Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 37 (2026) 1-24 https://doi.org/10.56947/amcs.v37.985

ISOTROPIC DIRICHLET PROBLEM WITH MEASURE DATA

BOUREIMA SAWADOGO[1], STANISLAS OUARO[3*] AND ADAMA KABORE[3]

Abstract. We study in this paper an isotropic problem with non-homogeneous Dirichlet boundary condition and Radon diffuse measure data. We prove, by using monotone operators in Banach spaces, approximation methods, the existence and uniqueness of entropy solution.

1. Introduction

We consider the following elliptic problem with nonlinear nonhomogenous Dirichlet boundary condition.

$$\begin{cases} -div(a(x,\nabla u)) + |u|^{p(x)-2}u = \mu \text{ in } \Omega \\ u = constant \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a rectangular-like domain, $\partial\Omega$ represents the boundary of Ω, $a(.,.) : \bar{\Omega}\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Caratheodory function. Note that μ is a bounded Radon diffuse measure (that is μ does not charge the sets of zero $p(.)$-capacity). The study of problems involving variable exponent are motivated by their applications to the mathematical analysis and in technology. Indeed, in technology, they can model various phenomena such as study of elastic mechanics (see [1]), image restoration (see[4]), electrorheological fluids [5, 14, 18, 19], mathematical biology (see[8]). The study of PDEs (Partial Differential Equations) is based on the theory of standard Sobolev spaces $W^{1,p(.)}(\Omega)$ in order to find weak solution. These spaces were studied by several mathematicians such as Hudzik (see [10]), Kovacik and Rakosnik (see [12]) and Musielak (see[15]). In the literature, there are many works analogous to problem (1.1). Note that, Prignet (see([17]), studied the following problem

$$-div\,(A(x,u,\nabla u)) = f \text{ in } \mathcal{D}'(\Omega),$$

Date: Received: Oct 10, 2026; Revised: Nov 9, 2026; Accepted: Nov 14, 2026.
[2] Corresponding author
© The Author(s) 2026. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of the licence, visit https://creativecommons.org/licenses/by-nc-nd/4.0/.
2010 *Mathematics Subject Classification.* 36J60, 35J65, 35J20, 35J25.
Key words and phrases. Sobolev spaces, variable exponent, weak solution, entropy solution, measure data.

where Ω is a smooth open bounded set of $\mathbb{R}^N$, f a Radon measure in the case of non-homogenous boundary conditions of Neumann, Fourier and Dirichlet types with constant exponent by showing the existence and uniqueness of entropy solution. In [21], one has studied problem (1.1) for the cases where the source term belongs to $L^\infty(\Omega)$ and to $L^1(\Omega)$. One has proved the existence and uniqueness of weak and entropy solutions respectively. We also have other works (see[11]) and (see [22]) in PDEs that have greatly motivated us. For example, in [11] the authors study the convergence of a sequence of solutions for a degenerate elliptic problems with variable coercivity and growth exponents. In [22], the authors studied a nonlinear multivalued homogeneous Robin boundary $p(u)$-Laplacian problem by showing the existence and partial uniqueness result of renormalized solution.
In this paper, we study problem (1.1) with the aim to prove the existence and uniqueness of entropy solution. To achieve this aim, one uses the decomposition of the measure result ([16]). One also uses an approximation methods, Minty theory and operators theory in Sobolev spaces with variable exponent. The remaining part of this paper is organized as follows: In Section 2, we introduce some useful preliminary results. In Section 3, we present the necessary assumptions on the data in order to address problem (1.1). Finally in Section 4 we prove the existence and uniqueness of entropy solution.

2. **Preliminary results**

We denote $C_+(\bar{\Omega}) = \left\{ p(.) \in C(\bar{\Omega}; \mathbb{R}) : \inf_{x\in\bar{\Omega}} p(x) > 1 \right\}$.
$\forall p \in C_+(\bar{\Omega})$, we denote

$$p^+ = \sup_{x\in\bar{\Omega}} p(x) \text{ and } p^- = \inf_{x\in\bar{\Omega}} p(x).$$

Let $p(.) \in C_+(\bar{\Omega})$. We set the variable exponent Lebesgue space denoted by $L^{p(\cdot)}(\Omega)$ as follow.

$$L^{p(\cdot)}(\Omega) = \left\{ u : u \text{ mesurable with real values such that } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

This space is equipped with the following Luxemburg norm.

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

The space $\left(L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}(\Omega)}\right)$ is a reflexive and separable Banach space if $p \in C_+(\bar{\Omega})$ and $p^+ < \infty$.
We also define the convex modular as follows.

$$\rho_{p(.)}(u) = \int_\Omega |u|^{p(x)} dx,$$

with $u : \Omega \longrightarrow \mathbb{R}$ a measurable function.

Proposition 2.1. [12] *If $\Omega \subset \mathbb{R}^N$ is a bounded open subset and p_1, $p_2 \in C_+(\bar{\Omega})$ such as $p_1 \leq p_2$ in Ω, then the following injection $L^{p_2(.)}(\Omega) \hookrightarrow L^{p_1(.)}(\Omega)$ is continuous.*

We denote by $L^{p'(.)}(\Omega)$ the conjugate space of $L^{p(.)}(\Omega)$ with $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$, $\forall\ x \in \Omega$.

Theorem 2.2. [12] *$\forall u \in L^{p(.)}(\Omega)$ and $\forall v \in L^{p'(.)}(\Omega)$, the following Hölder type inequality holds.*

$$\left|\int_\Omega uvdx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(.)}(\Omega)} \|v\|_{L^{p'(.)}(\Omega)}.$$

In particular, $\left|\int_\Omega uvdx\right| \leq 2\|u\|_{L^{p(.)}(\Omega)} \|v\|_{L^{p'(.)}(\Omega)}$.

Let's set the variable exponent Sobolev space $W^{1,p(.)}(\Omega)$ as follows.

$$W^{1,p(.)}(\Omega) = \left\{u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega)\right\}.$$

We also define the associated norm and the convex modular to this space as follows.
If $u \in W^{1,p(.)}(\Omega)$,

$$\|u\|_{1,p(.)} = \|u\|_{L^{p(.)}(\Omega)} + \|\nabla u\|_{L^{p(.)}(\Omega)}$$

and

$$\rho_{1,p(.)}(u) = \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx.$$

Proposition 2.3. [12] *The space $\left(W^{1,p(.)}(\Omega); \|.\|_{W^{1,p(.)}(\Omega)}\right)$ is a reflexive and separable Banach space for all $p(.) \in C_+(\bar{\Omega})$ and $p^+ < \infty$.*

Proposition 2.4. [7] *$\forall\ u \in L^{p(.)}(\Omega)$ we have the following relations.*

(i) $\|u\|_{L^{p(.)}(\Omega)} < 1 (= 1; > 1) \Longrightarrow \rho_{p(.)}(u) < 1 (= 1; > 1)$;
(ii) $\|u\|_{L^{p(.)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(.)}(\Omega)} \leq \rho_{p(.)}(u) \leq \|u\|^{p^+}_{L^{p(.)}(\Omega)}$;
(iii) $\|u\|_{L^{p(.)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(.)}(\Omega)} \leq \rho_{p(.)}(u) \leq \|u\|^{p^-}_{L^{p(.)}(\Omega)}$;
(iv) $\|u_n\|_{L^{p(.)}(\Omega)} \longrightarrow 0 (respectively \longrightarrow \infty) \Leftrightarrow \rho_{p(.)}(u_n) \longrightarrow 0 (respectively \longrightarrow \infty)$.

Proposition 2.5. [7] *$\forall\ u \in W^{1,p(.)}(\Omega)$, we have the following relations.*

(i) $\|u\|_{1,p(.)} > 1 \Longrightarrow \|u\|^{p^-}_{1,p(.)} \leq \rho_{1,p(.)}(u) \leq \|u\|^{p^+}_{1,p(.)}$;
(ii) $\|u\|_{1,p(.)} < 1 \Longrightarrow \|u\|^{p^+}_{1,p(.)} \leq \rho_{1,p(.)}(u) \leq \|u\|^{p^-}_{1,p(.)}$.

For every $A \subset \Omega$, we denote

$$S_{p(.)}(A) = \{u \in W^{1,p(.)}_0(\Omega) \cap C_0(\Omega) : u = 1 \text{ on } A,\ u \geq 0 \text{ on } \Omega\}.$$

The $p(.)$-capacity of every subset A with respect to Ω is defined by

$$Cap_{p(.)}(A,\Omega) = \inf_{u\in S_p(.)(A)} \left\{ \int_\Omega |\nabla u|^{p(x)} dx \right\}.$$

In the case $S_{p(.)}(A) = 0$, we set $Cap_{p(.)}(\Omega,\Omega) = \infty$.
We denote by $\mathcal{M}_b(\Omega)$ the space of bounded Radon measure on Ω, equiped with its standard norm $\|.\|_{\mathcal{M}(\Omega)}$. Note that, if μ belongs to $\mathcal{M}_b(\Omega)$, then $|\mu|(\Omega)$ (the total variation of μ) is a bounded positive measure in Ω. Given $\mu \in \mathcal{M}_b(\Omega)$, we say that μ is diffuse with respect to the capacity $W_0^{1,p(.)}(\Omega)$ ($p(.)$-capacity for short) if $\mu(A) = 0$, for every set A such that $Cap_{p(.)}(A,\Omega) = 0$.

