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NOTES ON THE NORMAL SOFT INT-GROUPS OF A SOFT INT-GROUP

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ABSTRACT. In this study, we generalize the concept of normal soft int-groups. We introduce the concept of normal soft int-groups of a soft int-group and investigate some of their properties and structured characteristics. Also we introduce the concepts of homomorphisms and isomorphisms of soft int-groups, and use them to develop results concerning soft int-groups of a soft int-group.

1. INTRODUCTION

In 1999, Molodtsov [19] proposed an approach for modeling, vagueness and uncertainty, called soft set theory. A soft set is a parameterized family of subsets of the universal set. We can say that soft sets are neighborhood systems, and that they are a special case of context dependent fuzzy sets. In soft set theory the problem of setting the membership function, among other related problems, simply does not arise. This makes the theory very convenient and easy to applyin practice. Some researchers have applied soft sets theory to many different areas such as decison making [7, 12, 13], algebras [1, 2, 3, 4, 14, 18], topology [5, 11], fuzzy sets [9, 10, 23] and matrix theory [6, 8]. The author, investigated some properties of soft algebraic structures [15, 16, 20, 21, 22]. The purpose of this paper is to deal with the algebraic structure of normal soft int-groups, homomorphisms and isomorphisms of them . In Section 2, we summarize some basic concepts which will be used throughout the paper. In Section 3, we introduce the concept of normal soft int-groups of a soft int-group and investigate some of their basic properties. In Section 4, we define the concepts of homomorphisms and isomorphisms of soft int-groups and prove some important results.

2. PRELIMINARIES

This section contains some basic definitions and preliminary results which will be needed in the sequal. Throughout this work, U refers to an initial universe set, E is a set of parameters and P(U) is the power set of U.

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Definition 2.1. ([19]) For any subset A of E, a soft set f_A over U is a set, defined by a function f_A, representing a mapping $f_A : E \to P(U)$, such that $f_A(x) = \emptyset$ if $x \notin A$. A soft set over U can also be represented by the set of ordered pairs $f_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\}$. Note that the set of all soft sets over U will be denoted by $S(U)$. From here on, "soft set" will be used without over U.

Definition 2.2. ([7]) Let $f_A \in S(U)$. Then,
(1) f_A is called an empty soft set, denoted by Φ_A, if $f_A(x) = \emptyset$ for all $x \in E$,
(2) f_A is called a A-universal soft set, denoted by $f_{\tilde{A}}$, if $f_A(x) = U$ for all $x \in E$,
(3) f_A is called a universal soft set, denoted by $f_{\tilde{E}}$, if $f_A(x) = U$ for all $x \in E$,
(4) the set $Im(f_A) = \{f_A(x) : x \in A\}$ is called image of f_A and if $A = E$, then $Im(f_E)$ is called image of E under f_A.

Definition 2.3. ([7]) Let $f_A, f_B \in S(U)$. Then,
(1) f_A is a soft sub set of f_B, denoted by $f_A \tilde{\subseteq} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$,
(2) f_A and f_B are soft equal, denoted by $f_A = f_B$, if and only if $f_A(x) = f_B(x)$ for all $x \in E$,
(3) the set $(f_A \tilde{\cup} f_B)(x) = f_A(x) \cup f_B(x)$ for all $x \in E$ is called union of f_A and f_B,
(4) the set $(f_A \tilde{\cap} f_B)(x) = f_A(x) \cap f_B(x)$ for all $x \in E$ is called intersection of f_A and f_B.

As an illustration, let us consider the following example.

Example 2.4. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be an initial universe set and $E = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of parameters. Let $A = \{x_1, x_2, x_3\}, B = \{x_3, x_4, x_5\}, C = \{x_1, x_2\}, D = \{x_3, x_4\}$. Define

$$f_A = \{(x_1, \{u_1, u_2\}), (x_2, \{u_3\}), (x_3, \{u_4, u_5\})\},$$

$$f_B = \{(x_3, \{u_1, u_4\}), (x_4, U), (x_5, \{u_3\})\},$$

$$f_C = \{(x_1, U), (x_2, U)\},$$

$$f_D = \{(x_3, \{\}), (x_4, \{\})\}.$$

Then for all $x \in E$ we have

$$(f_A \tilde{\cup} f_B)(x) = \{(x_1, \{u_1, u_2\}), (x_2, \{u_3\}), (x_3, \{u_4, u_5, u_1), (x_4, U), (x_5, \{u_3\})\}$$

and

$$(f_A \tilde{\cap} f_B)(x) = \{(x_3, \{u_4)\}.$$

Also $f_C = f_{\tilde{C}}$ and $f_D = \Phi_D$.

