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K-ANALOGUE OF THE RIEMANN ZETA FUNCTION AND SOME ASSOCIATED PROPERTIES

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Abstract. In this paper, we explore some analytical properties of the k-analogue of the Riemann zeta function. Particularly, we establish properties of the function such as inequalities, monotonicity, complete monotonicity, convexity, subadditivity, among others. By doing so, some known results in the literature are recovered as particular cases. Finally, we introduce a k-analogue of the Dirichlet eta function.

Keywords. Riemann zeta function; k-analogue; k-gamma function; Dirichlet eta function; inequality.
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1. Introduction

The celebrated Riemann zeta function is defined by any of the following equivalent forms.

$$\zeta(z) = \sum_{m=1}^{\infty} \frac{1}{m^z}, \quad z > 1, \tag{1.1}$$

$$= \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{r^{z-1}}{e^r - 1} dr, \quad z > 1, \tag{1.2}$$

$$= \frac{2^{z-1}}{\Gamma(z+1)} \int_0^{\infty} \frac{r^z}{\sinh^2(z)} dr, \quad z > 1, \tag{1.3}$$

where $\Gamma(z)$ is the gamma function. It features regularly in areas such as mathematical analysis, analytic number theory, statistics, probability theory, mathematical physics and engineering. As a result of its diverse areas of applications, as well as its connections with other special functions, many researchers have investigated its properties in diverse ways. We refer to [2, 3, 4, 5, 6, 7, 10, 14, 16, 18, 24] and the references in there.

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In 2006, Lafogia and Natalini [20] obtained the remarkable inequality

$$\frac{z}{z+1} \leq \frac{\zeta(z)\zeta(z+2)}{\zeta^2(z+1)}, \quad z>1, \tag{1.4}$$

by applying the integral representation (1.2), a generalized form of the Cauchy-Schwarz integral inequality, and the functional equation of the gamma function. Subsequently, other researchers independently obtained a refinement of (1.4) by establishing that

$$1 \leq \frac{\zeta(z)\zeta(z+2)}{\zeta^2(z+1)}, \quad z>1. \tag{1.5}$$

Specifically, Cerone and Dragomir obtained it in (2.4) of [12]. Ahsan et al. obtained it in (2) of [1]. Srivastava et al. obtained it in Theorem 5 of [27]. Cerone and Dragomir further recovered it in (3.5) of [13]. Additionally, the inequality (1.5) has been generalized in Proposition 3.1 of [13] and Theorem 2.1 of [23].

In 2007, Díaz and E. Pariguan [17] introduced a generalization of the gamma function which they called k-gamma function. It is defined as

$$\Gamma_k(z) = \int_0^\infty r^{z-1}e^{-\frac{r^k}{k}}dr, \tag{1.6}$$

$$= \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{z}{k}-1}}{(z)_{n,k}}, \tag{1.7}$$

$$= q^{\frac{z}{k}} \int_0^\infty r^{z-1}e^{-\frac{r^k}{k}q}dr, \quad q\in\mathbb{R}, \tag{1.8}$$

for $z>0$ and $k>0$, where $(z)_{n,k} = z(z+k)(z+2k)\dots(z+(n-1)k)$ is the Pochhammer k-symbol, and it satisfies the basic relations

$$\Gamma_k(z+k) = z\Gamma_k(z), \tag{1.9}$$

$$\Gamma_k(k) = 1, \tag{1.10}$$

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma(\frac{z}{k}). \tag{1.11}$$

The k-analogue (or k-generalization) of the classical Riemann zeta function is defined as [8, 26]

$$\zeta_k(z) = \sum_{m=1}^\infty \frac{1}{m^{\frac{z}{k}}}, \quad z>k, \tag{1.12}$$

$$= \frac{1}{\Gamma_k(z)} \int_0^\infty \frac{r^{z-1}}{e^{\frac{r^k}{k}}-1}dr, \quad z>k, \tag{1.13}$$

$$= \frac{1}{\Gamma_k(z)} \int_0^\infty \frac{(kr)^{\frac{z}{k}-1}}{e^r-1}dr, \quad z>k. \tag{1.14}$$

It is clear from (1.1) and (1.12) that

$$\zeta_k(z) = \zeta\left(\frac{z}{k}\right), \quad z>k. \tag{1.15}$$

Moreover, the limit

$$\lim_{z\to\infty} \zeta_k(z) = 1, \quad z>k, \tag{1.16}$$

holds. The relationship between (1.12) and (1.13) can readily be obtained as follows.

