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SUBLINEAR OPERATORS ON CONTINUAL HERZ–MORREY SPACES

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Abstract. We introduce a new class of function spaces, called continual Herz–Morrey spaces, in which all three parameters are variable. Under suitable assumptions, we establish the boundedness of sublinear operators on these spaces.

Keywords. Variable exponents, sublinear operators, continual Herz-Morrey spaces
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1. Introduction

In [19], Diening established the boundedness of the maximal operators on Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent. Variable exponent Lebesgue and Sobolev spaces were introduced to study nonlinear partial differential equations with nonstandard growth conditions, electrorheological fluids, and elasticity theory, see [20].

Boundedness of some sublinear operators and their commutators were also studied for generalized local Morrey spaces (see [17]), B_σ-Morrey-Campanato spaces (see [16]), generalized Morrey spaces (see [15]), Herz spaces (see [18]), new Hardy Spaces of Musielak–Orlicz type (see [21]), weighted anisotropic Hardy spaces (see [22]), Herz spaces with variable exponent (see [23]), weighted Herz-type spaces (see [24]), weighted anisotropic product Hardy spaces (see [25]), H Herz-Hardy spaces with variable exponents (see [26]), Herz-type Hardy spaces (see [27]), Hardy spaces on RD-spaces (see [28]), weighted multi-parameter local Hardy spaces (see [29]).

The boundedness of various sublinear operators and their commutators has also been studied for variable exponent Morrey-type spaces (see [30]), generalized mixed Morrey spaces (see [31]), block-type spaces (see [32]), mixed-norm Hardy spaces with variable exponents (see [33]), generalized Morrey spaces on quasi-metric measure spaces under the growth condition (see [34]), continual Herz spaces with variable exponents (see [10]).

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As is well known, Herz introduced Herz spaces while studying totally convergent Fourier transforms in 1968 by introducing a new class of function spaces that describe specific features of the functions. As a result, the theory of Herz spaces has advanced considerably, and these spaces have become important tools in harmonic analysis. The boundedness of sublinear operators is a significant issue on Herz spaces. The authors established the boundedness of sublinear operators in the context of variable exponent Herz spaces in [4]. The authors of [3] investigated the boundedness of a broad class of sublinear operators on variable Herz spaces, including maximum, potential, and Calderón-Zygmund operators. Observe that Herz-Morrey spaces with variable exponent are the generalizations of Herz spaces with variable exponent. The boundedness of higher-order commutators of fractional integrals on Herz-Morrey spaces can be checked in [8].

In [6], authors obtained the BMO estimate for the higher order commutators of Marcinkiewicz integral operator in grand variable Herz-Morrey spaces. The concept of grand variable Herz-Morrey spaces was first presented in [5, 13]. These function spaces further expanded the framework of Herz-Morrey spaces by incorporating the variable exponent setting.

In addition to their use in harmonic analysis, function spaces with a variable exponent prove to be an effective analytical tool when dealing with mathematical models where the conditions of integrability and growth depend on the spatial variable. Function spaces with a variable exponent appear naturally in the analysis of nonlinear partial differential equations with non-standard growth, including the study of electrorheological fluids and problems in the theory of elasticity. The new spaces of continually localized Herz-Morrey type introduced in this paper combine the variable exponent and radial localization together with scale-dependent norms and thus allow us to measure the degree of both localization and scale-dependent behavior of functions. The boundedness of sublinear operators on those spaces may be helpful in the study of integral operators, estimates of potentials, and regularity of solutions to appropriate nonlinear equations. Besides, such estimates can become tools for the analysis of various numerical schemes. However, particular applications to differential equations, iterative algorithms, or data-driven modeling require extra conditions and additional studies. The primary purpose of this paper is to develop the theory of function spaces needed for such applications. For more results in variable exponent analysis see [41, 43, 51, 52, 50, 49, 44, 42, 40, 39, 36, 38, 37, 35, 45, 46, 47, 48, 53, 54, 55, 56].

Let $H \subset \mathbb{R}^n$ be a measurable set and a measurable function $r(\cdot)$ defined on H and taking values in the interval $[1, \infty)$. Let us assume that $r_- := \operatorname{ess\,inf}_{h\in H} r(h)$, and $r_+ := \operatorname{ess\,sup}_{h\in H} r(h)$, then we have the following inequality

$$1 \leq r_-(H) \leq r(h) \leq r_+(H) < \infty, \tag{1.1}$$

We use the following notations:

(i) $\mathcal{P}^{\log} = \mathcal{P}^{\log}(H)$ denotes the class of functions satisfying the following conditions. The class $\mathcal{P}(H)$ consists of those exponents which satisfy (1.1) and (2.1);