Remark 2.5. $f_A \tilde{\subseteq} f_B$ does not imply that every element of f_A is an element of f_B. Then the difinition of classical subset is not valid for the soft subset. For example, let $U = \{u_1, u_2, u_3, u_4, u_5\}, E = \{x_1, x_2, x_3, x_4\}, A = \{x_1, x_2\}, B = \{x_1, x_2, x_4\}$. If $f_A = \{(x_1, \{u_1, u_2\}), (x_2, \{u_4\})\}$ and $f_B = \{(x_1, \{u_1, u_2, u_3\}), (x_2, \{u_4, u_5\}), (x_4, \{u_1\})\}$, then $f_A \tilde{\subseteq} f_B$ but $f_A \nsubseteq f_B$ as classical subset.

Definition 2.6. ([2]) Let φ be a function from A into B and $f_A, f_B \in S(U)$. Then soft image $\varphi(f_A)$ of f_A under φ is defined by

$$\varphi(f_A)(y) = \begin{cases} \cup\{f_A(x) \mid x \in A, \varphi(x) = y\} & \text{if } \varphi^{-1}(y) \neq \emptyset \\ \emptyset & \text{if } \varphi^{-1}(y) = \emptyset \end{cases}$$

and soft pre-image (or soft inverse image) of f_B under φ is $\varphi^{-1}(f_B)(x) = f_B(\varphi(x))$ for all $x \in A$.

The following Proposition extends the algebraic structures of soft sets.

Proposition 2.7. *Let φ be a mapping from A into B and ψ a mapping from B into C. Then the following assertions hold:*
(1) Let $(f_A)_i \in S(U), i \in I$, where I is a nonempty index set, then $\varphi(\tilde{\cup}_{i\in I} f_A)_i = \cup_{i\in I}\varphi((f_A)_i)$. Also for any $(f_A)_1, (f_A)_2 \in S(U)$, if $(f_A)_1 \tilde{\subseteq} (f_A)_2$, then $\varphi(f_A)_1 \subseteq \varphi(f_A)_2$.
(2) For any $(f_B)_i \in S(U), i \in I$, where I is a nonempty index set,

$$\varphi^{-1}((\tilde{\cup}_{i\in I} f_B)_i) = \cup_{i\in I}\varphi^{-1}((f_B)_i),$$

$$\varphi^{-1}((\tilde{\cap}_{i\in I} f_B)_i) = \cap_{i\in I}\varphi^{-1}((f_B)_i),$$

also for any $(f_B)_1, (f_B)_2 \in S(U)$, if $(f_B)_1 \tilde{\subseteq} (f_B)_2$, then $\varphi^{-1}(f_B)_1 \subseteq \varphi^{-1}(f_B)_2$.
(3) Let $f_A \in S(U)$, then $\varphi^{-1}(\varphi(f_A)) \supseteq f_A$. In particular, if φ is an injection, then $\varphi^{-1}(\varphi(f_A)) = f_A$, and hence, $f_A \to \varphi(f_A)$ is an injection and $f_B \to \varphi^{-1}(f_B)$ is a surjection.
(4) If $f_B \in S(U)$, then $\varphi(\varphi^{-1}(f_B) \subseteq f_B$. In particular, if φ is an a surjection,, then $\varphi(\varphi^{-1}(f_B) = f_B$, and hence, $f_A \to \varphi(f_A)$ is an surjection and $f_B \to \varphi^{-1}(f_B)$ is a injection.
(5) Let $f_A, f_B \in S(U)$. Then $\varphi(f_A) \subseteq f_B$ if and only if $f_A \subseteq \varphi^{-1}(f_B)$.
(6) Let $f_A, f_C \in S(U)$. Then $\psi(\varphi(f_A)) = (\psi\varphi)(f_A)$ and $\varphi^{-1}(\psi^{-1}(f_C)) = (\psi\varphi)^{-1}(f_C)$.

