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A WEIGHTED SHAPIRO-TYPE INEQUALITY FOR BOUNDED VARIABLES

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Abstract. We establish a weighted extension of the Shapiro inequality for positive variables restricted to a bounded interval. Combining a reduction strategy with the Titu lemma, the Lagrange identity, and the Cauchy-Schwarz inequality, we derive a threshold on the bounding parameter depending on the number of variables and weights. This threshold guarantees the validity of the weighted cyclic inequality whenever the primary weight dominates the secondary weight, generalizing classical unweighted bounds. Finally, numerical experiments illustrate the theoretical results and demonstrate the breakdown of the inequality when the weight dominance condition is violated.

Keywords. Cyclic inequalities, Shapiro inequality, Titu lemma, Lagrange identity, bounded variables.

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1. Introduction

1.1. **On the Shapiro inequality.** The study of cyclic inequalities constitutes a classical branch of mathematical analysis. A historical benchmark in this field is the Nesbitt inequality [8], proposed in 1903. It states that, for any real variables $x_1, x_2, x_3 > 0$ with $x_4 = x_1$ and $x_5 = x_2$, we have

$$\sum_{k=1}^{3} \frac{x_k}{x_{k+1} + x_{k+2}} \geq \frac{3}{2}. \tag{1.1}$$

In 1954, Shapiro [10] formulated a natural n-variable generalization. Specifically, for any real numbers $x_1, x_2, \ldots, x_n > 0$ indexed cyclically modulo n (with $x_{n+k} = x_k$ for all $k \in \mathbb{Z}$), he conjectured that

$$S(n) = \sum_{k=1}^{n} \frac{x_k}{x_{k+1} + x_{k+2}} \geq \frac{n}{2}.$$

This conjecture spurred extensive research to identify the precise values of n for which the inequality holds. Early progress was made by Diananda [4, 5],

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who established partial results for specific values of n. An important advance was achieved by Drinfel'd [6], who proved that the Shapiro inequality holds for any $n \leq 12$ when n is even, while exhibiting asymptotic bounds for large n. The complete settlement of the Shapiro inequality was finalized by Troesch [11, 12], who established that $S(n) \geq n/2$ holds if and only if $n \leq 12$ for even n, and $n \leq 23$ for odd n. Later, alternative analytic methods were developed by Bushell [1] and Bushell and McLeod [2], particularly addressing the even n regime. For a comprehensive review and pedagogical account of these developments, see Khrabrov [7].

1.2. **Some modern results.** When n exceeds the thresholds determined by Troesch [11, 12], counterexamples exist; the Shapiro inequality fails in general. This has motivated the development of alternative or refined approaches.

Below, we present two modern results on this subject.

One natural approach involves restricting the variables $x_1, x_2, \ldots, x_n$ to a bounded interval $[1/q_n, q_n]$ for some $q_n > 1$. Within this framework, Círtoaje [3] and Khrabrov [7] established the following baseline result.

Theorem 1.1. [3] [7, Solved problem 1.10 a)] *Let $n \geq 3$ be an integer and let $x_1, x_2, \ldots, x_n > 0$ be real variables indexed cyclically modulo n (with $x_{n+k} = x_k$ for all $k \in \mathbb{Z}$) satisfying*

$$x_1, x_2, \ldots, x_n \in \left[\frac{1}{q_n}, q_n\right]$$

for some $q_n > 1$. Then the inequality

$$\sum_{k=1}^{n} \frac{x_k}{x_{k+1} + x_{k+2}} \geq \frac{n}{2}$$

holds provided $1 < q_n \leq r_n$, where

$$r_n = \frac{n-1}{\sqrt{n^2 - 2n - 1}}.$$

Another approach introduces positive weights into the denominators of the cyclic terms. In this direction, Tuan and Thuong [13] examined weighted cyclic sums of the form

$$P(n, \alpha, \beta) = \sum_{k=1}^{n} \frac{x_k}{\alpha x_{k+1} + \beta x_{k+2}}$$

for weights $\alpha, \beta > 0$. In particular, they proved the theorem below for the special case $n = 4$.

Theorem 1.2. [13, Theorem 2.1] *Let $x_1, x_2, x_3, x_4 > 0$ be real variables and let $\alpha, \beta \geq 0$ be weights such that $\alpha + \beta > 0$. Then the inequality*

$$P(4, \alpha, \beta) = \frac{x_1}{\alpha x_2 + \beta x_3} + \frac{x_2}{\alpha x_3 + \beta x_4} + \frac{x_3}{\alpha x_4 + \beta x_1} + \frac{x_4}{\alpha x_1 + \beta x_2} \geq \frac{4}{\alpha + \beta}$$

holds provided $\alpha \geq \beta$, and fails for $\alpha < \beta$.

The case $n = 5$, more technical, is also discussed in [13, Theorem 2.3].

An open question raised in [13, Page 5] reads as follows: determine conditions on $\alpha, \beta > 0$ such that the inequality

$$\sum_{k=1}^{n} \frac{x_k}{\alpha x_{k+1} + \beta x_{k+2}} \geq \frac{n}{\alpha + \beta} \tag{1.2}$$

holds for any integer $n \geq 4$.

1.3. **Contributions.** In this paper, we resolve this question under variable interval constraints by extending the techniques of Círtoaje [3] and Khrabrov [7] to the weighted Shapiro sum investigated in [13]. Specifically, for any integer $n \geq 3$ and weights $\alpha \geq \beta \geq 0$ such that $\alpha + \beta > 0$, we determine a threshold $r_n(\alpha, \beta) > 1$ such that whenever $x_1, x_2, \ldots, x_n \in [1/r_n(\alpha, \beta), r_n(\alpha, \beta)]$ with $1 < q_n \leq r_n(\alpha, \beta)$, the weighted cyclic inequality (1.2) holds unconditionally for any n. In the symmetric case $\alpha = \beta$, our bound $r_n(\alpha, \alpha)$ reduces precisely to the threshold r_n obtained in Theorem 1.1 by Círtoaje [3] and Khrabrov [7]. Our second contribution is computational: through numerical experiments, we demonstrate the validity of $r_n(\alpha, \beta)$ and provide empirical evidence that the inequality can fail when $\alpha < \beta$.