$$\begin{aligned}\frac{r^{z-1}}{e^{\frac{r^k}{k}}-1} &= \frac{e^{-\frac{r^k}{k}}r^{z-1}}{1-e^{-\frac{r^k}{k}}}\\ &= e^{-\frac{r^k}{k}}r^{z-1}\sum_{m=0}^{\infty}e^{-\frac{r^k}{k}m}\\ &= \sum_{m=0}^{\infty}e^{-\frac{r^k}{k}(m+1)}r^{z-1} = \sum_{m=1}^{\infty}e^{-\frac{r^k}{k}m}r^{z-1}\end{aligned}$$

which implies that

$$\begin{aligned}\int_0^{\infty}\frac{r^{z-1}}{e^{\frac{r^k}{k}}-1}dr &= \int_0^{\infty}\sum_{m=1}^{\infty}e^{-\frac{r^k}{k}m}r^{z-1}dr\\ &= \sum_{m=1}^{\infty}\int_0^{\infty}e^{-\frac{r^k}{k}m}r^{z-1}dr\\ &= \sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{k}}}.m^{\frac{z}{k}}\int_0^{\infty}e^{-\frac{r^k}{k}m}r^{z-1}dr\\ &= \sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{k}}}.\Gamma_k(z)\end{aligned}$$

as a result of (1.8). The representation (1.14) is obtained by using (1.15) and (1.2).

In the present work, we explore some analytical properties of the function $\zeta_k(z)$. We particularly focus on properties such as inequalities, monotonicity, complete monotonicity, convexity, subadditivity, among others. Through these, some known results in the literature are recovered as particular cases. In the end, we introduce a k-analogue of the Dirichlet eta function.

2. Results

In this section, we present the main findings of the paper together with some discussions.

Theorem 2.1. *Let $z > k$, $y > k$, $\lambda > 1$, $\mu > 1$ and $1/\lambda + 1/\mu = 1$. Then the inequality*

$$\frac{[\zeta_k(z)]^{\frac{1}{\lambda}}[\zeta_k(y)]^{\frac{1}{\mu}}}{\zeta_k(\frac{z}{\lambda}+\frac{y}{\mu})} \geq \frac{\Gamma_k(\frac{z}{\lambda}+\frac{y}{\mu})}{[\Gamma_k(z)]^{\frac{1}{\lambda}}[\Gamma_k(y)]^{\frac{1}{\mu}}} \tag{2.1}$$

holds.

Proof. We employ the integral form of Hölder's inequality which is stated as

$$\int_u^v P(r)Q(r)dr \leq \left(\int_u^v (P(r))^{\lambda}dr\right)^{\frac{1}{\lambda}}\left(\int_u^v (Q(r))^{\mu}dr\right)^{\frac{1}{\mu}}$$

where $\lambda > 1$, $\mu > 1$ and $1/\lambda + 1/\mu = 1$. In view of (1.13), we have

$$\begin{aligned}
\zeta_k\left(\frac{z}{\lambda}+\frac{y}{\mu}\right)\zeta_k\left(\frac{z}{\lambda}+\frac{y}{\mu}\right) &= \int_0^\infty \frac{r^{\frac{z}{\lambda}+\frac{y}{\mu}-(\frac{1}{\lambda}+\frac{1}{\mu})}}{e^{\frac{r^k}{k}}-1}dr \\
&= \int_0^\infty \frac{r^{\frac{z-1}{\lambda}}}{\left(e^{\frac{r^k}{k}}-1\right)^{\frac{1}{\lambda}}}\frac{r^{\frac{y-1}{\mu}}}{\left(e^{\frac{r^k}{k}}-1\right)^{\frac{1}{\mu}}}dr \\
&\leq \left(\int_0^\infty \frac{r^{z-1}}{e^{\frac{r^k}{k}}-1}dr\right)^{\frac{1}{\lambda}}\left(\int_0^\infty \frac{r^{y-1}}{e^{\frac{r^k}{k}}-1}dr\right)^{\frac{1}{\mu}} \\
&= \left(\zeta_k(z)\Gamma_k(z)\right)^{\frac{1}{\lambda}}\left(\zeta_k(y)\Gamma_k(y)\right)^{\frac{1}{\mu}}
\end{aligned}$$

which gives (2.1) after rearranging. □

Remark 2.2. If $\lambda = \mu = 2$, $z = z + r$ and $y = z + s$ in (2.1) , then we get

$$\frac{\zeta_k(z+r)\zeta_k(z+s)}{\zeta_k^2(z+\frac{r+s}{2})} \geq \frac{\Gamma_k^2(z+\frac{r+s}{2})}{\Gamma_k(z+r)\Gamma_k(z+s)}. \tag{2.2}$$