Proof. It is clearly that the assertions (1) and (2) hold.
(3) For any $x_1 \in X, f_A \in S(U)$ we have that

$$\varphi^{-1}(\varphi(f_A))(x_1) = \varphi(f_A)(\varphi(x_1) = \cup\{f_A(x_2) \mid x_2 \in A, \varphi(x_1) = \varphi(x_2)\} \supseteq \varphi(x_1).$$

In particular, when φ is an injection $\varphi^{-1}(\varphi(f_A))(x_1) = \varphi(x_1)$.
(4) For any $y \in Y, f_B \in S(U)$ we have that

$$\begin{aligned} \varphi(\varphi^{-1}(f_B)(y) &= \cup\{\varphi^{-1}(f_B)(x) \mid x \in A, \varphi(x) = y\} \\ &= \cup\{\varphi^{-1}(f_B)(\varphi(x)) \mid x \in A, \varphi(x) = y\} \\ &= \cup\{f_B(\varphi(x)) \mid x \in A, \varphi(x) = y\} \\ &= \begin{cases} f_B(y) & \text{if } y \in \varphi(A) \\ \emptyset & \text{if } y \notin \varphi(A) \end{cases} \\ &\subseteq f_B(y). \end{aligned}$$

If φ is a surjection, then $\varphi(\varphi^{-1}(f_B) = f_B$.
(5) Assertion is an immediate consequence of the four preceding assertions.

(6) Let $f_A \in S(U), c \in C$. Then

$$\begin{aligned}\psi(\varphi(f_A))(c) &= \cup\{\varphi(f_A)(y) \mid y \in B, \psi(y) = c\}\\ &= \cup\{\cup\{f_A(x) \mid x \in A, \varphi(x) = y\} \mid y \in B, \psi(y) = c\}\\ &= \cup\{f_A(x) \mid x \in A, (\psi\varphi)(x) = y\}\\ &= (\psi\varphi)(f_A)(c).\end{aligned}$$

Also if $f_C \in S(U), x \in A$, then

$$((\psi\varphi)^{-1}(f_C))(x) = f_C(\psi(\varphi(x))) = \psi^{-1}(f_C)(\varphi(x)) = \varphi^{-1}(\psi^{-1}(f_C)(x).$$

□

Definition 2.8. ([2]) Let G be a group and $f_G \in S(U)$. Then, f_G is called a soft intersection group (int-group) over U if $f_G(xy) \supseteq f_G(x) \cap f_G(y)$ and $f_G(x^{-1}) = f_G(x)$ for all $x, y \in G$.
Throughout this paper, G denotes an arbitrary group with identity element e and the set of all soft int-groups with parameter set G over U will be denoted by $S_G(U)$. For short, instead of " f_G is a soft int-group with the parameter set G over U " we say " f_G is a soft int-group."

Definition 2.9. ([17]) Let G be a group and $A, B \subseteq G$. Then, the soft product of soft sets f_A and f_B is defined as $(f_A o f_B)(x) = \cup\{f_A(u) \cap f_B(v) \mid uv = x, u, v \in G\}$ for all $x \in G$.

Definition 2.10. ([2]) Let G be a group. Then, the soft set f_G is called normal soft int-group if $f_G(xyx^{-1}) = f_G(y)$ for all $x, y \in G$. Let $NS_G(U)$ denotes the set of all normal soft int-groups in G.

Definition 2.11. ([2]) Let $f_A \in S(U)$ and $\alpha \in P(U)$. Then, α-inclusion of the soft set f_A, denoted by f_A^α , is defined as $f_A^\alpha = \{x \in A \mid f_A(x) \supseteq \alpha\}$. If $\alpha = \emptyset$ then the set $f_A^* = \{x \in A \mid f_A(x) \neq \emptyset\}$ is called support of f_A.

Theorem 2.12. *([17]) Let $f_G \in S(U)$ and $\alpha \in P(U)$. Then $f_G \in S_G(U)$ if and only if f_G^α is a subgroup G, whenever it is nonempty.*

Definition 2.13. ([2]) Let $f_G \in S_G(U)$. Then, e-set of f_G, denoted by e_{f_G}, is defined as $e_{f_G} = \{x \in G \mid f_G(x) = f_G(e)\}$.

Theorem 2.14. *([2]) Let $f_G \in S_G(U)$. Then $f_G(e) \supseteq f_G(x)$ for all $x \in G$.*

3. NORMAL SOFT INT-GROUPS OF A SOFT INT-GROUP

In this section, we define normal soft int-groups of a soft int-group and study some of their basic properties.