The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(.)}(\Omega)$. We use the following decomposition of bounded Radon measure.

Theorem 2.6. [16] *Let $p(.) : \bar{\Omega} \longrightarrow (1,\infty)$ be a continuous function and $\mu \in \mathcal{M}_b(\Omega)$. Then $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$ if and only if $\mu \in L^1(\Omega) + W^{-1,p'(.)}(\Omega)$.*

Remark 2.7. Since $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$, Theorem 2.6 implies that there exist $f \in L^1(\Omega)$ and $F \in (L^{p'(.)}(\Omega))^N$ such that

$$\mu = f - divF.$$

Proposition 2.8. [6] *Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with smooth boundary and $p \in C_+(\bar{\Omega})$. If $s \in C(\bar{\Omega},\mathbb{R})$ satisfies the condition $1 \leq s(x) < p^*(x)$, $\forall x \in \bar{\Omega}$, then the embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{s(.)}(\Omega)$ is compact, where p^* denotes, as usual, the critical exponent given by*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & if\ p(x) < N, \\ \infty & if\ p(x) \geq N. \end{cases}$$

We denote by V the following space.

$$V = \left\{ u \in W^{1,p(.)}(\Omega) : u|_{\partial\Omega} = constant \right\}$$

and V^* its dual space.

Theorem 2.9. [3]
(i) *The space V is a closed subspace of $W^{1,p(.)}(\Omega)$.*

(ii) *The space V equipped with the norm $\|.\|_{1,p(.)}$ is a reflexive and separable Banach space, $\forall\ p \in C_+(\bar{\Omega})$ with $p^+ < \infty$.*

3. Assumptions on the data

We make the following assumptions on the data.
(H_1) $p : \bar{\Omega} \longrightarrow (1,\infty)$ with $1 < p^- \leq p^+ < \infty$ and p verifies the log-Hölder continuity condition

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|}, \quad \forall x,y \in \Omega, 0 < |x-y| \leq \frac{1}{2}. \tag{3.1}$$

For the vector fields $a(.,.)$, we assume that $a(x,\xi) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is the continuous derivative with respect to ξ of the mapping $A : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $A = A(x,\xi)$, i.e. $a(x,\xi) = \nabla_\xi A(x,\xi)$ such that
(H_2)

$$A(x,0) = 0, \tag{3.2}$$

$\forall\ x \in \Omega$.
(H_3) There exists a constant C_1 such that

$$|a(x,\xi)| \leq C_1 \left(j(x) + |\xi|^{p(x)-1}\right), \tag{3.3}$$

$\forall x \in \Omega$ and $\forall \xi \in \mathbb{R}^N$, with j a positive function in $L^{p'(.)}(\Omega)$.

(H_4) The following monotonicity condition holds.

$$\left(a\left(x,\xi_1\right) - a\left(x,\xi_2\right)\right).\left(\xi_1 - \xi_2\right) > 0, \tag{3.4}$$

$\forall x \in \Omega$ and $\forall \xi_1, \xi_2 \in \mathbb{R}^N$, with $\xi_1 \neq \xi_2$.
(H_5) The following inequalities holds.

$$|\xi|^{p(x)} \leq a(x,\xi).\xi \leq p(x)A(x,\xi), \tag{3.5}$$

$\forall x \in \Omega$ and $\forall\ \xi \in \mathbb{R}^N$.
(H_6) The mapping $a(.,.)$ is odd with respect to its second variable, that is,

$$a(x,-\xi) = -a(x,\xi), \tag{3.6}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.

As examples of models with respect to above assumptions, we can give the following.
(i) If $a(x,\xi) = |\xi|^{p(x)-2}\xi$, then, one has $A(x,\xi) = \dfrac{1}{p(x)}|\xi|^{p(x)}$ which is the $p(.)$-Laplacian operator.
(ii) Moreover, if one takes $a(x,\xi) = (1+|\xi|^2)^{\frac{(p(x)-2)}{2}}\xi$, one has $A(x,\xi) = \dfrac{1}{p(x)}\left[(1+|\xi|^2)^{\frac{p(x)}{2}} - 1\right]$ which is the generalized mean curvative operator.

4. **Existence and uniqueness result**

We first recall some notations.
For any $u \in W^{1,p(.)}(\Omega)$, we denote by $\mathcal{T}^{1,p(.)}(\Omega)$ the following set.

$$\mathcal{T}^{1,p(.)}(\Omega) = \{u : \Omega \longrightarrow \mathbb{R},\ \text{measurable such that } T_t(u) \in W^{1,p(.)}(\Omega), \text{for any } t > 0\}.$$

We define the space $\mathcal{T}^{1,p(.)}_{tr}(\Omega)$ as the set of function $u \in \mathcal{T}^{1,p(.)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p(.)}(\Omega)$ satisfying

$$u_n \longrightarrow u \text{ a.e. in } \Omega,$$

$$\nabla T_t(u_n) \longrightarrow \nabla T_t(u) \text{ in } L^1(\Omega)\ , \forall\ t > 0$$

and

$$u_n \longrightarrow \text{constant a.e. on } \partial\Omega.$$

Definition 4.1. A measurable function u is an entropy solution of problem (1.1) if $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$, $|u|^{p(x)-2}u \in L^1(\Omega)$ and

$$\int_\Omega a(x,\nabla u).\nabla T_t(u-v)dx+\int_\Omega |u|^{p(x)-2}uT_t(u-v)dx \leq \int_\Omega f(x)T_t(u-v)dx+\int_\Omega F.\nabla T_t(u-v)dx, \tag{4.1}$$

$\forall\ v \in W^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$, $\forall\ t>0$.

Theorem 4.2. *Assume $(H_1)-(H_5)$. Then, for any $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$, the problem (1.1) admits a unique entropy solution u in V.*

Proof. The proof of Theorem 4.2 is divided into several steps.

A-Proof of the existence of entropy solutions

Step 1: Approximate problem

One consider the following approximate problem.

$$\begin{cases} -div(a(x,\nabla u_n))+|u_n|^{p(x)-2}u_n=\mu_n & in\ \Omega \\ u_n = constant & \text{on } \Omega, \end{cases} \tag{4.2}$$

where $f_n = T_n(f) \in L^\infty(\Omega)$ and $\mu_n = f_n - divF$.

Note that

$$f_n \longrightarrow f \text{ in } L^\infty(\Omega)$$

and

$$\|f_n\|_{L^1(\Omega)} = \int_\Omega |f_n|dx \leq \int_\Omega |f|dx = \|f\|_{L^1(\Omega)}.$$

Definition 4.3. A measurable function $u_n \in V$ is a weak solution of (4.2) if

$$\int_\Omega a(x,\nabla u_n).\nabla v dx+\int_\Omega |u_n|^{p(x)-2}u_n v dx=\int_\Omega v d\mu_n, \tag{4.3}$$

$\forall\ v\ \in W^{1,p(.)}(\Omega)$.

Let us prove the following Lemma.

Lemma 4.4. *There exists at least one weak solution of the problem (4.2).*

Proof. One defines the operators G, L and B as follow.

$$\begin{aligned} G : V &\longrightarrow V^*, \text{where} \\ \langle Gu, v\rangle &= \int_\Omega a(x,\nabla u).\nabla v dx, \end{aligned} \tag{4.4}$$

for all $u \in V$ and $v \in W^{1,p(.)}(\Omega)$,

$$\begin{aligned} B : V &\longrightarrow V^*, \text{where} \\ \langle Bu, v\rangle &= \int_\Omega |u|^{p(x)-2}uv dx, \end{aligned} \tag{4.5}$$

for all $u \in V$ and $v \in W^{1,p(.)}(\Omega)$ and

$$L = G + B.$$

One has

$$\begin{aligned}\big|\langle Gu,v\rangle\big| &= \left|\int_\Omega a(x,\nabla u).\nabla v dx\right| \\ &\le \int_\Omega |a(x,\nabla u).\nabla v|dx \\ &\le C_1\int_\Omega (j(x)+|\nabla u|^{p(x)-1})|\nabla v|dx \\ &\le 2C_1\|j(.)\|_{L^{p'(.)}(\Omega)}\|\nabla v\|_{L^{p(.)}(\Omega)} \\ &+ 2C_1\||\nabla u|^{p(.)-1}\|_{L^{p'(.)}(\Omega)}\|\nabla v\|_{L^{p(.)}(\Omega)}. \end{aligned} \tag{4.6}$$

As $j\in L^{p'(.)}(\Omega)$, then, the operator G is bounded.
One has

$$\begin{aligned}\big|\langle Bu,v\rangle\big| &= \left|\int_\Omega |u_n|^{p(x)-2}u_n v dx\right| \\ &\le \int_\Omega ||u_n|^{p(x)-1}||v|dx \\ &\le 2\||u|^{p(.)-1}\|_{L^{p'(.)}(\Omega)}\|v\|_{L^{p(.)}(\Omega)}. \end{aligned} \tag{4.7}$$

From (4.6) and (4.7) , one deduces that L is bounded.