1.4. **Paper organization.** The remainder of this paper is organized as follows. Section 2 presents our main theorem along with its detailed proof, incorporating the Titu lemma, the Lagrange identity, and a cyclic pairwise difference bound. Section 3 presents a numerical study on the validity of this theorem. Finally, Section 4 offers concluding remarks and outlines directions for future research.

2. Main results, proof and remarks

2.1. **The weighted Shapiro-type inequality for n variables.** The theorem below resolves the open question posed in [13, Page 5] under variable interval constraints. Specifically, it determines a precise upper bound on the interval parameter q_n to guarantee the validity of the weighted Shapiro-type inequality for an arbitrary number of variables (i.e., $n \geq 3$).

Theorem 2.1. *Let $n \geq 3$ be an integer, let $\alpha \geq \beta \geq 0$ be weights such that $\alpha + \beta > 0$, and let $x_1, x_2, \ldots, x_n > 0$ be real variables indexed cyclically modulo n satisfying*

$$x_1, x_2, \ldots, x_n \in \left[\frac{1}{q_n}, q_n\right] \tag{2.1}$$

for some $q_n > 1$. Then the inequality

$$\sum_{k=1}^{n} \frac{x_k}{\alpha x_{k+1} + \beta x_{k+2}} \geq \frac{n}{\alpha + \beta} \tag{2.2}$$

holds provided $1 < q_n \leq r_n(\alpha, \beta)$, where

$$r_n(\alpha, \beta) = \frac{n-1}{\sqrt{(n-1)^2 - \frac{\alpha+\beta}{\alpha}}}.$$

Proof. By hypothesis (2.1), for any $i \in \{1, 2, \ldots, n\}$, we have $1/q_n \leq x_i \leq q_n$. For any pair of indices $a, b \in \{1, 2, \ldots, n\}$, we derive

$$\frac{x_a}{x_b} \leq \frac{\max_{i\in\{1,2,\ldots,n\}} x_i}{\min_{i\in\{1,2,\ldots,n\}} x_i} \leq \frac{q_n}{\frac{1}{q_n}} = q_n^2,$$

which implies $q_n^2 x_b - x_a \geq 0$. In particular, for any $k \in \{1, 2, \ldots, n\}$ and $\ell \in \{1, 2\}$, we get

$$q_n^2 x_k - x_{k+\ell} \geq 0. \tag{2.3}$$

Now, for any $k \in \{1, 2, \ldots, n\}$, define

$$y_k = \alpha x_k + \beta x_{k+1}, \quad N_k = (\alpha + \beta) q_n^2 x_k - y_{k+1},$$

(noting that $y_1, y_2, \ldots, y_n > 0$ are indexed cyclically modulo n). Thus, we have

$$\sum_{k=1}^{n} \frac{N_k}{y_{k+1}} = \sum_{k=1}^{n} \frac{(\alpha + \beta) q_n^2 x_k - y_{k+1}}{y_{k+1}} = (\alpha + \beta) q_n^2 \sum_{k=1}^{n} \frac{x_k}{y_{k+1}} - n.$$

Hence, the target inequality (2.2) is equivalent to

$$\sum_{k=1}^{n} \frac{N_k}{y_{k+1}} \geq (\alpha + \beta) q_n^2 \frac{n}{\alpha + \beta} - n = n(q_n^2 - 1). \tag{2.4}$$

To establish a lower bound for this sum, note that, for any $k \in \{1, 2, \ldots, n\}$,

$$N_k = (\alpha + \beta) q_n^2 x_k - (\alpha x_{k+1} + \beta x_{k+2}) = \alpha(q_n^2 x_k - x_{k+1}) + \beta(q_n^2 x_k - x_{k+2}). \tag{2.5}$$

Since $\alpha \geq \beta \geq 0$ with $\alpha + \beta > 0$, combining (2.3) with (2.5) yields $N_k \geq 0$. Applying the Titu lemma to the non-negative variables $N_1, N_2, \ldots, N_n$ and $N_1 y_2, N_2 y_3, \ldots, N_n y_1$ gives

$$\sum_{k=1}^{n} \frac{N_k}{y_{k+1}} = \sum_{k=1}^{n} \frac{N_k^2}{N_k y_{k+1}} \geq \frac{A^2}{B}, \tag{2.6}$$

where

$$A = \sum_{k=1}^{n} N_k, \quad B = \sum_{k=1}^{n} N_k y_{k+1}.$$

Thus, by (2.4) and (2.6), it suffices to show that $A^2/B \geq n(q_n^2 - 1)$, or equivalently,

$$A^2 - n(q_n^2 - 1) B \geq 0. \tag{2.7}$$

Let us now evaluate A and B. The cyclic symmetry implies $\sum_{k=1}^{n} x_{k+1} = \sum_{k=1}^{n} x_k$ and $\sum_{k=1}^{n} y_{k+1} = \sum_{k=1}^{n} y_k$. Therefore, we have

$$\sum_{k=1}^{n} y_k = \sum_{k=1}^{n} (\alpha x_k + \beta x_{k+1}) = \alpha \sum_{k=1}^{n} x_k + \beta \sum_{k=1}^{n} x_{k+1} = (\alpha + \beta) \sum_{k=1}^{n} x_k.$$

This gives

$$A = \sum_{k=1}^{n} N_k = \sum_{k=1}^{n} \left((\alpha+\beta)q_n^2 x_k - y_{k+1}\right) = (\alpha+\beta)q_n^2 \sum_{k=1}^{n} x_k - \sum_{k=1}^{n} y_{k+1}$$
$$= (\alpha+\beta)q_n^2 \frac{1}{\alpha+\beta} \sum_{k=1}^{n} y_k - \sum_{k=1}^{n} y_k = (q_n^2 - 1) \sum_{k=1}^{n} y_k. \tag{2.8}$$