Definition 3.1. Let G be a group,$A \subseteq G, \alpha \in P(U)$. We define $\alpha_A \in S(U)$ as follows:

$$\alpha_A(x) = \begin{cases} \alpha & \text{for } x \in A \\ \emptyset & \text{for } x \in G \setminus A \end{cases}$$

In particular, if A is a singleton, say, y, then α_A is denoted by $\alpha_{\{y\}}$. We refer to 1_A as the characteristic function of A as follows:

$$1_A(x) = \begin{cases} U & \text{for } x \in A \\ \emptyset & \text{for } x \in G \setminus A \end{cases}$$

Definition 3.2. Let $f_G, g_G \in S_G(U)$ and $f_G \tilde{\subseteq} g_G$. Then f_G is called a normal soft int-group of g_G, written $f_G \tilde{\triangleright} g_G$, if $f_G(xyx^{-1}) \supseteq f_G(y) \cap g_G(x)$ for all $x, y \in G$.

Corollary 3.3. *The following statements are immediate from Definition.*
(1) If G_1 and G_2 are two subgroups of G, then G_1 is a normal subgroup of G_2 if and only if 1_{G_1} is a normal soft int-group of 1_{G_2}.
(2) Let $f_G \in NS_G(U)$, $g_G \in S_G(U)$ and $f_G \tilde{\subseteq} g_G$. Then $f_G \tilde{\triangleright} g_G$.
(3) Every soft int-group is a normal soft int-group of itself.
(4) $f_G \in NS_G(U)$ if and only if $f_G \tilde{\triangleright} 1_G$.

Proposition 3.4. *Let $f_G, g_G \in S_G(U)$ and $f_G \tilde{\subseteq} g_G$. Then the following assertions are equivalent:*
(1) $f_G \tilde{\triangleright} g_G$.
(2) $f_G(yx) \supseteq f_G(xy) \cap g_G(y)$ for all $x, y \in G$.
(3) $f_G(e)_{\{x\}} o f_G \supseteq (f_G o f_G(e)_{\{x\}}) \cap g_G$ for all $x \in G$.

Proof. (1) $\Rightarrow$ (2) : Assume $x, y \in G$. Then

$$f_G(yx) = f_G(yxyy^{-1}) \supseteq f_G(xy) \cap g_G(y).$$

(2) $\Rightarrow$ (3) : let $x, y \in G$. Then

$$\begin{aligned}(f_G(e)_{\{x\}} o f_G)(y) &= \cup\{f_G(e)_{\{x\}}(y_1) \cap f_G(y_2) \mid y_1 y_2 = y, y_1, y_2 \in G\} \\ &= \cup\{f_G(e) \cap f_G(y_2) \mid x y_2 = y, x, y_2 \in G\} \\ &= f_G(y_2) \\ &= f_G(x^{-1}y)\end{aligned}$$

and also $(\mathrm{f}_G o f_G(e)_{\{x\}})(y) = f_G(yx^{-1})$. Now (2) completes the proof. .
(3) $\Rightarrow$ (1) : Let $x, y \in G$. By condition (3) we have that $f_G(x^{-1}y) \supseteq f_G(yx^{-1}) \cap g_G(y)$. Now

$$f_G(xyx^{-1}) = f_G((xy)x^{-1}) \supseteq f_G(x^{-1}(xy)) \cap g_G(x^{-1}) = f_G(y) \cap g_G(x).$$

□

Proposition 3.5. *Let $f_G, g_G \in S_G(U)$. Then f_G is a normal soft int-group of g_G if and only if f_G^α is a normal subgroup of g_G^α for all $\alpha \in P(U)$.*

Proof. Let f_G is a normal soft int-group of g_G and $\alpha \in P(U)$.
Then from Theorem 2.12 f_G^α is a subgroup of g_G^α. Assume that $x \in g_G^\alpha, y \in f_G^\alpha$ then $g_G(x), f_G(y) \supseteq \alpha$ and so $f_G(xyx^{-1}) \supseteq f_G(y) \cap g_G(x) \supseteq \alpha \cap \alpha = \alpha$. Hence $xyx^{-1} \in f_G^\alpha$ and f_G^α is a normal subgroup of g_G^α.
Conversely, suppose that f_G^α is a normal subgroup of g_G^α for all $\alpha \in P(U)$ so by Theorem 2.12 f_G is a soft int-group of g_G. Let $x, y \in G, \alpha, \beta \in P(U)$ such that $f_G(y) = \alpha, g_G(x) = \beta$. If $\gamma = \alpha \cap \beta$, then $y \in f_G^\gamma$ and so $xyx^{-1} \in f_G^\gamma$. Thus $f_G(xyx^{-1}) \supseteq \gamma = \alpha \cap \beta = f_G(y) \cap g_G(x)$ □

Proposition 3.6. *Let $f_G, g_G \in S_G(U)$ and f_G is a normal soft int-group of g_G. Then*
(1) If $f_G \tilde{\subseteq} g_G$, then e_{f_G} is a normal subgroup of e_{g_G}.
(2) If for all $\alpha \neq \emptyset \neq \beta$ in $P(U)$, $\alpha \cap \beta \neq \emptyset$, then f_G^* is a normal subgroup of g_G^*.

