p(s,x)-2}\nabla\left(v+\int_0^s hdW\right).\nabla\int_0^s hdwdxds\\
&+\frac{1}{2}\|v(t)\|^2_{L^2(D)}+\int_{\mathcal{Q}_t}\left|\nabla(v+\int_0^t h\,dW)\right|^{p(s,x)-2}\nabla(v+\int_0^s h\,dW).\nabla vdxds\\
&=\frac{1}{2}\|v(t)\|^2_{L^2(D)}+\int_{\mathcal{Q}_t}\left|\nabla\Big[v+\int_0^s hdW\Big]\right|^{p(s,x)}dxds.
\end{aligned}\tag{3.57}$$

By property of convexity of $\|.\|_{L^2(D)}$ and (3.57), one obtains

$$\frac{1}{2}\limsup_{\varepsilon}\|v^{\varepsilon}\|^2_{L^2(D)}\leq\frac{1}{2}\|v(t)\|^2_{L^2(D)}\leq\frac{1}{2}\liminf_{\varepsilon}\|v^{\varepsilon}\|^2_{L^2(D)}\Leftrightarrow\limsup_{\varepsilon}\|v^{\varepsilon}\|^2_{L^2(D)}=\liminf\|v^{\varepsilon}(t)\|^2_{L^2(D)}.\tag{3.58}$$

Since $v_{\varepsilon}(t)\rightharpoonup v(t)$ weakly in $L^2(D)$, (3.58), implies the strong convergence (see [5, 9]).

$$\lim_{\varepsilon\to0}\left\|v^{\varepsilon}(t)\right\|_{L^2(D)}\to\left\|v(t)\right\|_{L^2(D)}\Rightarrow\lim_{\varepsilon\to0}\|v^{\varepsilon}(t)-v(t)\|_{L^2(D)}=0.$$

As $(v^{\varepsilon})_{\varepsilon>0}$ is bounded in $C([0,T],L^2(D))$, one has

$$\begin{aligned}
\|v^{\varepsilon}(t)-v(t)\|^2_{L^2(D)}&\leq2\|v^{\varepsilon}(t)\|^2_{L^2(D)}+2\|v(t)\|^2_{L^2(D)}\\
&\leq2\sup_{\varepsilon}\sup_{t}\|v^{\varepsilon}(t)\|^2_{L^2(D)}+2\sup_{t}\|v(t)\|^2_{L^2(D)}\\
&\leq4\sup_{t}\|v(t)\|^2_{L^2(D)}=:K_1\in L^1(0,T).
\end{aligned}\tag{3.59}$$

Consequently, using Lebesgue dominated convergence theorem one deduces that

$$\lim_{\varepsilon\to0}\int_0^T\int_D|(v^{\varepsilon}-v)(t,x)|^2\,dxdt=\int_0^T\lim_{\varepsilon\to0}\int_D|((v^{\varepsilon}-v)(t,x))|^2\,dxdt=0.$$

Hence, $v^{\varepsilon}(t)$ converges to $v(t)$ in $L^2(\mathcal{Q}_T)$ and in particular in the space X_T.
Since every convergent subsequence of v^{ε} has the same limit v, knowing to the uniqueness of the solution to problem (3.6), it follows that the whole sequence v^{ε} converges to v.
We have already established that $v^{\varepsilon}(\omega)\to v(\omega)$ in $L^2(Q_T)$ for P-almost every $\omega\in\Omega$ as $\varepsilon\downarrow0$.
Integrating inequality (3.33) over $[0,t]$, one has

$$\begin{aligned}
&\int_0^t\frac{1}{2}\frac{d}{d\sigma}\|v^{\varepsilon}\|^2_{L^2(D)}\,d\sigma+\varepsilon\int_{\mathcal{Q}_t}\frac{1}{q}\left|\nabla\Big[v^{\varepsilon}+\int_0^{\sigma}hdW\Big]\right|^q dxd\sigma\\
&+\int_{\mathcal{Q}_t}\frac{1}{p(\sigma,x)}\left|\nabla\Big[v^{\varepsilon}+\int_0^{\sigma}hdW\Big]\right|^{p(\sigma,x)}dxd\sigma\leq\int_0^t\left(\|u_0\|^2_{L^2(D)}+R(\sigma)+\int_0^T R(s)ds\right)e^T d\sigma.
\end{aligned}\tag{3.60}$$

We deduce that this inequality holds for all $t \in [0,T]$, P-almost surely.

$$\begin{aligned}\frac{1}{2}\left\|v^{\varepsilon}(t)\right\|^2_{L^2(D)} + \frac{\varepsilon}{q}\int_{\mathcal{Q}_t}\left|\nabla\Big[v^{\varepsilon}+\int_0^{\sigma} hdW\Big]\right|^q dxd\sigma + \int_{\mathcal{Q}_t}\frac{1}{p(\sigma,x)}\left|\nabla\Big[v^{\varepsilon}+\int_0^{\sigma} hdW\Big]\right|^{p(\sigma,x)} dxd\sigma \\ \leq e^T\int_0^T\left(\left\|u_0\right\|^2_{L^2(D)} + R(\sigma) + \int_0^T R(s)ds\right)d\sigma + \frac{1}{2}\left\|u_0\right\|^2_{L^2(D)} \\ = \left((T+1)\left\|u_0\right\|^2_{L^2(D)} + (T+1)\int_0^T R(t)dt\right)e^T.\end{aligned} \tag{3.61}$$