Moreover, still by cyclic symmetry, we have $\sum_{k=1}^{n} y_{k+1}^2 = \sum_{k=1}^{n} y_k^2$. We derive

$$B = \sum_{k=1}^{n} N_k y_{k+1} = \sum_{k=1}^{n} \left((\alpha+\beta)q_n^2 x_k - y_{k+1}\right) y_{k+1}$$
$$= (\alpha+\beta)q_n^2 \sum_{k=1}^{n} x_k y_{k+1} - \sum_{k=1}^{n} y_{k+1}^2 = (\alpha+\beta)q_n^2 \sum_{k=1}^{n} x_k y_{k+1} - \sum_{k=1}^{n} y_k^2. \tag{2.9}$$

To evaluate B, define

$$E = \sum_{k=1}^{n} y_k^2 - (\alpha+\beta) \sum_{k=1}^{n} x_k y_{k+1}. \tag{2.10}$$

Substituting $(\alpha+\beta)\sum_{k=1}^{n} x_k y_{k+1} = \sum_{k=1}^{n} y_k^2 - E$ into (2.9) gives

$$B = q_n^2 \left(\sum_{k=1}^{n} y_k^2 - E \right) - \sum_{k=1}^{n} y_k^2 = (q_n^2 - 1) \sum_{k=1}^{n} y_k^2 - q_n^2 E. \tag{2.11}$$

Substituting (2.8) and (2.11) into the left-hand side of (2.7) and using the Lagrange identity for $y_1, \ldots, y_n$, that is $\left(\sum_{k=1}^{n} y_k\right)^2 - n\sum_{k=1}^{n} y_k^2 = -\sum_{1\le i<j\le n}(y_i - y_j)^2$, we obtain

$$A^2 - n(q_n^2-1)B = (q_n^2-1)^2 \left(\sum_{k=1}^{n} y_k \right)^2 - n(q_n^2-1) \left((q_n^2-1) \sum_{k=1}^{n} y_k^2 - q_n^2 E \right)$$
$$= (q_n^2-1)^2 \left(\left(\sum_{k=1}^{n} y_k \right)^2 - n \sum_{k=1}^{n} y_k^2 \right) + n q_n^2 (q_n^2-1) E$$
$$= -(q_n^2-1)^2 \sum_{1\le i<j\le n} (y_i - y_j)^2 + n q_n^2 (q_n^2-1) E.$$

Thus, (2.7) holds if we have $n q_n^2 (q_n^2-1) E \ge (q_n^2-1)^2 \sum_{1\le i<j\le n}(y_i - y_j)^2$, or equivalently,

$$n q_n^2 E \ge (q_n^2 - 1) \sum_{1\le i<j\le n} (y_i - y_j)^2. \tag{2.12}$$

Now, let us examine an expansion for E. By cyclic symmetry, we have

$$\sum_{k=1}^{n} y_k^2 = \sum_{k=1}^{n} (\alpha x_k + \beta x_{k+1})^2 = \alpha^2 \sum_{k=1}^{n} x_k^2 + 2\alpha\beta \sum_{k=1}^{n} x_k x_{k+1} + \beta^2 \sum_{k=1}^{n} x_{k+1}^2$$
$$= (\alpha^2 + \beta^2) \sum_{k=1}^{n} x_k^2 + 2\alpha\beta \sum_{k=1}^{n} x_k x_{k+1}. \tag{2.13}$$

Furthermore, it is clear that

$$(\alpha + \beta) \sum_{k=1}^{n} x_k y_{k+1} = (\alpha + \beta) \sum_{k=1}^{n} x_k (\alpha x_{k+1} + \beta x_{k+2})$$
$$= \alpha(\alpha + \beta) \sum_{k=1}^{n} x_k x_{k+1} + \beta(\alpha + \beta) \sum_{k=1}^{n} x_k x_{k+2}. \tag{2.14}$$

Subtracting (2.14) from (2.13) per definition (2.10) gives

$$E = (\alpha^2 + \beta^2) \sum_{k=1}^{n} x_k^2 + 2\alpha\beta \sum_{k=1}^{n} x_k x_{k+1} - \alpha(\alpha + \beta) \sum_{k=1}^{n} x_k x_{k+1}$$
$$- \beta(\alpha + \beta) \sum_{k=1}^{n} x_k x_{k+2}$$
$$= (\alpha^2 + \beta^2) \sum_{k=1}^{n} x_k^2 - \alpha(\alpha - \beta) \sum_{k=1}^{n} x_k x_{k+1} - \beta(\alpha + \beta) \sum_{k=1}^{n} x_k x_{k+2}. \tag{2.15}$$

For any pair of indices $a, b \in \{1, 2, \ldots, n\}$, we have $2x_a x_b = x_a^2 + x_b^2 - (x_a - x_b)^2$. This identity and the cyclic symmetry gives, for any $\ell \in \{1, 2\}$,

$$\sum_{k=1}^{n} x_k x_{k+\ell} = \sum_{k=1}^{n} x_k^2 - \frac{1}{2} \sum_{k=1}^{n} (x_k - x_{k+\ell})^2.$$

Substituting these expansions into (2.15) yields

$$E = (\alpha^2 + \beta^2) \sum_{k=1}^{n} x_k^2 - \alpha(\alpha - \beta) \left(\sum_{k=1}^{n} x_k^2 - \frac{1}{2} \sum_{k=1}^{n} (x_k - x_{k+1})^2 \right)$$
$$- \beta(\alpha + \beta) \left(\sum_{k=1}^{n} x_k^2 - \frac{1}{2} \sum_{k=1}^{n} (x_k - x_{k+2})^2 \right)$$
$$= \big((\alpha^2 + \beta^2) - \alpha(\alpha - \beta) - \beta(\alpha + \beta) \big) \sum_{k=1}^{n} x_k^2$$
$$+ \frac{\alpha(\alpha - \beta)}{2} \sum_{k=1}^{n} (x_k - x_{k+1})^2 + \frac{\beta(\alpha + \beta)}{2} \sum_{k=1}^{n} (x_k - x_{k+2})^2$$
$$= \frac{\alpha(\alpha - \beta)}{2} \sum_{k=1}^{n} (x_k - x_{k+1})^2 + \frac{\beta(\alpha + \beta)}{2} \sum_{k=1}^{n} (x_k - x_{k+2})^2. \tag{2.16}$$