Remark 2.3. If $\lambda = \mu = 2$ and $z = z + 2k$ in (2.1) , then we get

$$\frac{\zeta_k(z)\zeta_k(z+2k)}{\zeta_k^2(z+k)} \geq \frac{\Gamma_k^2(z+k)}{\Gamma_k(z)\Gamma_k(z+2k)}. \tag{2.3}$$

Theorem 2.4. *The function $\zeta_k(z)$ is strictly logarithmically convex on (k, ∞).*

Proof. It is known from [15, Lemma 2.1] that the function $\zeta(z)$ is strictly logarithmically convex on $(1, \infty)$. This implies that

$$\zeta''(y)\zeta(y) - (\zeta'(y))^2 > 0$$

for all $y \in (1, \infty)$. In view of (1.15), we have

$$(\ln\zeta_k(z))'' = \left(\ln\zeta\left(\frac{z}{k}\right)\right)'' = \frac{1}{k^2}\left[\frac{\zeta''(\frac{z}{k})\zeta(\frac{z}{k}) - (\zeta'(\frac{z}{k}))^2}{[\zeta(\frac{z}{k})]^2}\right] > 0$$

which ends the proof. □

Remark 2.5. The log-convexity property of $\zeta_k(z)$ implies that the function

$$h(z) = \frac{\zeta_k'(z)}{\zeta_k(z)}$$

is strictly increasing on (k, ∞).

Theorem 2.6. *Let $z > k$, $k > u \geq 0$ and $s \geq 0$. Then the inequality*

$$\frac{z+u}{z+s+u} \leq \frac{\zeta_k(z+u)\zeta_k(z+s+k)}{\zeta_k(z+k)\zeta_k(z+s+u)} \tag{2.4}$$

holds.

Proof. For $z > k$ and $k > u \geq 0$, let $\mathcal{B}(z)$ be defined as

$$\mathcal{B}(z) = (z+u)\frac{\zeta_k(z+k)}{\zeta_k(z+u)}.$$

Then by differentiating rearranging, we obtain

$$\mathcal{B}'(z) = \mathcal{B}(z)\left[\frac{1}{z+u} + \frac{\zeta_k'(z+k)}{\zeta_k(z+k)} - \frac{\zeta_k'(z+u)}{\zeta_k(z+u)}\right] > 0$$

which implies that $\mathcal{B}(z)$ is strictly increasing on (k, ∞). Let $s \geq 0$. Then by the monotonicity of $\mathcal{B}(z)$, $z \leq z+s$ implies that $\mathcal{B}(z) \leq \mathcal{B}(z+s)$. That is

$$(z+u)\frac{\zeta_k(z+k)}{\zeta_k(z+u)} \leq (z+s+u)\frac{\zeta_k(z+s+k)}{\zeta_k(z+s+u)}$$

which gives the inequality (2.4). □

Remark 2.7. If $u = 0$ in Theorem 2.6, then we get

$$\frac{z}{z+s} \leq \frac{\zeta_k(z)\zeta_k(z+s+k)}{\zeta_k(z+k)\zeta_k(z+s)}. \tag{2.5}$$

Remark 2.8. If $u = 0$ and $s = k$ in Theorem 2.6, then we get

$$\frac{z}{z+k} < \frac{\zeta_k(z)\zeta_k(z+2k)}{\zeta_k^2(z+k)}. \tag{2.6}$$

Inequality (2.6) is the k-analogue of the inequality by Laforgia and Natalini (1.4). Therefore, Theorem 2.6 is a generalization or an extension of (1.4).

The following theorem is a refinement of Theorem 2.6 and at the same time a generalization of the inequality (1.5).

Theorem 2.9. *Let $z > k$, $k > u \geq 0$ and $s \geq 0$. Then the inequality*

$$1 \leq \frac{\zeta_k(z+u)\zeta_k(z+s+k)}{\zeta_k(z+k)\zeta_k(z+s+u)} \tag{2.7}$$

holds.