Hence, one has

$$\begin{aligned}&\frac{1}{2}\left\|v^{\varepsilon}(t)\right\|^2_{L^2(D)} + \frac{\varepsilon}{q}\int_{\mathcal{Q}_t}\left|\nabla\Big[v^{\varepsilon}+\int_0^{\sigma} hdW\Big]\right|^q dxd\sigma + \int_{\mathcal{Q}_t}\frac{1}{p^+}\left|\nabla\Big[v^{\varepsilon}+\int_0^{\sigma} hdW\Big]\right|^{p(\sigma,x)} dxd\sigma \\ &\leq \left((T+1)\left\|u_0\right\|^2_{L^2(D)} + (T+1)\left(\int_{\mathcal{Q}_T}\frac{\varepsilon}{q}\left|\nabla\int_0^t hdW\right|^q dxdt + \int_{\mathcal{Q}_T}\frac{1}{p^-}\left|\nabla\int_0^t hdW\right|^{p(t,x)} dxdt\right)\right)e^T \\ &+ (T+1)\left(\frac{1}{2}\|\beta(S)\|^2_{L^2(\mathcal{Q}_T)}\right)e^T \\ &\leq \max\left(1,\frac{\varepsilon}{q},\frac{1}{p^-}\right)\times \\ &(T+1)\,e^T\left(\left\|u_0\right\|^2_{L^2(D)} + \int_{\mathcal{Q}_T}\left|\nabla\int_0^t hdW\right|^q dxdt + \int_{\mathcal{Q}_T}\left|\nabla\int_0^t hdW\right|^{p(t,x)} dxdt + \|\beta(S)\|^2_{L^2(\mathcal{Q}_T)}\right).\end{aligned} \tag{3.62}$$

As $q > 2$,

$$\begin{cases}\displaystyle\int_{\mathcal{Q}_t}\left|\nabla\Big[v^{\varepsilon}+\int_0^{\sigma} hdW\Big]\right|^{p(\sigma,x)} dxd\sigma \leq \max\left(1,\frac{\varepsilon}{2},\frac{1}{p^-}\right)(T+1)\,e^T p^+\times \\ \left(\left\|u_0\right\|^2_{L^2(D)} + \displaystyle\int_{\mathcal{Q}_T}\left|\nabla\int_0^t hdW\right|^q dxdt + \int_{\mathcal{Q}_T}\left|\nabla\int_0^t hdW\right|^{p(t,x)} dxdt + \|\beta(S)\|^2_{L^2(\mathcal{Q}_T)}\right) \\ \text{and} \\ \left\|v^{\varepsilon}(t)\right\|^2_{L^2(D)} \leq \max\left(2,\varepsilon,\frac{2}{p^-}\right)(T+1)\,e^T\times \\ \left(\left\|u_0\right\|^2_{L^2(D)} + \displaystyle\int_{\mathcal{Q}_T}\left|\nabla\int_0^t hdW\right|^q dxdt + \int_{\mathcal{Q}_T}\left|\nabla\int_0^t hdW\right|^{p(t,x)} dxdt + \|\beta(S)\|^2_{L^2(\mathcal{Q}_T)}\right).\end{cases} \tag{3.63}$$

By passing to the lim sup in (3.63), one gets

$$\begin{cases} \displaystyle\int_{\mathcal{Q}_t}\left|\nabla\Big[v+\int_0^\sigma h dW\Big]\right|^{p(\sigma,x)} dxds \le \max\left(1,\frac{\varepsilon}{2},\frac{1}{p^-}\right)(T+1)\,e^T p^+\times \\ \left(\|u_0\|^2_{L^2(D)}+\displaystyle\int_{\mathcal{Q}_T}\left|\nabla\int_0^t h dW\right|^q dxdt+\int_{\mathcal{Q}_T}\left|\nabla\int_0^t h dW\right|^{p(t,x)} dxdt+\|\beta(S)\|^2_{L^2(\mathcal{Q}_T)}\right) \\ \text{and} \\ \|v(t)\|^2_{L^2(D)} \le \max\left(2,\varepsilon,\frac{2}{p^-}\right)(T+1)\,e^T\times \\ \left(\|u_0\|^2_{L^2(D)}+\displaystyle\int_{\mathcal{Q}_T}\left|\nabla\int_0^t h dW\right|^q dxdt+\int_{\mathcal{Q}_T}\left|\nabla\int_0^t h dW\right|^{p(t,x)} dxdt+\|\beta(S)\|^2_{L^2(\mathcal{Q}_T)}\right). \end{cases} \tag{3.64}$$

As $\varepsilon\to 0$, using (3.63) and (3.64), one deduces that

$$\begin{aligned} &\int_0^t \|v^\varepsilon - v\|^2_{L^2(D)}\,dt \le 2\int_0^t \|v^\varepsilon\|^2_{L^2(D)}\,dt + 2\int_0^t \|v\|^2_{L^2(D)}\,dt \\ &\le \max\left(8,\frac{8}{p^-}\right)(T+1)\,e^T\times \\ &\left(\|u_0\|^2_{L^2(D)}+\int_{\mathcal{Q}_T}\left|\nabla\int_0^t h dW\right|^q dxdt+\int_{\mathcal{Q}_T}\left|\nabla\int_0^t h dW\right|^{p(t,x)} dxdt+\|\beta(S)\|^2_{L^2(\mathcal{Q}_T)}\right)\in L^1(\Omega). \end{aligned} \tag{3.65}$$