Now, to simplify E, for any $k \in \{1, \dots, n\}$, define

$$u_k = x_k - x_{k+1},$$

(noting that $u_1, u_2, \dots, u_n$ are indexed cyclically modulo n). Noticing that $y_k - y_{k+1} = (\alpha x_k + \beta x_{k+1}) - (\alpha x_{k+1} + \beta x_{k+2}) = \alpha u_k + \beta u_{k+1}$ and using the cyclic symmetry, we get

$$\begin{aligned}
&\sum_{k=1}^{n} (y_k - y_{k+1})^2 = \sum_{k=1}^{n} (\alpha u_k + \beta u_{k+1})^2 \\
&= \alpha^2 \sum_{k=1}^{n} u_k^2 + 2\alpha\beta \sum_{k=1}^{n} u_k u_{k+1} + \beta^2 \sum_{k=1}^{n} u_{k+1}^2 \\
&= (\alpha^2 + \beta^2) \sum_{k=1}^{n} u_k^2 + 2\alpha\beta \sum_{k=1}^{n} u_k u_{k+1}. \qquad (2.17)
\end{aligned}$$

Similarly, noticing that $x_k - x_{k+2} = (x_k - x_{k+1}) + (x_{k+1} - x_{k+2}) = u_k + u_{k+1}$, we have

$$\sum_{k=1}^{n} (x_k - x_{k+2})^2 = \sum_{k=1}^{n} (u_k + u_{k+1})^2 = 2 \sum_{k=1}^{n} u_k^2 + 2 \sum_{k=1}^{n} u_k u_{k+1}.$$

Multiplying this identity by $\alpha\beta$ and subtracting it from (2.17) gives

$$\begin{aligned}
&\sum_{k=1}^{n} (y_k - y_{k+1})^2 - \alpha\beta \sum_{k=1}^{n} (x_k - x_{k+2})^2 \\
&= (\alpha^2 + \beta^2) \sum_{k=1}^{n} u_k^2 + 2\alpha\beta \sum_{k=1}^{n} u_k u_{k+1} - 2\alpha\beta \sum_{k=1}^{n} u_k^2 - 2\alpha\beta \sum_{k=1}^{n} u_k u_{k+1} \\
&= (\alpha^2 + \beta^2 - 2\alpha\beta) \sum_{k=1}^{n} u_k^2 = (\alpha - \beta)^2 \sum_{k=1}^{n} (x_k - x_{k+1})^2.
\end{aligned}$$

Rearranging yields

$$\sum_{k=1}^{n} (x_k - x_{k+2})^2 = \frac{1}{\alpha\beta} \left(\sum_{k=1}^{n} (y_k - y_{k+1})^2 - (\alpha - \beta)^2 \sum_{k=1}^{n} (x_k - x_{k+1})^2 \right).$$

Substituting this into (2.16) gives

$$\begin{aligned}
E &= \frac{\alpha(\alpha-\beta)}{2}\sum_{k=1}^{n}(x_k-x_{k+1})^2 \\
&+ \frac{\beta(\alpha+\beta)}{2}\cdot\frac{1}{\alpha\beta}\left(\sum_{k=1}^{n}(y_k-y_{k+1})^2-(\alpha-\beta)^2\sum_{k=1}^{n}(x_k-x_{k+1})^2\right) \\
&= \frac{\alpha+\beta}{2\alpha}\sum_{k=1}^{n}(y_k-y_{k+1})^2+\left(\frac{\alpha(\alpha-\beta)}{2}-\frac{(\alpha+\beta)(\alpha-\beta)^2}{2\alpha}\right)\sum_{k=1}^{n}(x_k-x_{k+1})^2 \\
&= \frac{\alpha+\beta}{2\alpha}\sum_{k=1}^{n}(y_k-y_{k+1})^2+\frac{\beta^2(\alpha-\beta)}{2\alpha}\sum_{k=1}^{n}(x_k-x_{k+1})^2. \qquad (2.18)
\end{aligned}$$

Since $\alpha \geq \beta \geq 0$ with $\alpha+\beta > 0$, the second term in this expansion is non-negative, so

$$E \geq \frac{\alpha+\beta}{2\alpha}\sum_{k=1}^{n}(y_k-y_{k+1})^2.$$

Hence, (2.12) holds if

$$nq_n^2\left(\frac{\alpha+\beta}{2\alpha}\right)\sum_{k=1}^{n}(y_k-y_{k+1})^2 \geq (q_n^2-1)\sum_{1\leq i<j\leq n}(y_i-y_j)^2. \qquad (2.19)$$

Using a telescoping sum and applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\sum_{1\leq i<j\leq n}(y_i-y_j)^2 = \sum_{1\leq i<j\leq n}\left(\sum_{k=i}^{j-1}(y_k-y_{k+1})\right)^2 \\
&\leq \sum_{1\leq i<j\leq n}\left((j-i)\sum_{k=i}^{j-1}(y_k-y_{k+1})^2\right) \leq \sum_{1\leq i<j\leq n}\left((n-1)\sum_{k=1}^{n}(y_k-y_{k+1})^2\right) \\
&= (n-1)\left(\sum_{1\leq i<j\leq n}1\right)\sum_{k=1}^{n}(y_k-y_{k+1})^2 = (n-1)\binom{n}{2}\sum_{k=1}^{n}(y_k-y_{k+1})^2 \\
&= \frac{n(n-1)^2}{2}\sum_{k=1}^{n}(y_k-y_{k+1})^2.
\end{aligned}$$

Thus, (2.19) holds if

$$nq_n^2\left(\frac{\alpha+\beta}{2\alpha}\right)\sum_{k=1}^{n}(y_k-y_{k+1})^2 \geq (q_n^2-1)\frac{n(n-1)^2}{2}\sum_{k=1}^{n}(y_k-y_{k+1})^2,$$

which simplifies to

$$q_n^2\left(\frac{\alpha+\beta}{\alpha}\right) \geq (q_n^2-1)(n-1)^2.$$

Regrouping terms with respect to q_n^2 gives

$$q_n^2\left((n-1)^2-\frac{\alpha+\beta}{\alpha}\right)\leq(n-1)^2.$$

Since $n\geq 3$ and $\alpha\geq\beta\geq 0$ with $\alpha+\beta>0$, we obtain $(n-1)^2\geq 4>2\geq(\alpha+\beta)/\alpha$, ensuring $(n-1)^2-(\alpha+\beta)/\alpha>0$. Thus, we derive

$$q_n\leq\frac{n-1}{\sqrt{(n-1)^2-\frac{\alpha+\beta}{\alpha}}}=r_n(\alpha,\beta).$$

This completes the proof of Theorem 2.1. □

2.2. **Remarks.** Several technical remarks concerning Theorem 2.1 and its proof are provided below.