Proof. We use the same technique as in the proof of Theorem 2.6. For $z > k$ and $k > u \geq 0$, let $\mathcal{H}(z)$ be defined as

$$\mathcal{H}(z) = \frac{\zeta_k(z+k)}{\zeta_k(z+u)}.$$

Then by differentiating, we obtain

$$\mathcal{H}'(z) = \mathcal{H}(z)\left[\frac{\zeta_k'(z+k)}{\zeta_k(z+k)} - \frac{\zeta_k'(z+u)}{\zeta_k(z+u)}\right] > 0$$

which implies that $\mathcal{H}(z)$ is strictly increasing on (k, ∞). Then for $z \leq z+s$ where $s \geq 0$, we have $\mathcal{H}(z) \leq \mathcal{H}(z+s)$ which gives the inequality (2.7). □

Remark 2.10. If $u = 0$ and $s = k$ in Theorem 2.9, then we get

$$1 < \frac{\zeta_k(z)\zeta_k(z+2k)}{\zeta_k^2(z+k)} \tag{2.8}$$

which is a refinement of (2.6).

Remark 2.11. It is known in [17] that $\Gamma_k(z)$ is strictly logarithmically convex on $(0,\infty)$. This is equivalent to

$$\Gamma_k\left(\frac{z}{\lambda}+\frac{y}{\mu}\right) < [\Gamma_k(z)]^{\frac{1}{\lambda}}\,[\Gamma_k(y)]^{\frac{1}{\mu}} \tag{2.9}$$

where $z>0$, $y>0$, $\lambda>1$, $\mu>1$ and $1/\lambda+1/\mu=1$. If $\lambda=\mu=2$ and $y=z+2k$, then we obtain

$$\frac{\Gamma_k^2(z+k)}{\Gamma_k(z)\Gamma_k(z+2k)} < 1. \tag{2.10}$$

Combining (2.8) and (2.10) gives

$$\frac{\zeta_k(z)\zeta_k(z+2k)}{\zeta_k^2(z+k)} > 1 > \frac{\Gamma_k^2(z+k)}{\Gamma_k(z)\Gamma_k(z+2k)}, \quad z>k, \tag{2.11}$$

which refines (2.3).

Theorem 2.12. *The function $\zeta_k(z)$ is strictly completely monotonic on (k,∞).*

Proof. Recall that the function $h:(0,\infty)\to\mathbb{R}$ is said to be completely monotonic on $(0,\infty)$ if [21]

$$(-1)^r h^{(r)}(z)\geq 0$$

for all $z\in(0,\infty)$ and $r\in\mathbb{N}$. Now by using (1.12), we have

$$\begin{aligned}
\zeta_k'(z) &= -\frac{\ln m}{k}\sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{k}}},\\
\zeta_k''(z) &= \left(\frac{\ln m}{k}\right)^2\sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{k}}},\\
\zeta_k'''(z) &= -\left(\frac{\ln m}{k}\right)^3\sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{k}}},\\
\vdots &= \quad\vdots\\
\zeta_k^{(r)}(z) &= (-1)^r\left(\frac{\ln m}{k}\right)^r\sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{k}}}.
\end{aligned}$$

Hence

$$(-1)^r\zeta_k^{(r)}(z) = \left(\frac{\ln m}{k}\right)^r\sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{k}}} > 0$$

and this ends the proof. □

Remark 2.13. Theorem 2.12 implies that $\zeta_k(z)$ is strictly decreasing and strictly convex on (k,∞). The convexity of $\zeta_k(z)$ implies that

$$\zeta_k\left(\frac{z}{\lambda}+\frac{y}{\mu}\right) < \frac{\zeta_k(z)}{\lambda}+\frac{\zeta_k(y)}{\mu}$$

where $z>k$, $y>k$, $\lambda>1$, $\mu>1$ and $1/\lambda+1/\mu=1$. Letting $\lambda=\mu=2$ and $y=z+2k$ yields

$$\zeta_k(z+k) < \frac{\zeta_k(z)+\zeta_k(z+2k)}{2}. \tag{2.12}$$

Theorem 2.14. *The function $1/\zeta_k(z)$ is strictly concave on (k,∞).*

Proof. It is known from [11] that the function $1/\zeta(y)$ is strictly concave on $(1,\infty)$. This is equivalent to the inequality