Using Lebesgue dominated convergence theorem and the regularity of h, β, one deduces that

$$\lim_{\varepsilon\to 0}\int_\Omega\int_0^T\int_D |(v^\varepsilon - v)(w,t,x)|^2\,dxdtdP(w) = \int_\Omega \lim_{\varepsilon\to 0}\int_0^T\int_D |(v^\varepsilon - v)(w,t,x)|^2\,dxdtdP(w) = 0,$$

which is equivalent to say that $v^\varepsilon \to v \quad \text{in } L^2(\Omega\times\mathcal{Q}_T)$.
It is already established that, a.s,

$$\|\nabla(v^\varepsilon - v)\|_{L^{p(\cdot)}(\mathcal{Q}_T)} \longrightarrow 0 \quad (\varepsilon\to 0) \text{ (see [9])}.$$

Furthermore, (3.63) and (3.64) guarantees the existence of a constant $C(p^+,p^-)>0$ such that

$$\begin{aligned} &\int_{\mathcal{Q}_T}|\nabla(v^\varepsilon(\omega)-v(\omega))|^{p(t,x)}\,dxdt = \int_{\mathcal{Q}_T}\left|\nabla(v^\varepsilon(\omega)+\int_0^t h\,dW-\int_0^t h\,dW-v(\omega))\right|^{p(t,x)} dx\,dt \\ &\le 2^{p^+}\left(\int_{\mathcal{Q}_T}\left|\nabla v^\varepsilon(\omega)+\int_0^t h\,dW\right|^{p(t,x)} dx\,dt+\int_{\mathcal{Q}_T}\left|\nabla v(\omega)+\int_0^t h\,dW\right|^{p(t,x)} dx\,dt\right) \\ &\le C(p^+,p^-)\left(\int_{\mathcal{Q}_T}\left|\nabla\int_0^t h\,dW\right|^q dx\,dt+\int_{\mathcal{Q}_T}\left|\nabla\int_0^t h\,dW\right|^{p(t,x)} dx\,dt+\|u_0\|^2_{L^2(D)}+\|\beta(S)\|^2_{L^2(\mathcal{Q}_T)}\right) \\ &\in L^1(\Omega), \end{aligned} \tag{3.66}$$

where $C(p^+,p^-)=2^{p^+}\max\left(2,\dfrac{2}{p^-}\right)(T+1)\,e^T p^+$, for all $0<\varepsilon<1$.
Using Lebesgue dominated convergence theorem and the regularity of h, β, one deduces that

$$\lim_{\varepsilon\to 0}\int_\Omega\int_{\mathcal{Q}_T}|\nabla(v^\varepsilon(\omega)-v(\omega))|^{p(t,x)}\,dx\,dt\,dP(\omega)=\int_\Omega\int_{\mathcal{Q}_T}\lim_{\varepsilon\to 0}|\nabla(v^\varepsilon(\omega)-v(\omega))|^{p(t,x)}\,dx\,dt\,dP(\omega)=0. \tag{3.67}$$

Since u^ε converges to $u=v+\displaystyle\int_0^t h\,dW$ in

$$L^q(\Omega,L^q(0,T,W_0^{1,q}(D))\cap N_W^2(0,T,L^2(D))),$$

it follows that u is also a predictable $L^2(D)$-valued process.
Having established the above convergence results, we are now able to pass to the limit in (3.7) to get

$$\begin{aligned}u(w,t,x)=&u(w,s,x)+\int_s^t\operatorname{div}\left(|\nabla u(w,\sigma,x)|^{p(\sigma,x)-2}\,\nabla u(w,\sigma,x)\right)d\sigma\\&+\int_s^t\beta(S(w,\sigma,x))d\sigma+\int_s^t h(w,\sigma,x)dW(w,\sigma,x).\end{aligned} \tag{3.68}$$

4. Stability of the solution

Applying Itô's formula to $\|u^\varepsilon\|^2_{L^2(D)}$ together with (3.14) yields, for every $t\in[0,T]$,

$$\begin{aligned}&\|u^\varepsilon(t)\|^2_{L^2(D)}-\|u_0\|^2_{L^2(D)}+2\int_0^t\int_D\left[\varepsilon\,|\nabla u^\varepsilon|^q+|\nabla u^\varepsilon|^{p(\sigma,x)}\right]dxd\sigma\\&=2\int_0^t\int_D u^\varepsilon\beta(S)dxd\sigma+2\int_0^t\int_D u^\varepsilon h dxdW+\int_0^t\int_D h^2dxd\sigma.\end{aligned} \tag{4.1}$$

The mapping $\xi\mapsto|\xi|^{p(s,x)}$ is convex for every $(s,x)\in Q_t$, since $p(s,x)>1$. Consequently, the functional

$$v\longmapsto\int_{Q_t}|v(s,x)|^{p(s,x)}\,dx\,ds \text{ is weakly lower semicontinuous.}$$

As

$$\nabla u^\varepsilon\rightharpoonup\nabla u \text{ in } \left(L^{p(.,.)}(\mathcal{Q}_T)\right)^d \text{ then } \int_{Q_t}|\nabla u|^{p(s,x)}\,dx\,ds\le\liminf_{\varepsilon\to 0}\int_{Q_t}|\nabla u^\varepsilon|^{p(s,x)}\,dx\,ds.$$

This weak lower semicontinuity property allows us to pass to the limit in the nonlinear term. Using (4.1) and passing to the limit, ones deduces that

$$\begin{aligned}&\|u(t)\|^2_{L^2(D)}-\|u_0\|^2_{L^2(D)}+2\int_0^t\int_D|\nabla u|^{p(\sigma,x)}\,dxd\sigma\\&\le 2\int_0^t\int_D u\beta(S)dxd\sigma+2\int_0^t\int_D uh dxdW+\int_0^t\int_D h^2dxd\sigma.\end{aligned} \tag{4.2}$$