2.2.1. *Special cases and classical connections.*

Remark 2.2. For $n=3$, the theorem is valid but some restrictions can be removed: as demonstrated in [13, Equation (1.2)], the inequality holds for arbitrary variables $x_1,x_2,x_3>0$ and weights $\alpha,\beta>0$ without any interval or ordering constraints. For $n\geq 4$, the problem grows considerably more complex (see [13]), where our theorem provides a tractable analytical framework.

Remark 2.3. In the unweighted case $\alpha=\beta$, the threshold $r_n(\alpha,\beta)$ simplifies to

$$r_n(\alpha,\alpha)=\frac{n-1}{\sqrt{n^2-2n-1}},$$

corresponding to the bound r_n obtained in Theorem 1.1.

Remark 2.4. The target inequality (2.2) and the interval constraint (2.1) are invariant under homogeneous scaling $x_k\mapsto cx_k$ for any constant $c>0$. Consequently, the condition $x_1,x_2,\ldots,x_n\in[1/q_n,q_n]$ is fundamentally a restriction on the relative dispersion of the variables, namely,

$$\frac{\max_{i\in\{1,2,\ldots,n\}}x_i}{\min_{i\in\{1,2,\ldots,n\}}x_i}\leq q_n^2.$$

Thus, Theorem 2.1 guarantees that the weighted cyclic inequality holds whenever this maximal ratio does not exceed $r_n^2(\alpha,\beta)$.

Remark 2.5. The equality in (2.2) holds when $x_1=x_2=\cdots=x_n$. In this symmetric case, each term in the sum evaluates to $1/(\alpha+\beta)$. Furthermore, since $r_n(\alpha,\beta)>1$ for any $n\geq 3$ and $\alpha\geq\beta\geq 0$ with $\alpha+\beta>0$, setting $x_1=x_2=\cdots=x_n=1$, we have $x_1,x_2,\ldots,x_n\in[1/r_n(\alpha,\beta),r_n(\alpha,\beta)]$.

Remark 2.6. Setting $\beta=0$ reduces (2.2) to the classical unweighted scaled inequality

$$\sum_{k=1}^{n}\frac{x_k}{x_{k+1}}\geq n,$$

which holds for any variables $x_1,\ldots,x_n>0$ by the arithmetic mean-geometric mean inequality. Thus, the restriction $q_n\leq r_n(\alpha,\beta)$ specifically captures the instability introduced by secondary cyclic interactions (i.e., x_{k+2}).

2.2.2. *Role of weight dominance and parameter sensitivity.*

Remark 2.7. The condition $\alpha \geq \beta$ is used in the proof in the following two places.

(1) It ensures the non-negativity of $N_1, N_2, \ldots, N_n$, which is required to apply the Titu lemma.
(2) It yields a non-negative remainder term

$$\frac{\beta^2(\alpha-\beta)}{2\alpha}\sum_{k=1}^{n}(x_k - x_{k+1})^2 \geq 0$$

in (2.18), allowing us to bound E in terms of $(y_1, y_2, \ldots, y_n)$-differences.

Remark 2.8. Notice that $r_n(\alpha, \beta)$ depends on the parameter ratio $\gamma = \beta/\alpha \in (0, 1]$, since

$$r_n(\alpha, \beta) = \frac{n-1}{\sqrt{(n-1)^2 - (1+\gamma)}}.$$

Consequently, uniform scaling of weights $\lambda\alpha$ and $\lambda\beta$ for $\lambda > 0$ leaves the threshold $r_n(\alpha, \beta)$ invariant, even though the minimum bound $n/(\lambda(\alpha+\beta))$ scales inversely with λ.

2.2.3. *Structural decompositions.*

Remark 2.9. Equation (2.18) provides the following exact sum-of-squares decomposition:

$$E = \frac{\alpha+\beta}{2\alpha}\sum_{k=1}^{n}(y_k - y_{k+1})^2 + \frac{\beta^2(\alpha-\beta)}{2\alpha}\sum_{k=1}^{n}(x_k - x_{k+1})^2.$$

Because both quadratic forms are non-negative for $\alpha \geq \beta \geq 0$ such that $\alpha+\beta > 0$, $E \geq 0$ holds unconditionally for any variables $x_1, \ldots, x_n > 0$.

2.2.4. *Asymptotic analysis and bound tightness.*

Remark 2.10. As $n \to \infty$, a Taylor expansion of $r_n(\alpha, \beta)$ yields

$$r_n(\alpha, \beta) = 1 + \frac{\alpha+\beta}{2\alpha(n-1)^2} + \mathcal{O}\left(\frac{1}{n^4}\right).$$

This reveals that $r_n(\alpha, \beta) \to 1^+$ at a quadratic rate of $\mathcal{O}(n^{-2})$. Geometrically, as the number of variables increases, the interval $[1/q_n, q_n]$ must contract toward 1 at a quadratic rate of $\mathcal{O}(n^{-2})$ to suppress variations that would otherwise breach the lower bound.

Remark 2.11. The threshold $r_n(\alpha, \beta)$ is a *sufficient* bound rather than a strictly necessary one. The conservative nature of $r_n(\alpha, \beta)$ arises from the Cauchy-Schwarz inequality used to establish

$$\sum_{1\leq i<j\leq n}(y_i - y_j)^2 \leq \frac{n(n-1)^2}{2}\sum_{k=1}^{n}(y_k - y_{k+1})^2.$$

For specific boundary configurations, the true maximal threshold q_n^* permitting (2.2) may be strictly larger than $r_n(\alpha, \beta)$.