$$\zeta''(y)\zeta(y) - 2(\zeta'(y))^2 > 0,$$

for all $y > 1$. Now define $G(z)$ as

$$G(z) = \frac{1}{\zeta_k(z)} = \frac{1}{\zeta\left(\frac{z}{k}\right)}$$

for all $z > k$. Then

$$G''(z) = -\frac{1}{k^2}\left[\frac{\zeta''(\frac{z}{k})\zeta(\frac{z}{k}) - 2(\zeta'(\frac{z}{k}))^2}{\zeta^3(\frac{z}{k})}\right] < 0$$

which ends the proof. □

Remark 2.15. Theorem 2.14 is equivalent to the inequality

$$\zeta_k''(z)\zeta_k(z) - 2(\zeta_k'(z))^2 > 0$$

for all $z > k$.

Theorem 2.16. *For $z > k$, the inequality*

$$\zeta_k(z+k) < \frac{2\zeta_k(z)\zeta_k(z+2k)}{\zeta_k(z)+\zeta_k(z+2k)} \tag{2.13}$$

holds.

Proof. Concavity of the function $1/\zeta_k(z)$ on (k,∞) implies that, for $z > k$, $y > k$, $\lambda > 1$, $\mu > 1$ and $1/\lambda + 1/\mu = 1$, we have

$$\begin{aligned}\frac{1}{\zeta_k\left(\frac{z}{\lambda}+\frac{y}{\mu}\right)} &> \frac{1}{\lambda}\frac{1}{\zeta_k(z)} + \frac{1}{\mu}\frac{1}{\zeta_k(y)} \\ &= \frac{\frac{\zeta_k(z)}{\lambda}+\frac{\zeta_k(y)}{\mu}}{\zeta_k(z)\zeta_k(y)}\end{aligned}$$

which implies that

$$\zeta_k\left(\frac{z}{\lambda}+\frac{y}{\mu}\right) < \frac{\zeta_k(z)\zeta_k(y)}{\frac{\zeta_k(z)}{\lambda}+\frac{\zeta_k(y)}{\mu}}. \tag{2.14}$$

Then by letting $\lambda = \mu = 2$ and $y = z + 2k$ in (2.14), we arrive at (2.13). □

Lemma 2.17 ([9])**.** *A function $h : (0,\infty) \to \mathbb{R}$ is said to be subadditive on $(0,\infty)$ if*

$$h(r+s) \le h(r) + h(s)$$

holds for all $r, s \in (0,\infty)$. If the inequality is reversed, then h is said to be superadditive. If h is positive and $\frac{h(s)}{s}$ is decreasing (or increasing) on $(0,\infty)$, then h is respectively subadditive (or superadditive) on $(0,\infty)$.

Theorem 2.18. *The function $\zeta_k(z)$ is strictly subadditive on (k,∞).*

Proof. Let $u > 1$, $v > 1$ and $1/u + 1/v = 1$. Then, since $\zeta_k(z)$ is decreasing and convex, we obtain

$$\zeta_k(r+s) < \zeta_k\left(\frac{r}{u} + \frac{s}{v}\right) < \frac{\zeta_k(r)}{u} + \frac{\zeta_k(s)}{v} < \zeta_k(r) + \zeta_k(s)$$

for $r, s \in (k, \infty)$ and this gives the desired result. Alternatively, let $\phi(z) = \frac{\zeta_k(z)}{z}$ for $z \in (k, \infty)$. Then, since $\zeta_k(z)$ is decreasing, it follows readily that

$$z^2\phi'(z) = z\zeta_k'(z) - \zeta_k(z) < 0$$

which shows that $\phi(z)$ is strictly decreasing. The conclusion then from Lemma 2.17. □

Theorem 2.19. *Let $\alpha > 1$ and $z > k$. Then the inequality*

$$\alpha\zeta_k(z) > \zeta_k(\alpha z) \tag{2.15}$$

holds.

Proof. Let $A(z) = \alpha\zeta_k(z) - \zeta_k(\alpha z)$ for $z > k$. Then

$$A'(z) = \alpha\left[\zeta_k'(z) - \zeta_k'(\alpha z)\right] < 0$$

since $\zeta_k'(u)$ is strictly increasing. Thus, $A(z)$ is strictly decreasing. Hence for $z > k$, we have

$$A(z) > \lim_{z\to\infty} A(z) = \alpha - 1 > 0$$

which gives (2.15). □

Theorem 2.20. *Let $\alpha > 1$ and $z > k$. Then the function*

$$H(z) = \frac{(\zeta_k(z))^\alpha}{\zeta_k(\alpha z)} \tag{2.16}$$

is strictly decreasing and consequently the inequality

$$1 < \frac{(\zeta_k(z))^\alpha}{\zeta_k(\alpha z)} < \frac{(\zeta_k(y))^\alpha}{\zeta_k(\alpha y)} \tag{2.17}$$

holds for $k < y < z$.