$$\begin{aligned}\langle Gu-Gv,u-v\rangle &= \langle Gu,u-v\rangle+\langle Gv,v-u\rangle \\ &= \int_\Omega a(x,\nabla u).(\nabla u-\nabla v)dx+\int_\Omega a(x,\nabla v).(\nabla v-\nabla u)dx \\ &= \int_\Omega \left[a(x,\nabla u)-a(x,\nabla v).(\nabla u-\nabla v)\right]dx \\ &\ge 0, \quad by\ (H_4) \end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\langle Bu-Bv,u-v\rangle &= \langle Bu,u-v\rangle+\langle Bv,u-v\rangle \\ &= \int_\Omega |u|^{p(x)-2}u(u-v)dx+\int_\Omega |v|^{p(x)-2}v(v-u)dx \\ &= \int_\Omega (|u|^{p(x)-2}u-|v|^{p(x)-2}v)(u-v)dx \\ &\ge 0, \end{aligned} \tag{4.9}$$

since $\xi\longmapsto|\xi|^{p(.)-2}\xi$ is strictly monotone. From (4.8) and (4.9), one deduces that L is monotone.
One has

$$\langle Lu,u\rangle=\langle Gu,u\rangle+\langle Bu,u\rangle.$$

$$\langle Au,u\rangle=\int_\Omega a(x,\nabla u).\nabla u dx\ge\int_\Omega|\nabla u|^{p(x)}dx,\quad \text{from } (H_3)$$

and

$$\begin{aligned}\langle Bu,u\rangle &= \int_\Omega |u|^{p(x)-2}u^2dx\\ &= \int_\Omega |u|^{p(x)}dx.\end{aligned}$$

Then, one obtains

$$\begin{aligned}\langle Lu,u\rangle &\geq \int_\Omega |u|^{p(x)}dx+\int_\Omega |\nabla u|^{p(x)}dx\\ &= \rho_{1,p(.)}(u).\end{aligned}$$

For $\|u\|_{1,p(.)}>1$, it follows that

$$\langle Lu,u\rangle \geq \|u\|^{p^-}_{1,p(.)}.$$

Then,

$$\frac{\langle Lu,u\rangle}{\|u\|_{1,p(.)}}\geq \|u\|^{p^--1}_{1,p(.)}.$$

Hence

$$\lim_{\|u\|_{1,p(.)}\longrightarrow\infty}\frac{\langle Lu,u\rangle}{\|u\|_{1,p(.)}}=\infty,$$

since $p^->1$. Therefore, L is coercive in V.
We now show that L is hemicontinuous.
Let φ: $t\in\mathbb{R}\longmapsto\varphi(t)=\langle L(u+tv),w\rangle$ with u, $v\in V$ and $w\in W^{1,p(.)}(\Omega)$. Then,

$$\varphi(t)=\int_\Omega a(x,\nabla(u+tv)).\nabla w dx\int_\Omega |u+tv|^{p(x)-2}(u+tv)wdx.$$

Let $t\in\mathbb{R}$ be fixed and let $(t_n)_{n\in\mathbb{N}}\subset\mathbb{R}$ be a sequence converging to t in $\mathbb{R}$. Let also

$$\varphi(t_n)=\int_\Omega a(x,\nabla(u+t_nv)).\nabla w dx+\int_\Omega |u+t_nv|^{p(x)-2}(u+t_nv)wdx.$$

Since $a(.,.)$ is Carathéodory and $t_n\longrightarrow t$ in $\mathbb{R}$, one has

$$a(x,\nabla(u+t_nv)).\nabla w\longrightarrow a(x,\nabla(u+tv)).\nabla w,\ a.e.\ in\ \Omega.$$

One has

$$\begin{aligned}\left|a(x,\nabla(u+t_nv)).\nabla w\right| &= \left|a(x,\nabla(u+t_nv))\right||\nabla w|\\ &\leq C_1\left(j(x)+|\nabla(u+t_nv)|^{p(x)-1}\right)|\nabla w|\\ &\leq C_1\left(j(x)+|\nabla u+t_n\nabla v|^{p(x)-1}\right)|\nabla w|\\ &\leq C_1\left(j(x)+(|\nabla u|+M_1|\nabla v|)^{p(x)-1}\right)|\nabla w|\\ &\leq C_1\left(j(x)+(C_2)^{p^+-1}\left(|\nabla u|+|\nabla v|\right)^{p(x)-1}\right)|\nabla w|,\end{aligned}$$

where M_1 is a positive real constant and $C_2 = \max\{1, M_1\}$.
Note also that

$$\left(\sum_{i=1}^{N} b_i\right)^{\alpha} \leq \max\{1, N^{\alpha-1}\} \sum_{i=1}^{N} b_i^{\alpha}, \quad \forall\, b_i \geq 0,\ \alpha > 0.$$

Then,

$$\left|a(x, \nabla(u + t_n v))\nabla w\right| \leq C_1 \left(j(x) + (2C_2)^{p^+-1} \left(|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1}\right)\right) |\nabla w| \in L^1(\Omega).$$

By Lebesgue dominated convergence Theorem, one deduces that

$$\lim_{n\longrightarrow\infty} \int_\Omega a(x, \nabla(u + t_n v)).\nabla w dx = \int_\Omega a(x, \nabla(u + tv)).\nabla w dx. \tag{4.10}$$

As $\xi \longmapsto |\xi|^{p(x)-2}\xi$ is continuous and $t_n \longrightarrow t$ in $\mathbb{R}$, then,

$$|u + t_n v|^{p(x)-2}(u + t_n v)w \longrightarrow |u + tv|^{p(x)-2}(u + tv)w, \text{ a.e. in } \Omega.$$

Moreover,

$$\left||u|^{p(x)-2}u\right| \leq |u|^{p(x)-1}|v| \in L^1(\Omega).$$

By Lebesgue dominated convergence Theorem, it follows that

$$\lim_{n\longrightarrow\infty} \int_\Omega |u + t_n v|^{p(x)-2}(u + t_n v)w dx = \int_\Omega |u + tv|^{p(x)-2}(u + tv)w dx. \tag{4.11}$$

From (4.11) and (4.10), one obtains

$$\begin{aligned} \lim_{n\longrightarrow\infty} & \left(\int_\Omega a(x, \nabla(u + t_n v)).\nabla w dx + \int_\Omega |u + t_n v|^{p(x)-2}(u + t_n v)w dx\right) \\ = & \int_\Omega a(x, \nabla(u + tv)).\nabla w dx + \int_\Omega |u + tv|^{p(x)-2}(u + tv)w dx. \end{aligned}$$

Therefore,

$$\lim_{n\longrightarrow\infty} \varphi(t_n) = \varphi(t).$$

Thus, φ is continuous. Therefore, L is hemicontinuous.
Since L is monotone and hemicontinuous, then, it is of type (M). The operator L is of type (M), bounded and coercive, then, it is surjective from V into its dual V^*. Hence, from Lions's classical theorem (see [13], Section 1.3), there exists a function $u_n \in V$ solution of problem (4.2). □

Step 2: A priori estimates.
Assertion 1. $\nabla T_t(u_n)$ is bounded in $(L^{p^-}(\Omega))^N$, $\forall\, n \in \mathbb{N}$.

Proof. By choosing $T_t(u_n)$ in relation (4.3) as test function, one has

$$\int_\Omega a(x, \nabla u_n).\nabla T_t(u_n) dx + \int_\Omega |u_n|^{p(x)-2} u_n T_t(u_n) dx = \int_\Omega T_t(u_n) d\mu_n, \tag{4.12}$$

which implies that

$$\int_\Omega a(x, \nabla u_n).\nabla T_t(u_n) dx \leq \int_\Omega T_t(u_n) d\mu_n.$$

Then,

$$\int_\Omega a(x,\nabla T_t(u_n)).\nabla T_t(u_n)dx \leq t\int_\Omega d\mu_n.$$

Thus,

$$\int_\Omega |\nabla T_t(u_n)|^{p(x)}dx \leq t|\mu|(\Omega). \tag{4.13}$$

One has

$$\begin{aligned}\int_{\{|u_n|\leq t\}} |\nabla T_t(u_n)|^{p^-}dx &= \int_{\{|u_n|\leq t,\ |\nabla T_t(u_n)|>1\}} |\nabla T_t(u_n)|^{p^-}dx \\ &+ \int_{\{|u_n|\leq t,\ |\nabla T_t(u_n)|\leq 1\}} |\nabla T_t(u_n)|^{p^-}dx \\ &\leq \int_{\{|u_n|\leq t\}} |\nabla T_t(u_n)|^{p(x)}dx + meas\{\Omega\} \\ &\leq t|\mu|(\Omega) + meas\{\Omega\} \\ &\leq C_3(t+1),\ with\ C_3 = \max\{|\mu|(\Omega), meas\{\Omega\}\}.\end{aligned}$$

□

Assertion 2. The sequence $(T_t(u_n))_{n\in\mathbb{N}}$ is bounded in $W^{1,p(.)}(\Omega)$.