Remark 2.12. We have

$$\frac{\partial r_n(\alpha,\beta)}{\partial(\beta/\alpha)} = \frac{r_n(\alpha,\beta)^3}{2(n-1)^2} > 0.$$

This implies that strong weight asymmetry (i.e., $\beta \ll \alpha$) restricts the variables to tighter intervals around 1, whereas larger values of $\beta/\alpha < 1$ allow broader domains.

3. Numerical study

In this section, we present a numerical study of Theorem 2.1 and examine the necessity of the condition $\alpha \geq \beta \geq 0$ with $\alpha + \beta > 0$. All numerical experiments and evaluations are conducted using the R software environment [9].

3.1. **Validating the main inequality.** To begin, Table 1 presents numerical evaluations of $r_n(\alpha,\beta)$, $1/r_n(\alpha,\beta)$, and derives $[1/r_n(\alpha,\beta), r_n(\alpha,\beta)]$ across $n \in \{3,4,5,6,8,10\}$ for distinct α and β.

Table 1. Numerical analysis of $r_n(\alpha,\beta)$ and $[1/r_n(\alpha,\beta), r_n(\alpha,\beta)]$.

n	α	β	Threshold $r_n(\alpha,\beta)$	Bound $1/r_n(\alpha,\beta)$	$[1/r_n(\alpha,\beta), r_n(\alpha,\beta)]$
3	2.50	1.25	1.26491	0.79057	[0.79057, 1.26491]
3	3.20	0.80	1.20605	0.82916	[0.82916, 1.20605]
3	1.80	0.60	1.22474	0.81650	[0.81650, 1.22474]
4	3.50	2.10	1.10282	0.90676	[0.90676, 1.10282]
4	1.80	0.45	1.07763	0.92796	[0.92796, 1.07763]
5	2.00	1.50	1.05963	0.94373	[0.94373, 1.05963]
5	4.20	1.05	1.04151	0.96014	[0.96014, 1.04151]
6	0.80	0.40	1.03142	0.96954	[0.96954, 1.03142]
8	2.40	1.20	1.01567	0.98457	[0.98457, 1.01567]
10	5.00	1.25	1.00781	0.99225	[0.99225, 1.00781]

The results in Table 1 demonstrate the two following features.

(1) For a fixed n, the size of the admissible interval expands as the ratio β/α increases, confirming Remark 2.12. For example, let us focus on the case $n = 3$.

When $\beta/\alpha = 0.50$ (corresponding to $\alpha = 2.50$ and $\beta = 1.25$), we obtain $r_3(\alpha,\beta) \approx 1.26491$, yielding the widest interval $[0.79057, 1.26491]$.

As the weights become more asymmetric with $\beta/\alpha = 0.333$ (corresponding to $\alpha = 1.80$ and $\beta = 0.60$), $r_3(\alpha,\beta)$ drops to 1.22474.

Under stronger asymmetry where $\beta/\alpha = 0.25$ (corresponding to $\alpha = 3.20$ and $\beta = 0.80$), $r_3(\alpha,\beta)$ further contracts to 1.20605, restricting variables to the narrower interval $[0.82916, 1.20605]$.

This pattern persists for greater values of n. At $n = 4$, decreasing the ratio from $\beta/\alpha = 0.60$ to $\beta/\alpha = 0.25$ causes $r_4(\alpha,\beta)$ to drop from 1.10282 to 1.07763. Similarly, at $n = 5$, shifting from $\beta/\alpha = 0.75$ to

$\beta/\alpha = 0.25$ shrinks $r_5(\alpha,\beta)$ from 1.05963 to 1.04151. These numerical trends demonstrate that the primary-neighbor dominance (i.e., $\alpha \gg \beta$) demands tighter variable control around 1 to maintain the target lower bound.

(2) As the number of variables n increases, $r_n(\alpha,\beta)$ approaches 1 rapidly, causing the interval $[1/r_n(\alpha,\beta), r_n(\alpha,\beta)]$ to contract tightly around 1. For instance, with $\beta/\alpha = 0.50$, $r_n(\alpha,\beta)$ decreases from 1.26491 (corresponding to $n = 3$) to 1.03142 (corresponding to $n = 6$), 1.01567 (corresponding to $n = 8$), and reaches 1.00781 at $n = 10$. This decay highlights the quadratic contraction rate $\mathcal{O}(n^{-2})$ required to stabilize the cyclic interactions (see Remark 2.10).

Still in the framework of Theorem 2.1, to test whether the target lower bound

$$L(n,\alpha,\beta) = \frac{n}{\alpha+\beta}$$

holds for $x_1, x_2, \ldots, x_n \in [1/r_n(\alpha,\beta), r_n(\alpha,\beta)]$, we evaluate the cyclic sum

$$S(n,\alpha,\beta) = \sum_{k=1}^{n} \frac{x_k}{\alpha x_{k+1} + \beta x_{k+2}}$$

under distinct variable configurations operating at or near the interval boundary. We examine four distinct configurations. Under Configuration A (alternating profile), which maximizes contrast between consecutive indices, we consider

$$(x_1, x_2, \ldots, x_n) = \left(r_n(\alpha,\beta), \frac{1}{r_n(\alpha,\beta)}, r_n(\alpha,\beta), \frac{1}{r_n(\alpha,\beta)}, \ldots \right).$$

Next, under Configuration B (single peak spike), which isolates a single localized perturbation, we consider

$$(x_1, x_2, \ldots, x_n) = \left(r_n(\alpha,\beta), \frac{1}{r_n(\alpha,\beta)}, \frac{1}{r_n(\alpha,\beta)}, \ldots, \frac{1}{r_n(\alpha,\beta)} \right).$$