Proof. (1) Let $x \in e_{g_G}$ and $y \in e_{f_G}$, then $g_G(x) = g_G(e), f_G(y) = f_G(e)$. Now

$$f_G(xyx^{-1}) \supseteq f_G(y) \cap g_G(x) = f_G(e) \cap g_G(e) = f_G(e).$$

Also from Theorem 2.14 $f_G(e) \supseteq f_G(xyx^{-1})$. Then $f_G(xyx^{-1}) = f_G(e)$ and so e_{f_G} is a normal subgroup of e_{g_G}.
(2) Let $x \in g_G^*$ and $y \in f_G^*$. Then $f_G(y) \neq \emptyset \neq g_G(x)$. Since $f_G(xyx^{-1}) \supseteq f_G(y) \cap g_G(x) \neq \emptyset$ so $f_G(xyx^{-1}) \neq \emptyset$ and $f_G^* \trianglerighteq g_G^*$. □

Proposition 3.7. *Let $f_G \in NS_G(U)$ and $g_G \in S_G(U)$. Then $(f_G \tilde{\cap} g_G) \tilde{\triangleright} g_G$.*

Proof. Let $x, y \in G$. Then

$$\begin{aligned} (f_G \tilde{\cap} g_G)(xy) &= f_G(xy) \cap g_G(xy) \\ &\supseteq f_G(x) \cap f_G(y) \cap g_G(x) \cap g_G(y) \\ &= f_G(x) \cap g_G(x) \cap f_G(y) \cap g_G(y) \\ &= (f_G \tilde{\cap} g_G)(x) \cap (f_G \tilde{\cap} g_G)(y) \end{aligned}$$

thus $(f_G \tilde{\cap} g_G)(xy) \supseteq (f_G \tilde{\cap} g_G)(x) \cap (f_G \tilde{\cap} g_G)(y)$.
Also

$$(f_G \tilde{\cap} g_G)(x^{-1}) = f_G(x^{-1}) \cap g_G)(x^{-1}) = f_G(x) \cap g_G)(x) = (f_G \tilde{\cap} g_G)(x)$$

and so $(f_G \tilde{\cap} g_G) \in S_G(U)$. Now $f_G \tilde{\cap} g_G \subseteq g_G$ and

$$\begin{aligned} (f_G \tilde{\cap} g_G)(xyx^{-1}) &= f_G(xyx^{-1}) \cap g_G(xyx^{-1}) \\ &= f_G(y) \cap g_G(xyx^{-1}) \\ &\supseteq f_G(y) \cap g_G(x) \cap g_G(y) \cap g_G(x^{-1}) \\ &= (f_G \tilde{\cap} g_G)(y) \cap g_G(x). \end{aligned}$$

Hence $(f_G \tilde{\cap} g_G) \tilde{\triangleright} g_G$. □

Proposition 3.8. *Let $f_G, g_G, h_G \in S_G(U)$ such that $f_G, g_G \tilde{\triangleright} h_G$. Then $(f_G \tilde{\cap} g_G) \tilde{\triangleright} h_G$.*

Proof. Clearly, $(f_G \tilde{\cap} g_G) \in S_G(U)$ and $(f_G \tilde{\cap} g_G) \subseteq h_G$. If $x, y \in G$, then

$$\begin{aligned} (f_G \tilde{\cap} g_G)(xyx^{-1}) &= f_G(xyx^{-1}) \cap g_G(xyx^{-1}) \\ &\supseteq f_G(y) \cap h_G(x) \cap g_G(y) \cap h_G(x) \\ &= (f_G \tilde{\cap} g_G)(y) \cap h_G(x). \end{aligned}$$

Therefore $(f_G \tilde{\cap} g_G) \tilde{\triangleright} h_G$. □

Lemma 3.9. *Let $f_G \in S_G(U)$ and H be a group. Suppose that φ is an epimorphism of G onto H. Then $\varphi(f_G) \in S_H(U)$.*