(ii) If H is unbounded, $\mathcal{P}_\infty(H)$ and $\mathcal{P}_{0,\infty}(H)$ are subset of the class $\mathcal{P}(H)$ that consists of functions satisfying a certain condition, as indicated by equation (2.2). The values of the functions in this subset fall within the interval $[1,\infty)$, and satisfy both conditions (2.2) and (2.3) respectively;

(iii) Let $\mathbb{R}_{\mu+} := (\mu,\infty)$, where $\mu \geq 0$. Then $M_\infty(\mathbb{R}_{\mu+})$ denotes the class consisting of functions defined on the domain $\mathbb{R}\mu+$ that have certain properties. Specifically, these functions are of the form $g(t) = \text{constant} + g_0(t)$, where $g_0(t)$ is a function belonging to the class $\mathcal{P}\infty(\mathbb{R}_{\mu+})$, where $H = \mathbb{R}_{\mu+}, \mu \geq 0$.

(iv) $\mathcal{P}_\infty^{\log}(H)$ is the subset consists of exponents that satisfy the condition (2.1);

(v) If $H = \mathbb{R}_+$ (the case $\mu = 0$), $\mathcal{M}_{0,\infty}(\mathbb{R}+)$ is the class comprises functions defined on the positive real line $\mathbb{R}+$ that are also in the class $\mathcal{M}\infty(\mathbb{R}_+)$ and satisfy a decay condition at the origin ($y = 0$). The decay condition implies that these functions are bounded by a logarithmic term as y approaches zero. Specifically, for $|y| \leq \frac{1}{2}$, the function satisfies the inequality $|f(y) - f_0| \leq \frac{C}{\ln|y|}$ for some real numbers f_0 and C. We also write $f_0 = f(0)$, $f_\infty = f(\infty)$ in this case;

(vi) $\mathcal{P}_{0,\infty}(\mathbb{R}+)$ is a subclass of functions within the class $\mathcal{M}_{0,\infty}(\mathbb{R}+)$ that have values in the interval $[1,\infty)$. In other words, these are the functions from $\mathcal{M}_{0,\infty}(\mathbb{R}+)$ that satisfy the given conditions and have their values constrained within the specified range with values in $[1,\infty)$.

If $\mathbb{R}_{\mu+}, \mu \geq 0$ and $\frac{dt}{t}$ denotes the Haar measure, the norm in the variable Lebesgue space with respect to the Haar measure is defined by

$$\|g\|_{L^{q(\cdot)}(\mathbb{R}_{\mu+};\frac{dt}{t})} = \inf\left\{\lambda > 0 : \int_\mu^\infty \left|\frac{g(t)}{\lambda}\right|^{q(t)} \frac{dt}{t} \leq 1\right\}.$$

Let $\frac{d\theta}{\theta}$ denote the Haar measure, then the Mellin convolution operator with homogeneous kernel of order 0 is defined as

$$\Gamma\omega(j) = \int_0^\infty \mathcal{K}(\frac{j}{\theta})\omega(\theta)\frac{d\theta}{\theta}. \tag{1.2}$$

Let $\mathbf{1}_H$ denote the characteristic function of H and $\mathcal{R}_m := \mathcal{R}(2^{m-1}, 2^m)$ with $\mathbf{1}_{\tau,t}(y) = \mathbf{1}_{\mathcal{R}_{\tau,t}}(y)$. We define continual Herz-Morrey spaces with variable exponent $HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,\mu;(\gamma,\delta)}(\mathbb{R}^n)$ by its norm,

$$\|f\|_{HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,\mu;(\gamma,\delta)}(\mathbb{R}^n)} := \|f\|_{L^{p(\cdot)}(B(0,\gamma\mu+\theta))} + \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\|t^{\alpha(t)}\|f\mathbf{1}_{\mathcal{R}_{\gamma t,\delta t}}\|_{L^{p(\cdot)}}\|_{L^{q(\cdot)}((\gamma\mu,k_0);\frac{dt}{t}} < \infty. \tag{1.3}$$

2. Preliminaries

We denote by $B(y,s)$, the ball centered at y with radius s. We define the spherical annulus $\mathcal{R}(\tau,t)$ as $\mathcal{R}(\tau,t) := B(0,t) \setminus B(0,\tau) = \{y \in \mathbb{R}^n : \tau < |y| < t\}$. Let $\mathbb{N}$ denote the set of natural numbers with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{Z}$ denote the set

of all integers. For two non-negative functions f and g, $f \leq g$ we mean $f \leq Cg$, where C is independent of the variables involved.