Proof. Recall that $\zeta_k'(z)/\zeta_k(z)$ is strictly increasing. Then for $\alpha > 1$ and $z > k$, we have

$$\frac{H'(z)}{H(z)} = \alpha\left[\frac{\zeta_k'(z)}{\zeta_k(z)} - \frac{\zeta_k'(\alpha z)}{\zeta_k(\alpha z)}\right] < 0$$

which means that $H(z)$ is strictly decreasing. Hence for $k < y < z$, we have

$$H(y) > H(z) > \lim_{z\to\infty} H(z) = 1$$

which gives (2.17). □

Remark 2.21. Bounds for ratio of type (2.16) concerning the ordinary Riemann zeta function have been studied in [6] and [25]. Suppose that $y = 2k$ in Theorem 2.20. Then we obtain

$$1 < \frac{(\zeta_k(z))^\alpha}{\zeta_k(\alpha z)} < \frac{(\zeta_k(2k))^\alpha}{\zeta_k(2\alpha k)} \tag{2.18}$$

for $z > 2k$. If $k = 1$, then (2.18) reduces to

$$1 < \frac{(\zeta(z))^{\alpha}}{\zeta(\alpha z)} < \frac{(\zeta(2))^{\alpha}}{\zeta(2\alpha)} \tag{2.19}$$

for $z > 2$. This gives another perspective for the ratio studied in [6] and [25]. Specifically, if $\alpha = 2$ then (2.19) becomes

$$1 < \frac{\zeta^2(z)}{\zeta(2z)} < \frac{5}{2} \tag{2.20}$$

for $z > 2$.

Remark 2.22. In [19], the authors gave a different (or modified) definition of the k-analogue of the Riemann zeta function as

$$\zeta_k^*(z) = \frac{1}{\Gamma_k(z)} \int_0^{\infty} \frac{r^{z-k}}{e^r - 1} dr, \quad z > k, \tag{2.21}$$

where $k > 0$. They then proceeded to prove that

$$(u+k)\frac{\zeta_k^*(u)}{\zeta_k^*(u+k)} \geq u\frac{\zeta_k^*(u+k)}{\zeta_k^*(u+2k)}, \tag{2.22}$$

for $u > k$ and

$$\frac{\Gamma_k\left(\frac{u}{p} + \frac{v}{q}\right)}{[\Gamma_k(u)]^{\frac{1}{p}}[\Gamma_k(v)]^{\frac{1}{q}}} \leq \frac{[\zeta_k^*(u)]^{\frac{1}{p}}[\zeta_k^*(v)]^{\frac{1}{q}}}{\zeta_k^*\left(\frac{u}{p} + \frac{v}{q}\right)}, \tag{2.23}$$

for $u > k$, $v > k$, $1/p + 1/q = 1$ and $u/p + u/q > k$. It is worth noting that $\zeta_k^*(z) \neq \zeta_k(z)$. Therefore, in this regard, inequalities (2.22) and (2.23) are not the same as their respective counterparts (2.6) and (2.1).

We now proceed to introduce a k-analogue of the Dirichlet eta function. Recall that the classical Dirichlet eta function is defined as

$$\eta(z) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^z}, \tag{2.24}$$

$$= \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{r^{z-1}}{e^r + 1} dr, \tag{2.25}$$

for $z > 0$. See [22] and the references therein.