Furthermore, under Configuration C (geometric profile), which forms a finite geometric sequence from $1/r_n(\alpha,\beta)$ to $r_n(\alpha,\beta)$, for any $k \in \{1,2,\ldots,n\}$, we set

$$x_k = r_n(\alpha,\beta)^{-1+\frac{2(k-1)}{n-1}}.$$

Finally, under Configuration D (smooth cosinusoidal oscillation), which captures smooth cyclic wave profiles strictly inside $[1/r_n(\alpha,\beta), r_n(\alpha,\beta)]$, for any $k \in \{1,2,\ldots,n\}$, we set

$$x_k = \frac{1}{2}\left(r_n(\alpha,\beta) + \frac{1}{r_n(\alpha,\beta)} \right) + \frac{1}{2}\left(r_n(\alpha,\beta) - \frac{1}{r_n(\alpha,\beta)} \right) \cos\left(\frac{2\pi(k-1)}{n} \right).$$

Table 2 summarizes the calculated values for $S(n,\alpha,\beta)$, $L(n,\alpha,\beta)$, and the gap

$$\Delta(n,\alpha,\beta) = S(n,\alpha,\beta) - L(n,\alpha,\beta).$$

TABLE 2. Numerical verification of the inequality $S(n,\alpha,\beta) \geq L(n,\alpha,\beta)$ in Theorem 2.1.

n	(α,β)	$L(n,\alpha,\beta)$	Config.	$[1/r_n(\alpha,\beta), r_n(\alpha,\beta)]$	$S(n,\alpha,\beta)$	$\Delta(n,\alpha,\beta)$	Status
3	$(2.50, 1.25)$	0.80000	A	$[0.79057, 1.26491]$	0.82698	+0.02698	Valid
			B	$[0.79057, 1.26491]$	0.83937	+0.03937	Valid
			C	$[0.79057, 1.26491]$	0.82670	+0.02670	Valid
			D	$[0.79057, 1.26491]$	0.81841	+0.01841	Valid
4	$(3.50, 2.10)$	0.71429	A	$[0.90676, 1.10282]$	0.71641	+0.00213	Valid
			B	$[0.90676, 1.10282]$	0.71824	+0.00396	Valid
			C	$[0.90676, 1.10282]$	0.71740	+0.00311	Valid
			D	$[0.90676, 1.10282]$	0.71737	+0.00309	Valid
5	$(4.20, 1.05)$	0.95238	A	$[0.96014, 1.04151]$	0.95383	+0.00145	Valid
			B	$[0.96014, 1.04151]$	0.95325	+0.00087	Valid
			C	$[0.96014, 1.04151]$	0.95301	+0.00063	Valid
			D	$[0.96014, 1.04151]$	0.95293	+0.00055	Valid
6	$(0.80, 0.40)$	5.00000	A	$[0.96954, 1.03142]$	5.00213	+0.00213	Valid
			B	$[0.96954, 1.03142]$	5.00182	+0.00182	Valid
			C	$[0.96954, 1.03142]$	5.00147	+0.00147	Valid
			D	$[0.96954, 1.03142]$	5.00146	+0.00146	Valid
8	$(2.40, 1.20)$	2.22222	A	$[0.98457, 1.01567]$	2.22246	+0.00024	Valid
			B	$[0.98457, 1.01567]$	2.22237	+0.00015	Valid
			C	$[0.98457, 1.01567]$	2.22235	+0.00012	Valid
			D	$[0.98457, 1.01567]$	2.22233	+0.00011	Valid

The results in Table 2 demonstrate the following three features.

(1) Across all combinations of n, weights α and β, and configurations, the gap $\Delta(n,\alpha,\beta)$ remains non-negative, validating Theorem 2.1.
(2) The gap $\Delta(n,\alpha,\beta)$ decreases as n increases due to the quadratic shrinking of the allowable interval $[1/r_n(\alpha,\beta), r_n(\alpha,\beta)]$.
(3) Configuration D (i.e., smooth cosinusoidal oscillation) consistently yields tight gap values, illustrating how smooth continuous transitions between $1/r_n(\alpha,\beta)$ and $r_n(\alpha,\beta)$ closely track the theoretical lower bound $L(n,\alpha,\beta)$.

3.2. **Breakdown of the inequality for $\alpha < \beta$.** We now conduct a numerical study demonstrating that, when the condition $\alpha \geq \beta$ is violated (i.e., when $\alpha < \beta$), the target inequality

$$S(n,\alpha,\beta) \geq L(n,\alpha,\beta)$$

fails predominantly for $x_1, x_2, \ldots, x_n \in [1/r_n(\alpha,\beta), r_n(\alpha,\beta)]$.

From a theoretical perspective, the failure of the inequality for $\alpha < \beta$ can be explained by the behavior of the Hessian matrix of $S_n(\boldsymbol{x},\alpha,\beta) = S(n,\alpha,\beta)$ with $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ at the symmetric point $\boldsymbol{x}_0 = (1, 1, \ldots, 1)$. Indeed, expanding

$S_n(\boldsymbol{x},\alpha,\beta)-L(n,\alpha,\beta)$ along the alternating direction $\boldsymbol{v}=(1,-1,1,-1,\dots)$ gives

$$S_n(\boldsymbol{x}_0+\epsilon\boldsymbol{v},\alpha,\beta)-L(n,\alpha,\beta)=\frac{2n\alpha(\alpha-\beta)}{(\alpha+\beta)^3}\epsilon^2+\mathcal{O}(\epsilon^3).$$

For $\alpha<\beta$, the quadratic coefficient is strictly negative, turning $\boldsymbol{x}_0$ into a local maximum along high-frequency oscillatory trajectories. Consequently, any small perturbation along $\boldsymbol{v}$ immediately drives the sum below the lower bound $L(n,\alpha,\beta)$.

Table 3 presents numerical evaluations of $r_n(\alpha,\beta)$, $1/r_n(\alpha,\beta)$, and derives $[1/r_n(\alpha,\beta),r_n(\alpha,\beta)]$ across $n\in\{3,4,5,6,8,10\}$ for distinct α and β.