Proof. From (4.13) one has

$$\int_\Omega |\nabla T_t(u_n)|^{p(x)}dx \leq t|\mu|(\Omega).$$

$$\int_\Omega |T_t(u_n)|^{p(x)}dx = \int_{\{|u_n|\leq t\}} |T_t(u_n)|^{p(x)}dx + \int_{\{|u_n|>t\}} |T_t(u_n)|^{p(x)}dx. \tag{4.14}$$

One also has from (4.12)

$$\begin{aligned}\int_\Omega |T_t(u_n)|^{p(x)}dx &\leq \int_\Omega |u_n|^{p(x)-2}u_nT_t(u_n)dx \\ &\leq t|\mu|(\Omega)\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\int_\Omega |T_t(u_n)|^{p^-}dx &= \int_{\{|T_t(u_n)|\leq 1\}} |T_t(u_n)|^{p^-}dx + \int_{\{|T_t(u_n)|>1\}} |T_t(u_n)|^{p^-}dx \\ &\leq meas(\Omega) + \int_\Omega |T_t(u_n)|^{p(x)}dx \\ &\leq meas(\Omega) + t|\mu|(\Omega).\end{aligned} \tag{4.16}$$

From (4.13) and (4.16), one deduces that

$$\rho_{1,p(.)}(T_t(u_n)) \leq 2t|\mu|(\Omega) + meas(\Omega). \tag{4.17}$$

For $\|T_t(u_n)\|^{p^-}_{1,p(.)} \geq 1$, one has

$$\|T_t(u_n)\|^{p^-}_{1,p(.)} \leq \rho_{1,p(.)}(T_t(u_n)) \leq 2t|\mu|(\Omega) + meas(\Omega). \tag{4.18}$$

Using (4.18), one obtains

$$\|T_t(u_n)\|_{1,p(.)} \leq [2t|\mu|(\Omega) + meas(\Omega)]^{\frac{1}{p^-}}.$$

Therefore,

$$\|T_t(u_n)\|_{1,p(.)} \leq 1 + [2t|\mu|(\Omega) + meas(\Omega)]^{\frac{1}{p^-}}.$$

Hence,

$$\|T_t(u_n)\|_{1,p(.)} \leq 1 + C.$$

□

Proposition 4.5. *Let u_n be a weak solution to problem (4.2). Then,*

$$meas\{|u_n| > t\} \longrightarrow 0, \text{ as } t \longrightarrow \infty \tag{4.19}$$

and

$$meas\{|\nabla u_n| > t\} \longrightarrow 0, \text{ as } t \longrightarrow \infty. \tag{4.20}$$

Proof.

• Proof of (4.19)

From (4.12), one has

$$\int_\Omega |u_n|^{p(x)-2}u_n T_t(u_n)dx \leq \int_\Omega T_t(u_n)d\mu_n. \tag{4.21}$$

From (4.21), one obtains

$$\int_{\Omega\cap\{|u_n|\geq t\}} |u_n|^{p(x)-2}u_n T_t(u_n)dx \leq t|\mu|(\Omega),$$

which is equivalent to say

$$\int_{\Omega\cap\{u_n\geq t\}} t|u_n|^{p(x)-2}u_n dx - \int_{\Omega\cap\{u_n\leq -t\}} t|u_n|^{p(x)-2}u_n dx \leq t|\mu|(\Omega).$$

Therefore,

$$\int_{\Omega\cap\{|u_n|\geq t\}} |u_n|^{p(x)-1}dx \leq |\mu|(\Omega). \tag{4.22}$$

Finally, as $t \longrightarrow 0$ in (4.22), it follows by using Fatou's Lemma that

$$\left\| |u_n|^{p(.)-2}u_n \right\|_{L^1(\Omega)} = \left\| |u_n|^{p(.)-1} \right\|_{L^1(\Omega)} \leq |\mu|(\Omega). \tag{4.23}$$

Therefore,

$$\int_\Omega |u_n|^{p(x)-1}dx \leq |\mu|(\Omega). \tag{4.24}$$

One deduces that

$$\int_{\{|u_n|>t\}} |u_n|^{p(x)-1}dx \leq |\mu|(\Omega). \tag{4.25}$$

Moreover,

$$\int_{\{|u_n|>t\}} |u_n|^{p(x)-1}dx \geq \int_{\{|u_n|>t\}} t^{p(x)-1}dx. \tag{4.26}$$

For $t > 1$,

$$\int_{\{|u_n|>t\}} t^{p(x)-1}dx \geq \int_{\{|u_n|>t\}} t^{p^- -1}dx. \tag{4.27}$$

Therefore,

$$\int_{\{|u_n|>t\}} t^{p^- -1} dx \leq |\mu|(\Omega). \tag{4.28}$$

Thus,

$$meas\{|u_n| > t\} t^{p^- -1} \leq |\mu|(\Omega), \text{ for any } t > 1. \tag{4.29}$$

Therefore,

$$meas\{|u_n| > t\} \leq \frac{|\mu|(\Omega)}{t^{p^- -1}}, \text{ for any } t > 1. \tag{4.30}$$

Hence $meas\{|u_n| > t\}$ tend toward 0 as $t \longrightarrow \infty$.

- Proof of (4.20)

For the proof of (4.20) (see [2]). □

Assertion 3. $(u_n)_{n\in\mathbb{N}}$ converges in measure to a measurable function u.

Proof. The proof consists to show that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Indeed, $\forall\ \delta > 0$,

$$\begin{aligned} meas\{|u_n - u_m| > \delta\} &\leq meas\{|u_n| > t\} + meas\{|u_m| > t\} \\ &+ meas\{|T_t(u_n) - T_t(u_m)| > \delta\}. \end{aligned} \tag{4.31}$$

As $(T_t(u_n))_{n\in\mathbb{N}}$ is bounded in V, there exists $v_t \in V$ such that $T_t(u_n) \rightharpoonup v_t$ in V, as $n \longrightarrow \infty$. Therefore, $T_t(u_n) \longrightarrow v_t$ in $L^{p(.)}(\Omega)$ a.e. in Ω by using compact embedding $V \hookrightarrow L^{p(.)}(\Omega)$. Therefore, $(T_t(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Thus,

$$meas\{|T_t(u_n) - T_t(u_m)| > \delta\} \leq \frac{\varepsilon}{3}, \tag{4.32}$$

$\forall\ \varepsilon > 0$.
According to Proposition 4.5 , one has

$$meas\{|u_n| > t\} \leq \frac{\varepsilon}{3} \qquad and \qquad meas\{|u_m| > t\} \leq \frac{\varepsilon}{3},\ \forall \varepsilon > 0. \tag{4.33}$$

It follow that

$$meas\{|u_n - u_m| > \delta\} \leq \varepsilon,\ \forall \varepsilon > 0 \text{ and } \forall n, m \geq n_0(t, \varepsilon, \delta).$$

Therefore, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure and so converges to a measurable function u.
As for $t > 0$, T_t is continuous, then

$$T_t(u_n) \longrightarrow T_t(u) \text{ a.e. in } \Omega \text{ and } v_t = T_t(u).$$

One has $T_t(u_n) \longrightarrow T_t(u)$ in $L^{p(.)}(\Omega)$ *a.e.* in Ω and $L^{p(.)}(\Omega) \hookrightarrow L^{p^-}(\Omega)$, then,

$$T_t(u_n) \longrightarrow T_t(u) \text{ in } L^{p^-}(\Omega).$$

From Proposition 4.5, $(\nabla u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^{p(.)}(\Omega)$ and $L^{p(.)}(\Omega) \hookrightarrow L^{p^-}(\Omega)$, then, $(\nabla u_n)_{n\in\mathbb{N}}$ is also uniformly bounded in $L^{p^-(.)}(\Omega)$. Therefore,

$$T_t(u_n) \longrightarrow T_t(u) \text{ in } W^{1,p^-}(\Omega)$$

□

Lemma 4.6. [20] *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions. If $(u_n)_{n\in\mathbb{N}}$ converges in measure to u and is uniformly bounded in $L^{p(.)}(\Omega)$ for some $1 \ll p(.) \in L^{\infty}(\Omega)$, then $u_n \longrightarrow u$ strongly in $L^1(\Omega)$.*

Lemma 4.7. [9] *Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < \infty$. Consider a measurable function $\gamma : X \longrightarrow [0, \infty)$ such that*

$$\mu(\{x \in X : \gamma(x) = 0\}) = 0.$$

Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mu(A) < \varepsilon, \qquad for\ all\ A \in \mathcal{M}\ with \int_A \gamma d\mu < \delta.$$

Assertion 4. The sequence $(\nabla u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure.