Proof. Let $u, v \in H$. Then $u = f(x)$ and $v = f(y)$ for some $x, y \in G$. Now

$$\begin{aligned}
\varphi(f_G)(uv) &= \cup\{f_G(z) \mid z \in G, \varphi(z) = uv\} \\
&\supseteq \cup\{f_G(xy) \mid x, y \in G, u = f(x), v = f(y)\} \\
&\supseteq \cup\{f_G(x) \cap f_G(y) \mid x, y \in G, u = f(x), v = f(y)\} \\
&= (\cup\{f_G(x) \mid x \in G, u = f(x)\}) \cap (\cup\{f_G(y) \mid y \in G, v = f(y)\}) \\
&= \varphi(f_G)(u) \cap \varphi(f_G)(v).
\end{aligned}$$

Clearly, $\varphi(f_G)(u^{-1}) = \varphi(f_G)(u)$. Hence $\varphi(f_G) \in S_H(U)$. □

Proposition 3.10. *Let $f_G, g_G \in S_G(U)$ and $f_G \tilde{\triangleright} g_G$. If H be a group and φ a homomorphism from G into H, then $\varphi(f_G) \tilde{\triangleright} \varphi(g_G)$.*

Proof. From Lemma 3.9 and Proposition 2.7 we have that $\varphi(f_G), \varphi(g_G) \in S_H(U)$ and $\varphi(f_G) \subseteq \varphi(g_G)$ respectively. Let $x, y \in H$, then

$$\begin{aligned}
\varphi(f_G)(xyx^{-1}) &= \cup\{f_G(z) \mid z \in G, \varphi(u) = xyx^{-1}\} \\
&\supseteq \cup\{f_G(uvu^{-1})) \mid u, v \in G, x = \varphi(u), y = \varphi(v)\} \\
&\supseteq \cup\{f_G(v) \cap g_G(u) \mid u, v \in G, x = \varphi(u), y = \varphi(v)\} \\
&= (\cup\{f_G(v) \mid v \in G, y = \varphi(v)\}) \cap (\cup\{g_G(u) \mid u \in G, x = \varphi(u)\}) \\
&\supseteq \varphi(f_G(y)) \cap \varphi(g_G(x)).
\end{aligned}$$

Then $\varphi(f_G) \tilde{\triangleright} \varphi(g_G)$. □

Lemma 3.11. *Let H be a group and $g_H \in S_H(U)$. If φ is a homomorphism of G into H, then $\varphi^{-1}(g_H) \in S_G(U)$.*

Proof. Let $x, y \in G$. Then

$$\begin{aligned}
\varphi^{-1}(g_H)(xy) &= g_H(\varphi(xy)) \\
&= g_H(\varphi(x)\varphi(y)) \\
&\supseteq g_H(\varphi(x)) \cap g_H(\varphi(y)) \\
&= \varphi^{-1}(g_H)(x) \cap \varphi^{-1}(g_H)(y).
\end{aligned}$$

Also

$$\varphi^{-1}(g_H)(x^{-1}) = g_H(\varphi(x^{-1})) = g_H(\varphi^{-1}(x)) = g_H(\varphi(x)) = \varphi^{-1}(g_H)(x).$$

Then $\varphi^{-1}(g_H) \in S_G(U)$. □

Proposition 3.12. *Let H be a group. Let $f_H, g_H \in S_H(U)$ and $f_H \tilde{\triangleright} g_H$. Let φ be a homomorphism from G into H. Then $\varphi^{-1}(f_H) \tilde{\triangleright} \varphi^{-1}(g_H)$.*

Proof. By Lemma 3.11 and Proposition 2.7 we obtain $\varphi^{-1}(f_H), \varphi^{-1}(g_H) \in S_G(U)$ and $\varphi^{-1}(f_H) \subseteq \varphi^{-1}(g_H)$ respectively. Now for all $x, y \in G$ we have that

$$\begin{aligned}\varphi^{-1}(f_H)(xyx^{-1}) &= f_H(\varphi(xyx^{-1}))\\ &= f_H(\varphi(x)\varphi(y)\varphi(x^{-1}))\\ &= f_H(\varphi(x)\varphi(y)\varphi^{-1}(x))\\ &\supseteq f_H(\varphi(y)) \cap g_H(\varphi(x))\\ &= \varphi^{-1}(f_H)(y) \cap \varphi^{-1}(g_H)(x).\end{aligned}$$

Hence $\varphi^{-1}(f_H)\tilde{\trianglerighteq}\varphi^{-1}(g_H)$. □

4. HOMOMORPHISMS AND ISOMORPHISMS OF SOFT INT-GROUPS

Definition 4.1. Let G and H be groups and let $f_G \in S_G(U), g_H \in S_H(U)$.
(1) A homomorphism φ of G onto H is called a homomorphism of f_G onto g_H if $\varphi(f_G) = g_H$ and we write $f_G \approx g_H$.
(2) An isomorphism φ of G onto H is called an isomorphism of f_G onto g_H if $\varphi(f_G) = g_H$ and we write $f_G \cong g_H$.