TABLE 3. Numerical analysis of $r_n(\alpha,\beta)$ and $[1/r_n(\alpha,\beta),r_n(\alpha,\beta)]$.

n	α	β	Threshold $r_n(\alpha,\beta)$	Bound $1/r_n(\alpha,\beta)$	$[1/r_n(\alpha,\beta),r_n(\alpha,\beta)]$
3	0.80	1.60	2.00000	0.50000	[0.50000, 2.00000]
4	0.50	2.00	1.50000	0.66667	[0.66667, 1.50000]
4	0.30	1.80	2.12132	0.47140	[0.47140, 2.12132]
5	0.60	1.80	1.15470	0.86603	[0.86603, 1.15470]
6	0.40	2.00	1.14708	0.87178	[0.87178, 1.14708]
8	0.30	2.10	1.09322	0.91473	[0.91473, 1.09322]
10	0.50	4.50	1.06810	0.93624	[0.93624, 1.06810]

To demonstrate that the inequality is predominantly violated, we test the four variable configurations operating within $[1/r_n(\alpha,\beta),r_n(\alpha,\beta)]$ described below. Under Configuration A (standard alternating profile), we consider

$$(x_1,x_2,\dots,x_n)=\left(r_n(\alpha,\beta),\frac{1}{r_n(\alpha,\beta)},r_n(\alpha,\beta),\frac{1}{r_n(\alpha,\beta)},\dots\right).$$

Next, under Configuration B (asymmetric alternating wave), for any $k\in\{1,2,\dots,n\}$, we set

$$x_k=1-(-1)^k\cdot 0.50\cdot\frac{r_n(\alpha,\beta)-\frac{1}{r_n(\alpha,\beta)}}{2}.$$

Furthermore, under Configuration C (moderated high-frequency wave), for any $k\in\{1,2,\dots,n\}$, we set

$$x_k=1+(-1)^k\cdot 0.75\cdot\frac{r_n(\alpha,\beta)-\frac{1}{r_n(\alpha,\beta)}}{2}.$$

Finally, under Configuration D (smooth cosinusoidal oscillation), for any $k\in\{1,2,\dots,n\}$, we set

$$x_k=\frac{1}{2}\left(r_n(\alpha,\beta)+\frac{1}{r_n(\alpha,\beta)}\right)+\frac{1}{2}\left(r_n(\alpha,\beta)-\frac{1}{r_n(\alpha,\beta)}\right)\cos\left(\frac{2\pi(k-1)}{n}\right).$$

Table 4 summarizes the calculated values for $S(n,\alpha,\beta)$, $L(n,\alpha,\beta)$, and the gap

$$\Delta(n,\alpha,\beta)=S(n,\alpha,\beta)-L(n,\alpha,\beta).$$

TABLE 4. Numerical breakdown tests of the inequality $S(n,\alpha,\beta) \geq L(n,\alpha,\beta)$ for $\alpha < \beta$.

n	(α,β)	$L(n,\alpha,\beta)$	Config.	$[1/r_n(\alpha,\beta), r_n(\alpha,\beta)]$	$S(n,\alpha,\beta)$	$\Delta(n,\alpha,\beta)$	Status
4	$(0.50, 2.00)$	1.60000	A	$[0.66667, 1.50000]$	1.54000	-0.06000	Invalid
			B	$[0.66667, 1.50000]$	1.58307	-0.01693	Invalid
			C	$[0.66667, 1.50000]$	1.56113	-0.03887	Invalid
			D	$[0.66667, 1.50000]$	1.79300	$+0.19300$	Valid
4	$(0.30, 1.80)$	1.90476	A	$[0.47140, 2.12132]$	1.70635	-0.19841	Invalid
			B	$[0.47140, 2.12132]$	1.83234	-0.07242	Invalid
			C	$[0.47140, 2.12132]$	1.71983	-0.18493	Invalid
			D	$[0.47140, 2.12132]$	2.78665	$+0.88189$	Valid
6	$(0.40, 2.00)$	2.50000	A	$[0.87178, 1.14708]$	2.48958	-0.01042	Invalid
			B	$[0.87178, 1.14708]$	2.49736	-0.00264	Invalid
			C	$[0.87178, 1.14708]$	2.49405	-0.00595	Invalid
			D	$[0.87178, 1.14708]$	2.52810	$+0.02810$	Valid
8	$(0.30, 2.10)$	3.33333	A	$[0.91473, 1.09322]$	3.32837	-0.00496	Invalid
			B	$[0.91473, 1.09322]$	3.33209	-0.00125	Invalid
			C	$[0.91473, 1.09322]$	3.33053	-0.00281	Invalid
			D	$[0.91473, 1.09322]$	3.34456	$+0.01122$	Valid

The results in Table 4 demonstrate the following four features.

(1) Across all evaluated parameter sets with $\alpha < \beta$, 3 out of 4 configurations (75%) produce negative gaps.
(2) Sequences featuring alternating or modulated wave variations (Configurations A, B, and C) consistently drive $S(n,\alpha,\beta)$ below the target bound $L(n,\alpha,\beta)$.
(3) The empirical prevalence of invalid status outcomes confirms that $\alpha \geq \beta$ is a necessary condition for the global validity of the inequality in Theorem 2.1.
(4) The sufficient bound $r_n(\alpha,\beta)$ guarantees validity across all $x_1, x_2, \ldots, x_n \in [1/r_n(\alpha,\beta), r_n(\alpha,\beta)]$. For smooth configurations (such as Configuration D, i.e., smooth cosinusoidal oscillation), numerical evidence suggests that larger interval bounds might be admissible without violating the inequality.

4. CONCLUSION

In this paper, we established a threshold $r_n(\alpha,\beta)$ that guarantees the validity of a weighted Shapiro-type cyclic inequality for n positive variables restricted to a bounded interval. By combining a reduction strategy with the Titu lemma and the Lagrange identity, we derived sufficient conditions under which primary-neighbor weight dominance (i.e., $\alpha \geq \beta$) enforces the target lower bound. This framework generalizes classical unweighted cyclic inequalities, as well as recovering them as special cases. Furthermore, a comprehensive numerical study corroborated the utility of these interval constraints across diverse variable configurations, demonstrating the breakdown of the inequality when the weight dominance

condition is violated. Future research could explore the formulation of higher-order multi-variable cyclic inequalities utilizing localized ratio-form bounds, or investigate the integration of these discrete threshold methods into continuous frameworks.

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