Proof. The proof consists to show that

$$\forall \delta > 0, \forall \varepsilon > 0, \exists n_0 / \forall n \geq n_0, \forall m \geq n_0,\ \text{meas}\{|\nabla u_n - \nabla u_m| \geq \delta\} \leq \varepsilon.$$

Let's put $A_1 = \{|\nabla u_n| > k\}, A_2 = \{|\nabla u_m| > k\}, A_3 = \{|u_n - u_m| > t\}$, $A_4 = \{|\nabla u_n - \nabla u_m| \geq \delta, |\nabla u_n| \leq k, |\nabla u_m| \leq k, |u_n - u_m| \leq t\}$.
Then,

$$\{|\nabla u_n - \nabla u_m| \geq \delta\} \subset A_1 \cup A_2 \cup A_3 \cup A_4.$$

Thus,

$$\text{meas}\{|\nabla u_n - \nabla u_m| \geq \delta\} \leq \text{meas}(A_1) + \text{meas}(A_2) + \text{meas}(A_3) + \text{meas}(A_4). \tag{4.34}$$

According to Proposition 4.5, one has

$$meas(A_1) \leq \frac{\varepsilon}{4} \tag{4.35}$$

and

$$meas(A_2) \leq \frac{\varepsilon}{4}. \tag{4.36}$$

Moreover, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure by Assertion 3. Thus,

$$meas(A_3) \leq \frac{\varepsilon}{4}. \tag{4.37}$$

Let's now estimate $meas(A_4)$.
One has, according to (H_4) that

$$[a(x, \xi_1) - a(x, \xi_2)] . (\xi_1 - \xi_2) > 0,$$

for $\xi_1 - \xi_2 \neq 0$. Then, there exists a real valued function $\gamma : \Omega \longrightarrow [0, \infty)$ such that $meas(\{x \in \Omega : \gamma(x) = 0\}) = 0$ and

$$[a(x, \xi_1) - a(x, \xi_2)] . (\xi_1 - \xi_2) > \gamma(x),$$

for all $\xi_1, \xi_2 \in \mathbb{R}^N$ such that $|\xi_1| \leq k, |\xi_2| \leq k, |\xi_1 - \xi_2| \geq \delta$.
One has by taking $T_t(u_n - u_m)$ as test function in (4.3), one obtains

$$\begin{aligned}\int_\Omega a(x, \nabla u_n).\nabla T_t(u_n - u_m)dx &+ \int_\Omega |u_n|^{p(x)-2}u_n T_t(u_n - u_m)dx \\ &= \int_\Omega T_t(u_n - u_m)d\mu_n \leq t|\mu|(\Omega). \quad (4.38)\end{aligned}$$

In an analogous way, taking $T_t(u_m - u_n)$ as test function, one has

$$\begin{aligned}\int_\Omega a(x, \nabla u_m).\nabla T_t(u_m - u_n)dx &+ \int_\Omega |u_m|^{p(x)-2}u_m T_t(u_m - u_n)dx \\ &= \int_\Omega T_t(u_m - u_n)d\mu_n \leq t|\mu|(\Omega). \quad (4.39)\end{aligned}$$

By adding (4.38) and (4.39), one obtains

$$\begin{aligned}&\int_{\{|u_n-u_m|\}\leq t\}} \left[a(x, \nabla u_n) - a(x, \nabla u_m)\right].(\nabla u_n - \nabla u_m)dx \\ &+ \int_\Omega (|u_n|^{p(x)-2}u_n - |u_m|^{p(x)-2}u_m)T_t(u_n - u_m)dx \\ &\leq 2t|\mu|(\Omega). \quad (4.40)\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{A_4} \gamma dx &\leq \int_{A_4} [a(x, \nabla u_n) - a(x, \nabla u_m)].(\nabla u_n - \nabla u_m)dx \\ &\leq 2t|\mu|(\Omega) \leq \delta.\end{aligned}$$

Using Lemma 4.7, one obtains

$$meas(A_4) \leq \frac{\varepsilon}{4}. \quad (4.41)$$

From (4.34), (4.35), (4.36) and (4.41), one obtains,

$$\text{meas}\left\{|\nabla u_n - \nabla u_m| \geq \delta\right\} \leq \varepsilon.$$

Therefore, the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure.
This also implies that sequence $(\nabla u_n)_{n\in\mathbb{N}}$ converges in measure to a measurable function ν.

□

Assertion 5. for all $t > 0$, $(\nabla T_t(u_n))_{n\in\mathbb{N}}$ converges to $\nabla T_t(u)$ in $(L^1(\Omega))^N$, $(a(x, \nabla T_t(u_n)))_{n\in\mathbb{N}}$ converges to $a(x, \nabla T_t(u))$ strongly in $(L^1(\Omega))^N$ and weakly in $(L^{p'(.)}(\Omega))^N$.

Proof. Analogous to the proof of Proposition 13-(ii) and (iii) in [21] . □

Assertion 6. u_n converges to some constant a.e. on $\partial\Omega$.
By choosing $T_t(u_n)$ as test function in (4.3), one obtains

$$\int_\Omega a(x, \nabla u_n).\nabla T_t(u_n) + \int_\Omega |u_n|^{p(x)-2}u_n T_t(u_n)dx = \int_\Omega T_t(u_n)d\mu_n.$$

Therefore,

$$\int_\Omega |u_n|^{p(x)-2}u_n T_t(u_n)dx \leq t|\mu|(\Omega). \tag{4.42}$$

Moreover,

$$\int_\Omega |T_t(u_n)|^{p(x)}dx \leq \int_\Omega |u_n|^{p(x)-2}u_n T_t(u_n)dx.$$

Then,

$$\int_\Omega |T_t(u_n)|^{p(x)}dx \leq t|\mu|(\Omega).$$

One has

$$\begin{aligned}
\int_\Omega |T_t(u_n)|^{p^-}dx &= \int_{\{|T_t(u_n)|\leq 1\}} |T_t(u_n)|^{p^-}dx + \int_{\{|T_t(u_n)|>1\}} |T_t(u_n)|^{p^-}dx \\
&\leq meas(\Omega) + \int_{\{|T_t(u_n)|>1\}} |T_t(u_n)|^{p(x)}dx \\
&\leq meas(\Omega) + \int_\Omega |T_t(u_n)|^{p(x)}dx \\
&\leq meas(\Omega) + t|\mu|(\Omega)
\end{aligned} \tag{4.43}$$

and

$$\begin{aligned}
\int_\Omega |\nabla T_t(u_n)|^{p^-}dx &= \int_{\{|\nabla T_t(u_n)|\leq 1\}} |\nabla T_t(u_n)|^{p^-}dx + \int_{\{|\nabla T_t(u_n)|>1\}} |\nabla T_t(u_n)|^{p^-}dx \\
&\leq meas(\Omega) + \int_{\{|\nabla T_t(u_n)|>1\}} |\nabla T_t(u_n)|^{p(x)}dx \\
&\leq meas(\Omega) + \int_\Omega |\nabla T_t(u_n)|^{p(x)}dx \\
&\leq meas(\Omega) + t|\mu|(\Omega).
\end{aligned} \tag{4.44}$$

For all $1 \leq p^- < \infty$, the map

$$\tau : W^{1,p^-}(\Omega) \longrightarrow L^{p^-}(\partial\Omega);\ u \longmapsto \tau(u) = u|_{\partial\Omega}$$

is compact.
As $T_t(u_n) \longrightarrow T_t(u)$ in $W^{1,p^-}(\Omega)$, one obtains that $T_t(u_n)$ converges strongly to $T_t(u)$ in $L^{p^-}(\partial\Omega)$. Therefore, there exists a subsequence still denoted $(T_t(u_n))$ which converges to $T_t(u)$ a.e. on $\partial\Omega$.
Hence, there exists a set $C \subset \partial\Omega$ such that $T_t(u_n)$ converges to $T_t(u)$ on $\partial\Omega \setminus C$, with $\sigma(C) = 0$ where σ is a measure on $\partial\Omega$. By using (4.43), (4.44) and Hölder's inequality, one obtains

$$\int_\Omega |T_t(u_n)| \leq (meas(\Omega))^{\frac{1}{(p^-)'}} \left(meas(\Omega) + t|\mu|(\Omega)\right)^{\frac{1}{(p^-)'}} \tag{4.45}$$

and

$$\int_\Omega |\nabla T_t(u_n)| \leq (meas(\Omega))^{\frac{1}{(p^-)'}} \left(meas(\Omega) + t|\mu|(\Omega)\right)^{\frac{1}{(p^-)'}}. \tag{4.46}$$

By using Fatou's Lemma, one obtains from (4.45) and (4.46),

$$\int_\Omega |T_t(u)| \leq (meas(\Omega))^{\frac{1}{(p^-)'}} \, (meas(\Omega) + t|\mu|(\Omega))^{\frac{1}{(p^-)'}} \tag{4.47}$$

and

$$\int_\Omega |\nabla T_t(u)| \leq (meas(\Omega))^{\frac{1}{(p^-)'}} \, (meas(\Omega) + t|\mu|(\Omega))^{\frac{1}{(p^-)'}} \, . \tag{4.48}$$