Theorem 2.23. *For $z > 0$ and $k > 0$, the k-analogue Dirichlet eta function is defined as*

$$\eta_k(z) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{\frac{z}{k}}} \tag{2.26}$$

and satisfies the properties

$$\eta_k(z) = \frac{1}{\Gamma_k(z)} \int_0^\infty \frac{r^{z-1}}{e^{\frac{r^k}{k}} + 1} dr, \tag{2.27}$$

$$= \frac{1}{\Gamma_k(z)} \int_0^\infty \frac{(kr)^{\frac{z}{k}-1}}{e^r + 1} dr, \tag{2.28}$$

$$= \left(1 - \frac{2}{2^{\frac{z}{k}}}\right) \zeta_k(z), \quad z > 0,\ z \neq k. \tag{2.29}$$

Proof. By employing (1.8) we have

$$\frac{(-1)^{m+1}}{m^{\frac{z}{k}}} = \frac{(-1)^{m+1}}{\Gamma_k(z)} \int_0^\infty r^{z-1} e^{-\frac{r^k}{k} m} dr. \tag{2.30}$$

Note that

$$\sum_{m=1}^{\infty} (-1)^{m+1} s^m = \frac{s}{1+s} = \frac{1}{s^{-1}+1}. \tag{2.31}$$

Now by summing both sides of (2.30) from $m = 1$ to ∞ and applying (2.31), we obtain

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{\frac{z}{k}}} = \frac{1}{\Gamma_k(z)} \int_0^\infty r^{z-1} \sum_{m=1}^{\infty} (-1)^{m+1} e^{-\frac{r^k}{k} m} dr$$

$$= \frac{1}{\Gamma_k(z)} \int_0^\infty \frac{r^{z-1}}{e^{\frac{r^k}{k}} + 1} dr$$

which gives (2.27). Next, comparing (2.24) and (2.26) reveals that

$$\eta_k(z) = \eta\left(\frac{z}{k}\right). \tag{2.32}$$

Then by using (2.32) and (2.25) as well as (1.11), we obtain

$$\eta_k(z) = \frac{1}{\Gamma(\frac{z}{k})} \int_0^\infty \frac{r^{\frac{z}{k}-1}}{e^r + 1} dr$$

$$= \frac{k^{\frac{z}{k}-1}}{\Gamma_k(z)} \int_0^\infty \frac{r^{\frac{z}{k}-1}}{e^r + 1} dr$$

$$= \frac{1}{\Gamma_k(z)} \int_0^\infty \frac{(kr)^{\frac{z}{k}-1}}{e^r + 1} dr$$

which gives (2.28). Finally, we have

$$\begin{aligned}
\eta_k(z) &= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{\frac{z}{k}}} \\
&= 1 - \frac{1}{2^{\frac{z}{k}}} + \frac{1}{3^{\frac{z}{k}}} - \frac{1}{4^{\frac{z}{k}}} + \frac{1}{5^{\frac{z}{k}}} - \frac{1}{6^{\frac{z}{k}}} + \frac{1}{7^{\frac{z}{k}}} - \frac{1}{8^{\frac{z}{k}}} + \dots \\
&= \left(1 + \frac{1}{2^{\frac{z}{k}}} + \frac{1}{3^{\frac{z}{k}}} + \frac{1}{4^{\frac{z}{k}}} + \dots\right) - 2\left(\frac{1}{2^{\frac{z}{k}}} + \frac{1}{4^{\frac{z}{k}}} + \frac{1}{6^{\frac{z}{k}}} + \frac{1}{8^{\frac{z}{k}}} + \dots\right) \\
&= \sum_{m=1}^{\infty} \frac{1}{m^{\frac{z}{k}}} - \frac{2}{2^{\frac{z}{k}}} \sum_{m=1}^{\infty} \frac{1}{m^{\frac{z}{k}}} \\
&= \left(1 - \frac{2}{2^{\frac{z}{k}}}\right) \sum_{m=1}^{\infty} \frac{1}{m^{\frac{z}{k}}} \\
&= \left(1 - \frac{2}{2^{\frac{z}{k}}}\right) \zeta_k(z),
\end{aligned}$$

which gives (2.29). This completes the proof of Theorem 2.23. □

Remark 2.24. The relation (2.29) provides a connection between the k-analogue of the Dirichlet eta function and the k-analogue of the Riemann zeta function. In other words, it extends the domain of definition of the k-analogue of the Riemann zeta function from $z > k$ to $z > 0$ except $z = k$.

3. Conclusion

In this work, we considered some analytical properties of the k-analogue of the Riemann zeta function. Notably, we have established inequalities as well as monotonicity, complete monotonicity, convexity and subadditivity properties of the function. For the particular case where $k = 1$, the established results reduce to the corresponding results for the classical Riemann zeta function. Consequently, some known results in the literature are recovered as particular cases of the present results. We have also introduced a k-analogue of the Dirichlet eta function. In our next work, we shall explore the properties of this new function.

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