Let (S, h_k), $k = 1, 2$, be two sets of data associated with u_k^ε, respectively. Subtracting the equations satisfied by u_1^ε and u_2^ε, and then applying Itô's formula, we obtain

$$\begin{aligned}
&\left\|\left(u_1^\varepsilon - u_2^\varepsilon\right)(t)\right\|_{L^2(D)}^2 \\
&+ 2\varepsilon \int_0^t \int_D \left[|\nabla u_1^\varepsilon|^{q-2}\nabla u_1^\varepsilon - |\nabla u_2^\varepsilon|^{q-2}\nabla u_2^\varepsilon\right]\nabla\left[u_1^\varepsilon - u_2^\varepsilon\right] dxd\sigma \\
&+ 2\int_0^t \int_D \left[|\nabla u_1^\varepsilon|^{p(\sigma,x)-2}\nabla u_1^\varepsilon - |\nabla u_2^\varepsilon|^{p(\sigma,x)-2}\nabla u_2^\varepsilon\right]\nabla\left[u_1^\varepsilon - u_2^\varepsilon\right] dxd\sigma \\
&\leq 2\int_0^t \int_D (u_1^\varepsilon - u_2^\varepsilon)(h_1 - h_2)dxdW + \int_0^t \int_D (h_1 - h_2)^2 dxd\sigma.
\end{aligned} \tag{4.3}$$

Using (3.54) for $i \in \{1; 2\}$, one has

$$\left|\nabla\left(v_i^\varepsilon + \int_0^t h\, dW\right)\right|^{p(t,x)-2}\nabla\left(v_i^\varepsilon + \int_0^t h\, dW\right) \to \left|\nabla\left(v_i + \int_0^t h\, dW\right)\right|^{p(t,x)-2}\nabla\left(v_i + \int_0^t h\, dW\right)$$

and passing to the limit in (4.3) as $\varepsilon \to 0$, yields

$$\begin{aligned}
&\left\|\left(u_1 - u_2\right)(t)\right\|_{L^2(D)}^2 \\
&+ 2\int_0^t \int_D \left[|\nabla u_1|^{p(\sigma,x)-2}\nabla u_1 - |\nabla u_2|^{p(\sigma,x)-2}\nabla u_2\right]\nabla\left[u_1 - u_2\right] dxd\sigma \\
&\leq 2\int_0^t \int_D (u_1 - u_2)(h_1 - h_2)dxdW + \int_0^t \int_D (h_1 - h_2)^2 dxd\sigma.
\end{aligned} \tag{4.4}$$

Taking the expectation of the supremum in time and applying the Burkholder-Davis-Gundy inequality to the stochastic term, we obtain

$$2E\sup_{t\in[0,T]}\left|\int_0^t \langle h_1 - h_2, u_1 - u_2\rangle_{L^2(D)}\, dW\right| \leq 4C\, E\int_0^T \|h_1 - h_2\|_{L^2(D)}^2\, d\sigma. \tag{4.5}$$

Combining (4.4) and (4.5), we deduce the estimate

$$\begin{aligned}
&E\sup_{t\in[0,T]}\left\|\left(u_1 - u_2\right)(t)\right\|_{L^2(D)}^2 \\
&+ 2E\int_0^T \int_D \left[|\nabla u_1|^{p(\sigma,x)-2}\nabla u_1 - |\nabla u_2|^{p(\sigma,x)-2}\nabla u_2\right]\nabla\left[u_1 - u_2\right] dxd\sigma \\
&\leq (4C+1)E\|h_1 - h_2\|_{N_W^2(0,T;L^2(D))}^2.
\end{aligned} \tag{4.6}$$

4.1. **The addiive case for a general case.** In this section, we consider $h \in N_W^2(0, T; L^2(D))$. Since h is Lipschitz continuous with respect to its fourth variable whenever $S \in N_W^2(0, T; L^2(D))$, it follows that $h \in N_W^2(0, T; L^2(D))$. Thus, h can be expressed as a function of S. To this end, we consider the following

problem.

$$((P),h(.,S))\begin{cases}\partial_t\left[u-\displaystyle\int_0^t h(.,S)\,dW(w,\sigma,x)\right]-\Delta_{p(.)}u=\beta(S) \text{ in } \Omega\times(0,T)\times D\\ u(w,t,x)=0 \text{ on } \Omega\times(0,T)\times\partial D\\ u(w,0,.)=u_0(w,x)\in\Omega\times D.\end{cases}\tag{4.7}$$

Definition 4.1. For a given $(S,h)\in L^2(\Omega\times\mathcal{Q}_T)\times N_w^2(0,T;L^2(D))$, a weak solution of Problem (4.7) associated with (S,h) is defined as the unique element

$$u:=\overline{T}(h)\in N_w^2(0,T;L^2(D)),$$

where $\overline{T}$ denotes the unique continuous extension of T from the dense subspace $S_W^2(0,T;H_0^k(D))$ to $N_W^2(0,T;L^2(D))$.