Let's put $B_t := \{x \in \partial\Omega : |T_t(u)| < t\}$ for all $t > 0$ and $C' = \partial\Omega \setminus \cup_{t>0} B_t$. One has

$$\begin{aligned} \sigma(C') = \frac{1}{t}\int_{C'} |T_t(u)|dx &\leq \frac{1}{t}\int_{\partial\Omega} |T_t(u)|dx \\ &\leq \frac{C_4}{t}\|T_t(u)\|_{W^{1,1}(\Omega)} \\ &\leq \frac{C_4}{t}\|T_t(u)\|_{L^1(\Omega)} + \frac{C_4}{t}\|\nabla T_t(u)\|_{L^1(\Omega)}, \end{aligned}$$

where C_4 is a positive constant.
Using (4.47) and (4.48), one obtains by taking $t \longrightarrow \infty$ that $\sigma(C') = 0$.
Let's define the function v on $\partial\Omega$ by $v(x) := T_t(u(x))$ if $x \in B_t$.
$\forall\, x \in \partial\Omega \setminus (C \cup C')$, $\exists t > 0$ such that $x \in B_t$ and one has

$$u_n(x) - v(x) = u_n(x) - T_t(u_n(x)) + T_t(u_n(x)) - T_t(u(x)).$$

Since $x \in B_t$, one has $|T_t(u(x))| < t$, so $|T_t(u_n(x))| < t$. Therefore, $|u_n(x)| < t$.
Thus, $u_n(x) - v(x) = T_t(u_n(x)) - T_t(u(x)) \longrightarrow 0$ as $n \longrightarrow \infty$.
Consequently, u_n converges to v a.e. on $\partial\Omega$. As $u_n = constant$ on $\partial\Omega$, one deduces that $v = constant$.
Let's now show that the sequence $\left(|u_n|^{p(x)-2}u_n\right)_{n\in\mathbb{N}}$ converges to $|u|^{p(x)-2}u$ in $L^1(\Omega)$. Let's start by showing that the sequence $\left(|u_n|^{p(x)-2}u_n\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\Omega)$. As u_n is a weak solution of problem (4.2), then taking $\frac{1}{t}T_t(u_n - u_m)$ as test function in (4.3), yelds

$$\begin{aligned} &\int_\Omega \frac{1}{t}a(x,\nabla u_n).\nabla T_t(u_n - u_m)dx + \int_\Omega |u_n|^{p(x)-2}u_n\frac{1}{t}T_t(u_n - u_m)dx \\ = &\int_\Omega \frac{1}{t}f_n T_t(u_n - u_m)dx + \int_\Omega \frac{1}{t}F.\nabla T_t(u_n - u_m)dx. \end{aligned} \tag{4.49}$$

In a similar way for u_m a weak solution to problem (4.2), one take $\frac{1}{t}T_t(u_m - u_n)$ as test function in (4.3) to obtain

$$\begin{aligned} &\int_\Omega \frac{1}{t}a(x,\nabla u_m).\nabla T_t(u_m - u_n)dx + \int_\Omega |u_m|^{p(x)-2}u_m\frac{1}{t}T_t(u_m - u_n)dx \\ = &\int_\Omega \frac{1}{t}f_n T_t(u_m - u_n)dx + \int_\Omega \frac{1}{t}F.\nabla T_t(u_m - u_n)dx. \end{aligned} \tag{4.50}$$

Relation (4.49) and (4.50) give respectively.

$$\begin{aligned}&\int_\Omega \frac{1}{t}a(x,\nabla u_n).\nabla T_t(u_n-u_m)dx+\int_\Omega |u_n|^{p(x)-2}u_n\frac{1}{t}T_t(u_n-u_m)dx\\ =&\int_\Omega \frac{1}{t}f_nT_t(u_n-u_m)dx+\int_{\{|u_n-u_m|\le t\}}\frac{1}{t}F.\nabla(u_n-u_m)dx \qquad (4.51)\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega \frac{1}{t}a(x,\nabla u_m)\nabla T_t(u_m-u_n)dx+\int_\Omega |u_m|^{p(x)-2}u_m\frac{1}{t}T_t(u_m-u_n)dx\\ =&\int_\Omega \frac{1}{t}f_nT_t(u_m-u_n)dx-\int_{\{|u_n-u_m|\le t\}}\frac{1}{t}F.\nabla(u_u-u_m)dx. \qquad (4.52)\end{aligned}$$

By adding (4.51) and (4.52), one obtains

$$\begin{aligned}&\int_{\{|u_n-u_m|\le t\}}\frac{1}{t}\left[a(x,\nabla u_n)-a(x,\nabla u_m)\right].(\nabla u_n-\nabla u_m)dx\\ +&\int_\Omega\left(|u_n|^{p(x)-2}u_n-|u_m|^{p(x)-2}u_m\right)\frac{1}{t}T_t(u_n-u_m)dx\\ \le&\int_\Omega |f_n-f_m|dx. \qquad (4.53)\end{aligned}$$

Letting $t\longrightarrow 0$ and as the first and the second terms of the left-hand side of inequality (4.53) are nonnegative, one gets

$$\int_\Omega \left||u_n|^{p(x)-2}u_n-|u_m|^{p(x)-2}u_m\right|dx\le\int_\Omega |f_n-f_m|dx. \qquad (4.54)$$

Since $(f_n)_{n\in\mathbb{N}}$ converges in $L^1(\Omega)$, by (4.54), $\left(|u_n|^{p(x)-2}u_n\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\Omega)$. Also, $u_n\longrightarrow u$ in measure. As $L^1(\Omega)$ is a Banach space and $\xi\longmapsto|\xi|^{p(x)-2}\xi$ is continuous and strictly monotone, then

$$|u_n|^{p(x)-2}u_n\longrightarrow|u|^{p(x)-2}u \text{ in } L^1(\Omega), \text{ as } n\longrightarrow\infty. \qquad (4.55)$$

Assertion 7. u is an entropy solution of the problem (4.1).

Proof. Let $u_n\in V$ be a weak solution of problem (4.2). Then, taking $T_t(u_n-v)\in V$ as test function in relation (4.3) with $v\in W^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$, one obtains

$$\begin{cases}\displaystyle\int_\Omega a(x,\nabla u).\nabla\left(T_t(u_n-v)\right)dx+\int_\Omega|u_n|^{p(x)-2}u_nT_t(u_n-v)\,dx\\ \qquad\displaystyle=\int_\Omega f_n(x)T_t(u_n-v)\,dx+\int_\Omega F.\nabla T_t(u_n-v)dx,\end{cases} \qquad (4.56)$$

with $t>0$.
One has

$$\begin{aligned}\int_\Omega|u_n|^{p(x)-2}u_nT_t(u_n-v)dx&=\int_\Omega\Big(|u_n|^{p(x)-2}u_n)-|v|^{p(x)-2}v)\Big)T_t(u_n-v)dx\\ &+\int_\Omega|v|^{p(x)-2}vT_t(u_n-v)dx.\end{aligned}$$

The sequence $\left(|u_n|^{p(x)-2}u_n) - |v|^{p(x)-2}v\right) T_t(u_n - v)$ is nonnegative and $\forall\ \xi \in \mathbb{R}$, $\xi \longmapsto |\xi|^{p(x)-2}$ is continuous, so one obtains

$$\left(|u_n|^{p(x)-2}u_n) - |v|^{p(x)-2}v\right) T_t(u_n - v) \longrightarrow \left(|u|^{p(x)-2}u - |v|^{p(x)-2}v\right) T_t(u - v)$$

a.e. in Ω. Then, by Fatou's Lemma,

$$\begin{aligned} &\liminf_{n\longrightarrow\infty} \int_\Omega \left(|u_n|^{p(x)-2}u_n - |v|^{p(x)-2}v\right) T_t(u_n - v)dx \\ \geq\ &\int_\Omega \left(|u|^{p(x)-2}u - |v|^{p(x)-2}v\right) T_t(u - v)dx. \end{aligned} \tag{4.57}$$

Since $T_t(u_n - v)$ converges weably* to $T_t(u - v)$ in $L^\infty(\Omega)$, as $n \longrightarrow \infty$ and

$$\left||v|^{p(x)-2}v\right| \ =\ |v|^{p(x)-2}|v| \ =\ |v|^{p(x)-1} \in \ L^1(\Omega),$$

it follows by using the Lebesgue dominated convergence theorem that

$$\lim_{n\longrightarrow\infty} \int_\Omega |v|^{p(x)-2}vT_t(u_n - v)dx = \int_\Omega |v|^{p(x)-2}vT_t(u - v)dx. \tag{4.58}$$

One has (see[21])

$$\int_{\{|u-v|\}\leq t\}} a(x, \nabla u).\nabla u dx \leq \lim_{n\to\infty} \inf \int_{\{|u_n-v|\leq t\}} a(x, \nabla u_n).\nabla u_n dx. \tag{4.59}$$

From (4.57),(4.58) and (4.59), one obtains

$$\begin{cases} \int_\Omega a(x, \nabla u).\nabla T_t(u - v)dx + \int_\Omega |u|^{p(x)-2}uT_t(u - v)dx \leq \\ \qquad \lim_{n\to\infty} \inf \left(\int_\Omega a(x, \nabla u_n).\nabla(u_n - v)dx + \int_\Omega T_n(|u_n|^{p(x)-2}u_n)T_t(u_n - v)dx\right). \end{cases}$$

Moreover, the integral $\int_\Omega f_n(x)T_t(u_n - v)dx$ converges to $\int_\Omega f(x)T_t(u - v)dx$ since $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^1(\Omega)$ and $(T_t(u_n - v))_{n\in\mathbb{N}}$ converges weakly to $T_t(u - v)$ in $L^\infty(\Omega)$, as $n \longrightarrow \infty$.
The integral

$$\int_\Omega F.\nabla T_t(u_n - v)dx \ \text{ converges to } \ \int_\Omega F.\nabla T_t(u - v)dx,$$

as $n \longrightarrow \infty$, since $(\nabla T_t(u_n - v))_{n\in\mathbb{N}}$ converges to $\nabla T_t(u - v)$ in $\left(L^{p(.)}(\Omega)\right)^N$ and $F \in \left(L^{p'(.)}(\Omega)\right)^N$.
Thus, one obtains

$$\begin{cases} \int_\Omega a(x, \nabla u_n).\nabla T_t(u - v)dx + \int_\Omega |u|^{p(x)-2}uT_t(u - v)dx \leq \int_\Omega fT_t(u - v)dx + \int_\Omega F\nabla T_t(u - v)dx \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \int_\Omega T_t(u - v)d\mu, \end{cases}$$