Proposition 4.2. *Let $f_G \in S_G(U), N \trianglerighteq G$ and let $x \in G$ such that $[x]$ denotes the coset xN. Define $g : G/N \to P(U)$ as $g([x]) = \cup\{f_G(z) \mid z \in [x]\}$. Then $g \in S_{G/N}(U)$.*

Proof. Let $x, y \in G$. Then
(1)

$$\begin{aligned}g([x][y]) &= \cup\{f_G(z) \mid z \in [xy]\}\\ &= \cup\{f_G(uv) \mid u \in [x], v \in [y]\}\\ &\supseteq \cup\{f_G(u) \cap f_G(v) \mid u \in [x], v \in [y]\}\\ &= (\cup\{f_G(u) \mid u \in [x]\}) \cap (\cup\{f_G(v) \mid v \in [y]\})\\ &= g([x])g([y]).\end{aligned}$$

(2)

$$\begin{aligned}g([x]^{-1}) &= g([x^{-1}])\\ &= \cup\{f_G(z) \mid z \in [x^{-1}]\}\\ &= \cup\{f_G(w^{-1}) \mid w^{-1} \in [x^{-1}]\}\\ &= \cup\{f_G(w) \mid w \in [x]\}\\ &= g([x]).\end{aligned}$$

Therefore $g \in S_{G/N}(U)$. □

The soft int-group g defined in Proposition 4.2 is called the quotient soft int-group of f_G of G relative to the normal subgroup N of G and is denoted by f_G/N.

Definition 4.3. Let $f_G, g_G \in S_G(U)$ and $f_G \tilde{\triangleright} g_G$. If for all $\alpha \neq \emptyset \neq \beta$ in $P(U)$,$\alpha \cap \beta \neq \emptyset$, then from Proposition 3.6 f_G^* is a normal subgroup of g_G^* and so $g_G|_{g_G^*} \in S_{g_G^*}(U)$. Now by Proposition 4.2 the factor soft int-group of $g_G|_{g_G^*}$ relative to f_G^* exists. We denote this factor soft int-group by g_G/f_G and call it the quotient soft int-group of g_G relative to f_G.

Proposition 4.4. *Let $f_G \tilde{\triangleright} g_G$ and for all $\alpha \neq \emptyset \neq \beta$ in $P(U)$,$\alpha \cap \beta \neq \emptyset$. Then $g_G|_{g_G^*} \approx g_G/f_G$.*

Proof. Let φ be the natural homomorphism from g_G^* onto g_G^*/f_G^*. Let $x \in g_G^*$ and $[x] = xf_G^*$. Then $\varphi(g_G|_{g_G^*})([x]) = \cup\{g_G|_{g_G^*}(z) \mid \varphi(z) = [x]\} = \cup\{g_G(y) \mid y \in [x]\} = (g_G/f_G)([x])$.
Hence $g_G|_{g_G^*} \approx g_G/f_G$. □

Proposition 4.5. *Let G and H be two groups and $g_G \in S_G(U)$, $h_H \in S_H(U)$, $g_G \approx h_H$. If for all $\alpha \neq \emptyset \neq \beta$ in $P(U)$,$\alpha \cap \beta \neq \emptyset$, then there exists a normal soft int-group f_G of g_G such that $g_G/f_G \cong h_H|_{h_H^*}$.*

Proof. If $g_G \approx h_H$, then there exists an epimorphism φ of G onto H such that $\varphi(g_G) = h_H$. Let $x \in G$ and define $f_G : G \to P(U)$ as

$$f_G(x) = \begin{cases} g_G(x) & \text{for } x \in Ker\varphi \\ \emptyset & \text{for } x \in G \setminus Ker\varphi \end{cases}$$