Definition 4.2. A strong solution to problem (4.7) is a predictable process u such that

$$u\in L^2\left(\Omega;C\left([0,T];L^2(D)\right)\right)\cap X_T\cap N_w^2(0,T;L^2(D)),$$

satisfying $u(w,0,.)=u_0(w,x)\in L^2(\Omega\times D)$ with $\nabla u\in\left(L^{p(.,.)}(\Omega,\mathcal{Q}_T)\right)^d$, and for all $t\in[0,T]$, $x\in D$ and almost surely w in Ω,

$$\begin{aligned}u(w,t,x)=u(w,s,x)+&\int_s^t\left[\operatorname{div}\left(|\nabla u(w,\sigma,x)|^{p(\sigma,x)-2}\nabla u(w,\sigma,x)\right)+\beta(S(w,\sigma,x))\right]d\sigma\\&+\int_s^t h(.,S(w,\sigma,x))dW(w,\sigma,x).\end{aligned}\tag{4.8}$$

Lemma 4.3. *Assume that (H_1), (H_2) and (H_3) hold true. Then, for every $S\in N_W^2(0,T;L^2(D))$, there exists a unique solution u_S to Problem (4.7) associated with $h(.,S)$, in the sense of Definition 4.2 . Moreover, if $S_1,S_2\in N_W^2(0,T;L^2(D))$ and u_{S_1},u_{S_2} denote the corresponding solutions associated with $(h_1(.,S_1),\beta_1(S_1))$ and $(h_2(.,S_2),\beta_2(S_2))$, then for every $t\in(0,T)$,*

$$\begin{aligned}&E\|u_{S_1}-u_{S_2}\|^2_{C([0,T],L^2(D))}\\&+E\int_0^t\int_D\left(|\nabla u_{S_1}|^{p(\sigma,x)-2}\nabla u_{S_1}-|\nabla u_{S_2}|^{p(\sigma,x)-2}\nabla u_{S_2}\right)\cdot\nabla(u_{S_1}-u_{S_2})\,dx\,d\sigma\\&\le CE\int_0^T\left(\|\beta_1(S_1)-\beta_2(S_2)\|^2_{L^2(D)}+\|h_1(.,S_1)-h_2(.,S_2)\|^2_{L^2(D)}\right)d\sigma.\end{aligned}\tag{4.9}$$

Lemma 4.4. *Assume that (H_1), (H_2) and (H_3) hold true. Then, for every $h\in N_W^2(0,T;L^2(D))$, there exists a unique solution u to Problem (4.7), in the sense of Definition 4.1 .*

4.2. **Proof of Lemma 4.4.** By using the monotonicity of the $p(.,.)$-Laplacian and using (4.6), we obtain,

$$\|u_1 - u_2\|_{N^2_W(0,T;L^2(D))} \leq \sqrt{4C+1}\, \|h_1 - h_2\|_{N^2_W(0,T;L^2(D))}\,. \tag{4.10}$$

As a direct consequence of the preceding estimate (4.10), the functional

$$T : S^2_W(0,T;H^k_0(D)) \to N^2_W(0,T;L^2(D)), \quad h \mapsto u,$$

where u is the unique strong solution to (3.1), is Lipschitz continuous. Since $S^2_W(0,T,H^k_0(D))$ is dense in $N^2_W(0,T,L^2(D))$ (see [4]), T is Lipschitz continuous and thus uniformly continuous, and $N^2_W(0,T,L^2(D))$ is a Hilbert space, hence a complete Hausdorff uniform space. Therefore, it follows from the extension theorem for uniformly continuous mappings that T admits a unique uniformly continuous extension (see [4, 15] Theorem 26 p.195).

$$\overline{T} : N^2_W(0,T,L^2(D)) \longrightarrow N^2_W(0,T,L^2(D)).$$

Thus, for every $h \in N^2_W(0,T;L^2(D))$, we define the mild solution $u = \overline{T}(h)$ as the limit of the strong solutions $T(h_n)$ as $h_n \to h$ in $N^2_W(0,T;L^2(D))$.
The theory developed for $h \in S^2_W(0,T,H^k_0(D))$ allows, by extension of the solution operator, the definition of a weak solution (see Definition 4.1) for the problem associated with $h(.,S)$, $S \in N^2_W(0,T,L^2(D))$.

4.3. **Uniqueness of the weak solution.** Finally, let u_1 and u_2 be two weak solutions corresponding to the same initial datum u_0, obtained as continuous extensions from two sequences $(h^{(1)}_m)_{m\geq 1}$ and $(h^{(2)}_m)_{m\geq 1}$, respectively, both converging to the same limit h. Using (3.12),

$$\partial_t(u_1 - u_2) \in L^{q'}(\Omega, L^{q'}(0,T;W^{-1,q'}(D)))$$

almost surely, and

$$u_1 - u_2 \in L^q(\Omega, L^q(0,T;W^{1,q}_0(D))).$$

The duality pairing

$$\langle \partial_t(u_1 - u_2), u_1 - u_2 \rangle$$

is therefore well-defined, and the Lions-Magenes lemma (see [16]) with the pivot space $L^2(D)$ applies. Then, $u_1 - u_2$ satisfies, almost surely,

$$\begin{aligned} \partial_t(u_1 - u_2) - \left(\Delta_{p(.)}u_1 - \Delta_{p(.)}u_2\right) &= h(.,S) - h(.,S) + \beta(S) - \beta(S) \\ &= 0. \end{aligned} \tag{4.11}$$

Testing (4.11) with $u_1 - u_2$ yields

$$\|u_1(t) - u_2(t)\|^2_{L^2(D)} + \int_{Q_t} \left(|\nabla u_1|^{p(s,x)-2}\nabla u_1 - |\nabla u_2|^{p(s,x)-2}\nabla u_2\right)\cdot\nabla(u_1 - u_2)\, dx\, ds = 0, \tag{4.12}$$

which implies $u_1 = u_2$, thereby establishing the uniqueness of the weak solution. This completes the proof of Lemma (4.4).