$\forall\ v \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

□

B-Proof of the uniqueness of entropy solution

Remark 4.8. If (H_5) is satisfied, then,

$$\begin{aligned}\int_\Omega a(x,\nabla u).\nabla T_t(u)dx &= \int_{\{|u|\le t\}} a(x,\nabla u).\nabla T_t(u)dx \\ &= \int_{\{|u|\le t\}} a(x,\nabla T_t(u)).\nabla T_t(u)dx \\ &\ge \int_{\{|u|\le t\}} |\nabla T_t(u)|^{p(x)}dx \ \ge 0.\end{aligned}$$

Note also that

$$\begin{aligned}\int_\Omega |u|^{p(x)-2}uT_t(u)dx &= \int_{\{|u|\le t\}} |u|^{p(x)-2}uT_t(u)dx + \int_{\{|u|>t\}} |u|^{p(x)-2}uT_t(u)dx \\ &= \int_{\{|u|\le t\}} |u|^{p(x)-2}u^2dx + \int_{\{|u|>t\}} |u|^{p(x)-2}ut.\frac{u}{|u|}dx \\ &= \int_{\{|u|\le t\}} |u|^{p(x)}dx + t\int_{\{|u|>t\}} |u|^{p(x)-1}dx \ge 0,\ \forall\, t>0.\end{aligned}$$

Therefore,

$$\int_\Omega a(x,\nabla u).\nabla T_t(u)dx \ge 0 \ and \ \int_\Omega |u|^{p(x)-2}uT_t(u)dx \ge 0.$$

We deduce from (4.1) by taking $v=0$ that

$$\int_\Omega a(x,\nabla u).\nabla T_t(u)dx \le \int_\Omega T_t(u)d\mu \tag{4.60}$$

and

$$\int_\Omega |u|^{p(x)-2}uT_t(u)dx \le \int_\Omega T_t(u)d\mu. \tag{4.61}$$

Lemma 4.9. *Assume that $(H_1)-(H_6)$ hold true. Let u be an entropy solution of (1.1). Then,*

$$\int_{\{|u|\le t\}} |\nabla u|^{p^-}dx \le tC_6 \tag{4.62}$$

and

$$\||u|^{p(x)-2}u\|_{L^1(\Omega)} = \||u|^{p(x)-1}\|_{L^1(\Omega)} \le |\mu|(\Omega). \tag{4.63}$$

Proof. Using (4.60) one has

$$\begin{aligned}\int_\Omega a(x,\nabla u).\nabla T_t(u)dx &\le \int_\Omega T_t(u)d\mu \\ &\le t|\mu|(\Omega).\end{aligned} \tag{4.64}$$

Moreover, according to (H_5),

$$\int_\Omega a(x,\nabla u).\nabla T_t(u)dx = \int_{\{|u|\le t\}} a(x,\nabla u).\nabla u dx \ge \int_{\{|u|\le t\}} |\nabla u|^{p(x)}dx \tag{4.65}$$

From (4.64) and (4.65) we deduce that

$$\int_{\{|u|\leq t\}} |\nabla u|^{p(x)} dx \leq t|\mu|(\Omega),\ \forall\, t\ >\ 0;$$

which is equivalent to say

$$\int_{\Omega} |\nabla T_t(u)|^{p(x)}\, dx \leq t|\mu|(\Omega),\ \forall\, t > 0.$$

One has

$$\begin{aligned}
\int_{\Omega} |\nabla u|^{p^-} dx &= \int_{\{|u|\leq t\}} |\nabla u|^{p^-} dx \\
&= \int_{\{|u|\leq t, |\nabla u|\leq 1\}} |\nabla u|^{p^-} dx + \int_{\{|u|\leq t, |\nabla u|>1\}} |\nabla u|^{p^-} dx \\
&\leq meas(\Omega) + \int_{\{|u|\leq t, |\nabla u|>1\}} |\nabla u|^{p(x)} dx \\
&\leq tmeas(\Omega) + t|\mu|(\Omega) \quad (\text{for } t > 1) \\
&\leq C_5 t,
\end{aligned}$$

where $C_5 = \max\{meas(\Omega), |\mu|(\Omega)\}$.
One has according to (4.61)

$$\int_{\Omega} |u|^{p(x)-2} u T_t(u) dx \leq \int_{\Omega} T_t(u) d\mu \leq t|\mu|(\Omega). \tag{4.66}$$

From (4.66), one obtains

$$\int_{\Omega\cap\{|u|\geq t\}} |u|^{p(x)-2} u T_t(u) dx \leq t|\mu|(\Omega).$$

Which is equivalent to say

$$\int_{\Omega\cap\{u\geq t\}} t|u|^{p(x)-2} u dx - \int_{\Omega\cap\{u\leq -t\}} t|u|^{p(x)-2} u dx \leq t|\mu|(\Omega).$$

Therefore,

$$\int_{\Omega\cap\{|u|\geq t\}} |u|^{p(x)-1} dx \leq |\mu|(\Omega). \tag{4.67}$$

Finally, as $t \longrightarrow 0$ in (4.67), it follows by using Fatou's Lemma that

$$\| |u|^{p(x)-2} u \|_{L^1(\Omega)} = \| |u|^{p(x)-1} \|_{L^1(\Omega)} \leq |\mu|(\Omega). \tag{4.68}$$

□

Proposition 4.10. *Let u be an entropy solution to problem (1.1). Then,*

$$meas\{|u| > t\} \longrightarrow 0, \quad as\ t \longrightarrow \infty \tag{4.69}$$

and

$$meas\{|\nabla u| > t\} \longrightarrow 0, \quad as\ t \longrightarrow \infty. \tag{4.70}$$

Proof.
• Proof of (4.70)
From (4.68), one has

$$\int_{\Omega} |u|^{p(x)-1} dx \leq |\mu|(\Omega). \tag{4.71}$$

One deduces that

$$\int_{\{|u|>t\}} |u|^{p(x)-1} dx \leq |\mu|(\Omega). \tag{4.72}$$

Moreover

$$\int_{\{|u|>t\}} |u|^{p(x)-1} dx \geq \int_{\{|u|>t\}} t^{p(x)-1} dx. \tag{4.73}$$

For $t > 1$,

$$\int_{\{|u|>t\}} t^{p(x)-1} dx \geq \int_{\{|u|>t\}} t^{p^- -1} dx. \tag{4.74}$$

From (4.72), (4.73) and (4.74), one obtains

$$\int_{\{|u|>t\}} t^{p^- -1} dx \leq |\mu|(\Omega). \tag{4.75}$$

Thus,

$$meas\{|u| > t\} t^{p^- -1} \leq |\mu|(\Omega). \tag{4.76}$$

Therefore

$$meas\{|u| > t\} \leq \frac{|\mu|(\Omega)}{t^{p^- -1}}. \tag{4.77}$$

Hence, $meas\{|u| > t\}$ tends toward 0 as $t \longrightarrow \infty$.
• Proof of (4.70)
For the proof of (4.70), (see [2]).

□

□

Now, let $k > 0$, u_1 and u_2 two entropy solutions of problem (1.1).
One has

$$\int_{\{|u_1 - T_k(u_2)| \leq t\}} a\left(x, \nabla u_1\right) . \nabla T_t\left(u_1 - T_k(u_2)\right) dx + \int_{\Omega} |u_1|^{p(x)-2} u_1 T_t\left(u_1 - T_k(u_2)\right) dx \leq$$
$$\int_{\Omega} f(x) T_t\left(u_1 - T_k(u_2)\right) dx + \int_{\Omega} F . \nabla T_t(u_1 - T_k(u_2)) dx \tag{4.78}$$

and

$$\int_{\{|u_2 - T_k(u_1)| \leq t\}} a\left(x, \nabla u_2\right) . \nabla T_t\left(u_2 - T_k(u_1)\right) dx + \int_{\Omega} |u_2|^{p(x)-2} u_2 T_t\left(u_2 - T_k u_1\right) dx \leq$$
$$\int_{\Omega} f(x) T_t\left(u_2 - T_k(u_1)\right) dx + \int_{\Omega} F . \nabla T_t(u_2 - T_k(u_1)) dx. \tag{4.79}$$

By adding (4.78) and (4.79), one obtains

$$\begin{aligned}\int_{\{|u_1-T_k(u_2)|\leq t\}} & a\left(x,\nabla u_1\right).\nabla T_t\left(u_1-T_k(u_2)\right)dx+\int_{\{|u_2-T_k(u_1)|\leq t\}} a\left(x,\nabla u_2\right).\nabla T_t\left(u_2-T_k(u_1)\right)dx\\ &+\int_\Omega |u_1|^{p(x)-2}u_1T_t\left(u_1-T_k(u_2)\right)dx+\int_\Omega |u_2|^{p(x)-2}u_2T_t\left(u_2-T_k(u_1)\right)dx\\ &\leq \int_\Omega f(x)\left(T_t(u_1-T_k(u_2))+T_t(u_2-T_k(u_1))\right)dx\\ &+\int_{\{|u_1-T_k(u_2)|\leq t\}} F.\nabla(u_1-T_k(u_2))dx+\int_{\{|u_2-T_k(u_1)|\leq t\}} F.\nabla(u_2-T_k(u_1))dx\\ &\leq \int_\Omega f(x)\left(T_t(u_1-T_k(u_2))+T_t(u_2-T_k(u_1))\right)dx\\ &+\int_\Omega F.(\nabla(u_1-T_k(u_2))+\nabla(u_2-T_k(u_1)))dx. \end{aligned}\tag{4.80}$$