Clearly $f_G \subseteq g_G$ and let $x, y \in G$.
(1) If $x, y \in Ker\varphi$, then

$$f_G(xy) = g_G(xy) \supseteq g_G(x) \cap g_G(y) = f_G(x) \cap f_G(y)$$

and

$$f_G(x^{-1}) = g_G(x^{-1}) = g_G(x) = f_G(x).$$

(2) If $x \in Ker\varphi, y \in G \setminus Ker\varphi$, then

$$f_G(xy) = \emptyset \supseteq g_G(x) \cap \emptyset = f_G(x) \cap f_G(y)$$

and

$$f_G(x^{-1}) = g_G(x^{-1}) = g_G(x) = f_G(x).$$

(3) If $x, y \in G \setminus Ker\varphi$, then $f_G(xy) = f_G(x) = f_G(y) = \emptyset$.
Hence $f_G \in S_G(U)$. Now we prove $f_G \tilde{\triangleright} g_G$.
(1) If $x \in Ker\varphi, y \in G$, then $yxy^{-1} \in Ker\varphi$ and so

$$\begin{aligned} f_G(yxy^{-1}) &= g_G(yxy^{-1}) \\ &\supseteq g_G(y) \cap g_G(x) \cap g_G(y^{-1}) \\ &= g_G(x) \cap g_G(y) \\ &= f_G(x) \cap g_G(y). \end{aligned}$$

(2) If $x \in G \setminus Ker\varphi, y \in G$, then $f_G(x) = \emptyset$ and $f_G(yxy^{-1}) \supseteq \emptyset = f_G(x) \cap g_G(y)$.
Hence $f_G \tilde{\triangleright} g_G$.
Since $g_G \approx h_H$ then $\varphi(g_G^*) = h_G^*$. Let $\psi = \varphi|_{g_G^*}$ and so ψ is a homomorphism of

g_G^* onto h_G^* and $Ker\psi = f_G^*$. Then there exists an isomorphism η of g_G^*/f_G^* onto h_H^* such that for all $x \in g_G^*, z \in h_H^*$ we have $\eta([x]) = \psi(x) = \varphi(x)$ and so

$$\begin{aligned}\eta(g_G/f_G)(z) &= \cup\{(g_G/f_G)([x]) \mid x \in g_G^*, \eta([x]) = z\} \\ &= \cup\{\cup\{g_G(y) \mid y \in [x]\} \mid x \in g_G^*, \psi(x) = z\} \\ &= \cup\{g_G(y) \mid y \in g_G^*, \psi(y) = z\} \\ &= \cup\{g_G(y) \mid y \in G, \varphi(y) = z\} \\ &= h_H(z).\end{aligned}$$

So $g_G/f_G \cong h_H|_{h_H^*}$. □

Proposition 4.6. *Let $f_G, g_G, h_G \in S_G(U)$, $f_G \tilde{\trianglerighteq} g_G$ and $f_G, g_G \tilde{\trianglerighteq} h_G$. If for all $\alpha \neq \emptyset \neq \beta$ in $P(U)$,$\alpha \cap \beta \neq \emptyset$, then $(h_G/f_G)/(g_G/f_G) \cong h_G/g_G$.*

Proof. From Proposition 3.7 $f_G^* \trianglerighteq g_G^*$ and $f_G^*, g_G^* \trianglerighteq h_G^*$. Let $x \in h_G^*$ and define $\varphi : (h_G^*/f_G^*)/(g_G^*/f_G^*) \to (h_G^*/g_G^*)$ as: $\varphi(xf_G^*.(g_G^*/f_G^*)) = xg_G^*$. Then φ is an isomorphism of Groups. Now

$$\begin{aligned}\varphi((h_G/f_G)/(g_G/f_G))(xg_G^*) &= ((h_G/f_G)/(g_G/f_G))(xf_G^*.(g_G^*/f_G^*)) \\ &= \cup\{(h_G/f_G)(yf_G^*) \mid y \in h_G^*, yf_G^* \in xf_G^*.(g_G^*/f_G^*)\} \\ &= \cup\{\cup\{h_G(z) \mid z \in yf_G^*\} \mid y \in h_G^*, yf_G^* \in xf_G^*.(g_G^*/f_G^*)\} \\ &= \cup\{h_G(z) \mid z \in h_G^*, zf_G^* \in xf_G^*.(g_G^*/f_G^*)\} \\ &= \cup\{h_G(z) \mid z \in xf_G^*.(g_G^*/f_G^*)\} \\ &= \cup\{h_G(z) \mid z \in h_G^*, \varphi(z) \in xg_G^*\} \\ &= (h_G/f_G)(xg_G^*).\end{aligned}$$

Therefore $(h_G/f_G)/(g_G/f_G) \cong h_G/g_G$. □

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