4.4. **Proof of Lemma 4.3.** The aim of this section is to show that the weak solution obtained in Lemma(4.4) is in fact a strong solution, in the sense that it satisfies the regularity properties stated in Definition 4.2 .
By uniqueness of the weak solution, denote by u the solution associated with $h \in N_W^2(0,T,L^2(D))$, obtained as the limit of the sequence of strong solutions (u_m) associated with a sequence $(h_m) \subset S_W^2(0,T,H_0^k(D))$ converging to h, for a given $S \in N_W^2(0,T,L^2(D))$.
Using (4.6), one has

$$\begin{aligned} & E\left\|\left(u_n-u_m\right)(t)\right\|_{C([0,T],L^2(D))}^2 \\ & \quad+2E\int_0^T\int_D\left[\left|\nabla u_n\right|^{p(s,x)-2}\nabla u_n-\left|\nabla u_2\right|^{p(s,x)-2}\nabla u_m\right]\nabla\left[u_n-u_m\right]dxdt \quad (4.13) \\ & \quad\leq(4C+1)\,E\left(\left\|h_n-h_m\right\|_{L^2(\mathcal{Q}_T)}^2\right), \end{aligned}$$

which implies that

$$E\int_0^T\int_D\left[\left|\nabla u_n\right|^{p(t,x)-2}\nabla u_n-\left|\nabla u_2\right|^{p(t,x)-2}\nabla u_m\right]\nabla\left[u_n-u_m\right]dxdt\to 0$$

as n and m tend to infinity and that (u_m) is a Cauchy sequence in $L^2(\Omega,C([0,T],L^2(D)))$. Hence, the mild solution u is a continuous $L^2(D)$-valued process.
It remains to prove that u is a solution in the sense of Definition 4.2 . To this end, let $\mathcal{J}$ denote the proper, convex, continuous, and Gâteaux differentiable functional defined on

$$\mathcal{E}:=\left\{u\in N_W^2(0,T;L^2(D))\ ,\ \nabla u\in L^{p(\cdot)}(\Omega\times Q_T)\right\},$$

endowed with the norm $\|u\|_{\mathcal{E}}=\|u\|_{N_W^2(0,T;L^2(D))}+\|\nabla u\|_{L^{p(\cdot)}(\Omega\times Q_T)}$.
The functional $\mathcal{J}$ is given by

$$\mathcal{J}(u)=E\int_{Q_T}\left[\frac{u^2}{2}+\frac{1}{p(t,x)}|\nabla u|^{p(t,x)}\right]dx\,dt.$$

As previously noted in a related setting, the Gâteaux derivative $D\mathcal{J}(u)$ satisfies, for any

$$v\in\mathcal{E},\langle D\mathcal{J}(u),v\rangle=E\int_{Q_T}\left(uv+|\nabla u|^{p(t,x)-2}\nabla u\cdot\nabla v\right)dx\,dt.$$

This defines a single-valued maximal monotone operator. Moreover, the above estimate (4.13) guarantees that

$$\lim_{n,m\to\infty}\langle D\mathcal{J}(u_n)-D\mathcal{J}(u_m),u_n-u_m\rangle=0.$$

From 4.2 (following the same procedure as in (3.18) through (3.24)), one deduces that the sequence (u_n) is bounded in the reflexive Banach space $\mathcal{E}$. Consequently, there exists a subsequence (u_{n_l}) that converges weakly in $\mathcal{E}$ to a unique limit u, while $D\mathcal{J}(u_n)$ converges weakly in the dual space. By standard results on maximal monotone operators in reflexive spaces, the weak limit is necessarily $D\mathcal{J}(u)$, and the pairing $\langle D\mathcal{J}(u_n),u_n\rangle$ converges to $\langle D\mathcal{J}(u),u\rangle$ (see [4], [5] Proposition 2.5). Furthermore, the combination of weak convergence and convergence of the

modular in $L^{p(\cdot)}(\Omega \times Q_T)$ implies that u_{n_l} converges strongly to u in $\mathcal{E}$. The uniqueness of the limit then ensures that the entire sequence (u_n) converges to u. This allows passing to the limit in each term of the equation satisfied by u_n, showing that the mild solution u is also a solution in the sense of Definition 4.2.

4.5. **Stability of the solution.** Let h_1 and h_2 be two forcing terms associated with S_1 and S_2, respectively. Proceeding as in the derivation of (4.3), (4.4) and (4.5), we obtain

$$\begin{aligned}
&\|(u_1-u_2)(t)\|^2_{L^2(D)}\\
&+\int_0^t\int_D\left[|\nabla u_1|^{p(\sigma,x)-2}\nabla u_1-|\nabla u_2|^{p(\sigma,x)-2}\nabla u_2\right]\nabla[u_1-u_2]\,dxd\sigma\\
&\le (4C+1)E\|h_1(.,S_1)-h_2(.,S_2)\|^2_{N^2_W(0,T;L^2(D))}+2\int_0^t\int_D(u_1-u_2)(\beta_1(S_1)-\beta_2(S_2))dxd\sigma.
\end{aligned}\tag{4.14}$$