Moreover,

$$\begin{aligned}\int_\Omega F.(\nabla(u_1-T_k(u_2))+\nabla(u_2-T_k(u_1)))dx &= \int_\Omega F.(\nabla(u_1-T_k(u_1))+\nabla(u_2-T_k(u_2)))dx\\ &= \int_\Omega F.\nabla(u_1-T_k(u_1))dx+\int_\Omega F.\nabla(u_2-T_k(u_2))dx.\end{aligned}$$

One has

$$\begin{aligned}\int_\Omega F.\nabla(u_1-T_k(u_1))dx &= \int_{\{|u_1|>k\}} F.\nabla(u_1-ksgn(u_1))dx\\ &= \int_{\{|u_1|>k\}} F.\nabla u_1dx\\ &\leq \int_{\{|u_1|>k\}} |F||\nabla u_1|dx.\end{aligned}$$

One has $|F.\nabla u_1|=|F||\nabla u_1|\in L^1(\Omega)$ and $\lim\limits_{k\longrightarrow\infty} meas\{|u_1|>k\}\longrightarrow 0$. Thus,

$$\lim_{k\longrightarrow\infty}\int_\Omega F.\nabla(u_1-T_k(u_1))dx\leq 0.\tag{4.81}$$

In a similar way, one also obtains that

$$\lim_{k\longrightarrow\infty}\int_\Omega F.\nabla(u_2-T_k(u_2))dx\leq 0.\tag{4.82}$$

Therefore,

$$\lim_{k\longrightarrow\infty}\int_\Omega F.(\nabla(u_1-T_k(u_2))+\nabla(u_2-T_k(u_1)))dx\leq 0.\tag{4.83}$$

One has (see[21])

$$\lim_{k\longrightarrow\infty}\int_\Omega f(x)\left(T_t(u_1-T_k(u_2))+T_t(u_2-T_k(u_1))\right)=0.\tag{4.84}$$

By using (4.84) and (4.83) one deduces that

$$\lim_{k\longrightarrow\infty}\int_{\Omega}\left[f(x)\left(T_t(u_1-T_k(u_2))+T_t(u_2-T_k(u_1))\right)+F(\nabla(u_1-T_k(u_2))+\nabla(u_2-T_k(u_1)))\right]dx\leq 0.$$

Thus (see[21]),

$$\int_{A_1\cap A_2\cap A_3}\left[a(x,\nabla u_1)-a(x,\nabla u_2)\right].\nabla(u_1-u_2)dx$$
$$+\int_{\Omega}\left(|u_1|^{p(x)-2}u_1-|u_2|^{p(x)-2}u_2\right)T_t\left(u_1-u_2\right)dx\leq 0.$$

Therefore,

$$\int_{\Omega}\left(|u_1|^{p(x)-2}u_1-|u_2|^{p(x)-2}u_2\right)T_t\left(u_1-u_2\right)dx=0.$$

Thus,

$$u_1-u_2=0\ in\ \Omega.$$

References

1. S.N. Antontsev and J. F. Rodrigues, *On stationary thermo-rheological viscous flows*. Annal del Univ de Ferrara. 52 (2006), 19–36. https://doi.org/10.1007/s11565-006-0002-9
2. B. K. Bonzi and S. Ouaro; *Entropy solution for a doubly nonlinear elliptic problem with variable exponent*. J. Math. Appl. 370(2010), 392-405. https://doi.org/10.1016/j.jmaa.2010.05.022
3. M. M. Boureanu and D. N. Udrea; *Existence and multiplicity results for elliptic problems with p(.)-Growth conditions*. Nonlinear Anal., Real World Appl. 14, No. 4, 1829-1844 (2013). https://doi.org/10.1016/j.nonrwa.2012.12.001
4. Y. Chen, S. Levine and R. Rao; *Variable exponent, linear growth functionals in image restoration*. SIAM J. Appl. Math. 66, No. 4, 1383-1406 (2006). https://doi.org/10.1137/050624522
5. L. Diening; *Theoretical and numerical results for electrorheological fluids*. Freiburg im Breisgau: Univ. Freiburg im Breisgau, Mathematische Fakultät, 156 p. (2002). https://www.math.lmu.de/~diening/archive/diening_phd.pdf
6. X. Fan; *Solutions for p(x)–Laplacian Dirichlet problems with singular coefficients*. J. Math. Anal. Appl. 312, No. 2, 464-477 (2005). https://doi.org/10.1016/j.jmaa.2005.03.057
7. X. Fan and X. Han, *Existence and multiplicity of solutions for p(x)-Laplacian equations in* $\mathbb{R}^N$. Nonlinear Anal. 59 (2004) 173–188. https://doi.org/10.1016/j.na.2004.07.009
8. G. Fragnelli, *Positive periodic solutions for a system of anisotropic parabolic equations*. J. Math. Anal. Appl. 367 (2010) 204–228. https://doi.org/10.1016/j.jmaa.2009.12.039
9. P. Halmos and R. Paul; *Measure Theory*. Graduate Texts in Mathematics., vol. 18, Springer, New York, 1974. https://doi.org/10.1007/978-1-4684-9440-2
10. H. Hudzik; *"On generalized Orlicz-Sobolev space"* . Funct. Approximatio Comment. Math., vol. 4, pp. 37-51, 1976. https://zbmath.org/0355.46012
11. K. Kansié and S. Ouaro; *structural stability of p(x)-Laplacian kind problems with maximal monotone graphs and Neumann type boundary condition*. Ann. Math. Comput. Sci. Vol 21 (2024) 44-68. https://doi.org/10.56947/amcs.v21.247
12. O. Kovacik and J. Rakosnik; *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$. Czech. Math. J. 41(116), No. 4, 592-618 (1991). https://doi.org/10.21136/CMJ.1991.102493
13. J.L. Lions; *Quelques methodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris (1969). https://Etude/mathematiques/ISSN:1766-7526

14. M. Mihailescu and V. Radulescu; *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids*. Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. 467, No. 2134, 3033-3034 (2011). https://doi.org/10.1098/rspa.2005.1633
15. J. Musielak; *Orlicz Spaces and modular spaces*. Lecture Notes in Mathematics, vol. 1034 (1983), springer, Berlin. https://doi.org/10.1007/BFb0072210
16. I. Nyanquini, S. Ouaro and S. Soma; *"Entropy solution to nonlinear multivalued elliptic problem with variable exponents and measure data"*. Ann. Univ. Craiova. Ser. Mat. Inform., vol. 40, no.2, pp. 174-198, 2013. https://inf.ucv.ro/~ami/index.php/ami/article/viewFile/1619/863
17. A. Prignet; *Conditions aux limites non homogènes pour des problèmes elliptiques avec second membre mesure*. Ann. Fac. Sci. Toulouse, Math. 6 (1997), no. 2, 297–318. https://doi.org/10.5802/afst.867
18. K. Rajagopal and M. Ruzicka; *Mathematical modeling of electrorheological materials*. Contin. Mech. Thermodyn. 13, No. 1, 59-78 (2001). https://doi.org/10.1007/s001610100034
19. M. Ruzicka; *Electrorheological fluids: modeling and mathematical theory*. Lecture Notes in Mathematics. 1748. Berlin: Springer. xvi, 161 p. (2000). https://doi.org/10.1007/bfb0104029
20. M. Sanchon and J. M. Urbano; *Entropy solutions for the $p(x)$-Laplace equation*. Trans. Amer. Math. Soc. 361, No. 12, 6387-6405 (2009). https://doi.org/10.1090/s0002-9947-09-04399-2
21. B. Sawadogo, S. Ouaro and A. Kaboré; *Isotropic problem with non homogeneous Dirichlet boundary condition and L^1-data*. Bukovyn. Math. Zh. 14. No. 1, 70-89 (2026). https://doi.org/10.31861/bmj2026.01.07
22. N. Sawadogo and S. Ouaro; *nonlinear multivalued homogeneous Robin boundary $p(u)$-Laplacian problem*. Ann. Math. Comput. Sci. Vol 24 (2024) 57-84. https://doi.org/10.56947/amcs.v24.308

[1] Laboratoire de Mathématiques et d'Informatique, ED/ST, Université Joseph KI-ZERBO, 03 BP 7021 Ouagadougou 03, Burkina Faso.
Email address: boureimas808@gmail.com

[2] Laboratoire de Mathématiques et d'Informatique, ED/ST, Université Joseph KI-ZERBO, 03 BP 7021 Ouagadougou 03, Burkina Faso.
Email address: ouaro@yahoo.fr

[3] Laboratoire de Mathématiques et d'Informatique, Ecole Normale Supérieure 01 BP 1757 Ouagadougou 01, Burkina Faso.
Email address: kaboreadama59@yahoo.fr