From the Cauchy-Schwarz inequality and Young inequality, we obtain

$$\begin{aligned}
&E\,\|(u_1-u_2)(t)\|^2_{L^2(D)}\\
&+E\int_0^T\int_D\left[|\nabla u_1|^{p(\sigma,x)-2}\nabla u_1-|\nabla u_2|^{p(\sigma,x)-2}\nabla u_2\right]\cdot\nabla(u_1-u_2)\,dxd\sigma\\
&\quad\le 2E\int_0^T\|u_1-u_2\|_{L^2(D)}\|\beta_1(S_1)-\beta_2(S_2)\|_{L^2(D)}\,d\sigma\\
&+(4C+1)E\|h_1(.,S_1)-h_2(.,S_2)\|^2_{N^2_W(0,T;L^2(D))}\\
&\quad\le\left(E\int_0^T\|u_1-u_2\|^2_{L^2(D)}\,d\sigma+E\int_0^T\|\beta_1(S_1)-\beta_2(S_2)\|^2_{L^2(D)}\,d\sigma\right)\\
&+(4C+1)E\|h_1(.,S_1)-h_2(.,S_2)\|^2_{N^2_W(0,T;L^2(D))}\\
&\quad\le E\int_0^T\|\beta_1(S_1)-\beta_2(S_2)\|^2_{L^2(D)}\,d\sigma+(4C+1)E\|h_1(.,S_1)-h_2(.,S_2)\|^2_{N^2_W(0,T;L^2(D))}\\
&\qquad+\int_0^T E\,\|u_1-u_2\|^2_{L^2(D)}\,dt\\
&+\int_0^T E\int_0^\sigma\int_D\left[|\nabla u_1|^{p(s,x)-2}\nabla u_1-|\nabla u_2|^{p(s,x)-2}\nabla u_2\right]\cdot\nabla(u_1-u_2)\,dxdsd\sigma.
\end{aligned}\tag{4.15}$$

By applying Gronwall's lemma, one deduces that

$$\begin{aligned}
&E\sup_{t\in[0,T]}\|(u_1-u_2)(t)\|^2_{L^2(D)}+E\int_0^T\int_D\left[|\nabla u_1|^{p(\sigma,x)-2}\nabla u_1-|\nabla u_2|^{p(\sigma,x)-2}\nabla u_2\right]\cdot\nabla[u_1-u_2]\,dxd\sigma\\
&\quad\le\left(E\int_0^T\|\beta_1(S_1)-\beta_2(S_2)\|^2_{L^2(D)}\,d\sigma+(4C+1)E\|h_1(.,S_1)-h_2(.,S_2)\|^2_{N^2_W(0,T;L^2(D))}\right)\exp(T)\\
&\le E\int_0^T\left(\|\beta_1(S_1)-\beta_2(S_2)\|^2_{L^2(D)}+\|h_1(.,S_1)-h_2(.,S_2)\|^2_{L^2(D)}\right)d\sigma\exp(T)(4C+1).
\end{aligned}\tag{4.16}$$

5. The multiplicative case

5.1. **Definition and statement of the main theorem.** One first defines the notion of solution of problem (1.1).

Definition 5.1. A strong solution to problem (1.1) is a predictable process u such that

$$u \in L^2\left(\Omega; C\left([0,T]; L^2(D)\right)\right) \cap X_T \cap N_w^2(0,T;L^2(D)),$$

satisfying $u(w,0,.) = u_0(w,x) \in L^2(\Omega \times D)$ with $\nabla u \in \left(L^{p(.,.)}(\Omega, \mathcal{Q}_T)\right)^d$, and for all $t \in [0,T]$, $x \in D$ and almost surely w in Ω,

$$\begin{aligned} u(w,t,x) &= u(w,s,x) \\ &+ \int_s^t \left[\operatorname{div}\left(|\nabla u(w,\sigma,x)|^{p(\sigma,x)-2}\nabla u(w,\sigma,x)\right) + \beta(u(w,\sigma,x))\right] d\sigma \\ &+ \int_s^t h(.,u(w,\sigma,x))dW(w,\sigma,x). \end{aligned} \tag{5.1}$$

The main result of this paper is the following.

Theorem 5.2. *Assume that $(H_1), (H_2)$ and (H_3) hold true. Then, there exists a unique solution u to (1.1), in the sense of Definition 5.1 . Moreover there exists a constant $C > 0$, depending only on the problem data, such that for any two distinct source terms h_1 and h_2,*

$$\begin{aligned} &E\left(\|u_1 - u_2\|^2_{C([0,T],L^2(D))}\right) + E\int_0^T \int_D \left(|\nabla u_1|^{p(t,x)-2}\nabla u_1 - |\nabla u_2|^{p(t,x)-2}\nabla u_2\right).\nabla(u_1-u_2)\,dtdx \\ &\leq (4C+1)\exp(T)\,E\int_0^T \left(\|\beta_1(u_1) - \beta_2(u_2)\|^2_{L^2(D)} + \|h_1(.,u_1) - h_2(.,u_2)\|^2_{L^2(D)}\right) d\sigma. \end{aligned} \tag{5.2}$$

5.2. **Proof of Theorem 5.2.**

5.2.1. *Preliminary Observation.* The proof of Theorem 5.2 will be carried out in several gradual stages. The proof relies on the fact that u is a solution of

$$\begin{cases} du(w,t,x) - \Delta_{p(.)}u(w,t,x)\,dt = h(.,u(w,t,x))\,dW(w,t,x) + \beta(u(w,t,x))\,dt \\ \text{in } \ \Omega \times (0,T) \times D \\ u(w,t,x) = 0 \text{ on } \Omega \times (0,T) \times \partial D \\ u(w,0,.) = u_0(w,x) \in \Omega \times D \end{cases} \tag{5.3}$$

if and only if u is a fixed point of the functional $\mathcal{T}$, defined by

$$\mathcal{T} : N_w^2(0,T;L^2(D)) \longrightarrow N_w^2(0,T;L^2(D)), \quad S \longmapsto u_S, \tag{5.4}$$

where u_S is the solution to the auxiliary problem (4.7).
Indeed, if u is a solution of (5.3) by taking $S = u$ then u check (4.7) and therefore $u_S = u$. Thus, $\mathcal{T}(u) = u$. Conversely, let $u \in N_w^2(0,T;L^2(D))$ such that $\mathcal{T}(u) = u$, then $u = u_u$. Thus, u yields (4.7) with $S = u$ and therefore u yields (5.3).

5.2.2. *Contraction of the functional $\mathcal{T}$ and Fixed-Point Construction of the Solution.* The proof of Lemma 4.3 shows that $\mathcal{T}$ is well-defined. Moreover, if S_1 and S_2 are given in $N_w^2(0,T,L^2(D))$ and u_{S_1}, u_{S_2} are the solutions of $(P, h(.,S_1), \beta(S_1))$, $(P, h(.,S_2), \beta(S_2))$ respectively, then for all $t \in (0,T)$,

$$\begin{aligned} &E\left\|u_{S_1}-u_{S_2}\right\|^2_{L^2(D)} + E\int_0^T\int_D \left(|\nabla u_{S_1}|^{p(\sigma,x)-2}\nabla u_{S_1} - |\nabla u_{S_2}|^{p(\sigma,x)-2}\nabla u_{S_2}\right).\nabla(u_{S_1}-u_{S_2})\,d(\sigma,x) \\ &\quad \leq C'' E\int_0^T \left(\left\|\beta(S_1)-\beta(S_2)\right\|^2_{L^2(D)} + \left\|h(.,S_1)-h(.,S_2)\right\|^2_{L^2(D)}\right) d\sigma, \end{aligned} \tag{5.5}$$

where $C'' = \exp(T)\,(4C+1)$. Since

$$E\int_0^T\int_D \left(|\nabla u_{S_1}|^{p(\sigma,x)-2}\nabla u_{S_1} - |\nabla u_{S_2}|^{p(\sigma,x)-2}\nabla u_{S_2}\right).\nabla(u_{S_1}-u_{S_2})\,d(\sigma,x) \geq 0,$$

then,

$$E\left(\left\|(u_{S_1}-u_{S_2})(t)\right\|^2_{L^2(D)}\right) \leq C'' E\int_0^T \left(\left\|\beta(S_1)-\beta(S_2)\right\|^2_{L^2(D)} + \left\|h(.,S_1)-h(.,S_2)\right\|^2_{L^2(D)}\right) d\sigma.$$

Therefore, by the Lipschitz property,

$$\begin{aligned} E\left(\left\|(u_{S_1}-u_{S_2})(t)\right\|^2_{L^2(D)}\right) &\leq C''\left(L_h^2+L_\beta^2\right)\int_0^t E\left\|S_1-S_2\right\|^2_{L^2(D)}\,d\sigma \\ &\leq K\int_0^t E\left\|S_1-S_2\right\|^2_{L^2(D)}\,d\sigma, \end{aligned} \tag{5.6}$$

where $K := C''\left(L_h^2+L_\beta^2\right)$.
We fix $\alpha > 0$. Multiplying inequality (5.6) by the exponential factor $e^{-\alpha t}$ and integrating in time from 0 to T yields.

$$\int_0^T E\left(\left\|(u_{S_1}-u_{S_2})(t)\right\|^2_{L^2(D)}\right)e^{-\alpha t}dt \leq K\int_0^T \frac{d}{dt}\left(\frac{-1}{\alpha}e^{-\alpha t}\right)\int_0^t E\left\|S_1-S_2\right\|^2_{L^2(D)}\,d\sigma dt, \tag{5.7}$$

which is equivalent to say

$$\begin{aligned} \int_0^T E\left(\left\|(u_{S_1}-u_{S_2})(t)\right\|^2_{L^2(D)}\right)e^{-\alpha t}dt &\leq \frac{K}{\alpha}\left(1-e^{-\alpha T}\right)\int_0^T E\left\|S_1-S_2\right\|^2_{L^2(D)}\,e^{-\alpha t}dt \\ &\leq \frac{K}{\alpha}\int_0^T E\left\|S_1-S_2\right\|^2_{L^2(D)}\,e^{-\alpha t}dt. \end{aligned} \tag{5.8}$$

Choosing $\alpha > 0$ such that $\dfrac{K}{\alpha} < 1$ and using Lemma 2.21 , we obtain that $\mathcal{T}$ is a contraction. Applying Banach fixed-point theorem (see [19], Theorem 1.12), $\mathcal{T}$ therefore has only one fixed point. This fixed point is the solution to problem (1.1) .
Therefore, if u_i denotes the solution to problem (4.7) corresponding to the forcing

term $h_i(\cdot, S_i)$ and the reaction term β_i, then $u_i = S_i$, and

$$E\left(\|u_1 - u_2\|^2_{C([0,T],L^2(D))}\right) + E\int_0^T \int_D \left(|\nabla u_1|^{p(t,x)-2}\nabla u_1 - |\nabla u_2|^{p(t,x)-2}\nabla u_2\right).\nabla(u_1 - u_2)\, dtdx$$
$$\leq (C+1)\exp(T)\, E\int_0^T \left(\|\beta_1(u_1) - \beta_2(u_2)\|^2_{L^2(D)} + \|h_1(.,u_1) - h_2(.,u_2)\|^2_{L^2(D)}\right) dt. \tag{5.9}$$

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