Hölder's inequality in variable Lebesgue spaces states that

$$\|fg\|_{r(\cdot)} \leq \|f\|_{p(\cdot)}\|g\|_{q(\cdot)},$$

where we define r as $\frac{1}{r(i)} = \frac{1}{p(i)} + \frac{1}{q(i)}$ for every $i \in H$, and $p, q, r \in \mathcal{P}(\mathbb{R}^n)$

The set $\mathcal{B}(\mathbb{R}^n)$ consists of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ that satisfy the condition that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

We recall the well-known log-condition

$$|r(h_1) - r(h_2)| \leq \frac{C(r)}{-\ln|h_1 - h_2|}, \quad |h_1 - h_2| \leq \frac{1}{2}, \quad h_1, h_2 \ \in E, \tag{2.1}$$

where $C(r) > 0$. And the decay condition: there exists a number $r_\infty \in (1, \infty)$, such that

$$|r(h) - r_\infty| \leq \frac{C}{\ln(e + |h|)}, \tag{2.2}$$

and also decay condition

$$|r(h) - r_0| \leq \frac{C}{\ln|h|}, |h| \leq \frac{1}{2}, \tag{2.3}$$

holds for some $r_0 \in (1, \infty)$.

Lemma 2.1. [10] *Let $D > 1$ and $r \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$. Then*

$$\frac{1}{t_0} s^{\frac{n}{r(0)}} \leq \|\mathbf{1}_{\mathcal{R}_{s,D_s}}\|_{r(\cdot)} \leq t_0 s^{\frac{n}{r(0)}}, \text{ for } 0 < s \leq 1 \tag{2.4}$$

and

$$\frac{1}{t_\infty} s^{\frac{n}{r_\infty}} \leq \|\mathbf{1}_{\mathcal{R}_{s,D_s}}\|_{r(\cdot)} \leq t_\infty s^{\frac{n}{r_\infty}}, \text{ for } s \geq 1, \tag{2.5}$$

respectively, where $t_0 \geq 1$ and $t_\infty \geq 1$. These are constants that depend on a certain parameter denoted as B. Specifically, they might vary according to the value of B. It's noted that these constants t_0 and t_∞ do not depend on another parameter s.

Lemma 2.2. *(see [10]) Let $p \in \mathcal{P}_{0,\infty}(\mathbb{R}_+)$ and $p(0) = p(\infty)$. The operator Γ is bounded on $L^{p(\cdot)}(\mathbb{R}_+; \frac{dt}{t})$ if*

$$\int_0^\infty |\Gamma(t)|^s \frac{dt}{t} < \infty \ \text{ when } s = 1 \ \& \ s = s_0, \tag{2.6}$$

for $\frac{1}{s_0} = 1 - \frac{1}{p_-} + \frac{1}{p_+}$.

Lemma 2.3. *(see [10]) For every measurable function Ω, the following relations*

$$\int_{2b<|x|<j} |\Omega(x)|dx = \frac{1}{\ln 2} \int_b^t \frac{d\theta}{\theta} \int_{max(2b,\theta)<|x|<min(j,2\theta)} |\Omega(x)|dx, \ j > 2\alpha > 0, \tag{2.7}$$

and

$$\int_{|x|\geq 2j}|\Omega(x)|dx=\frac{1}{\ln 2}\int_t^\infty\frac{d\theta}{\theta}\int\limits_{max(\theta,2j)<|x|<2\theta}|\Omega(x)|dx,\ j>0, \tag{2.8}$$

hold if the integrals on the left-hand side of above relations exists.

These lemmas were proved in [10].

Lemma 2.4. *Let* $1\leq p_-\leq p(x)\leq p_+<\infty$ *holds,*

$$\|g\|_{H^{p(\cdot),q,\alpha(\cdot)}_{\mu,\delta}}\approx\|g\|_{L^{p(\cdot)}(B(0,\gamma\mu+\theta))}+\|t^{\alpha_\infty}\|g\mathbf{1}_{\mathcal{R}_{\gamma t,\delta t}}\|_{p(\cdot)}\|_{L^q(\mathbb{R}_{+.\mu};\frac{dt}{t})},\mu>0, \tag{2.9}$$

and

$$\|g\|_{H^{p(\cdot),q,\alpha(\cdot)}_{0,\delta}}\approx\|t^{\alpha(0)}(1+t)^{\alpha_\infty-\alpha(0)}\|g\mathbf{1}_{R_{\gamma t,\delta t}}\|_{p(\cdot)}\|_{L^{q(\cdot)}(\mathbb{R}_+;\frac{dt}{t})}), \tag{2.10}$$

then the above norm equivalences hold, if $\alpha\in M^{\log}_\infty(\mathbb{R}_{+,\mu})$ *for* (2.9) *and* $\alpha\in M^{\log}_{0,\infty}(\mathbb{R}_+)$ *in the case of* (2.10).

Lemma 2.5. *Let* $4\leq\mathbb{R}<\infty$ *and* $0<\rho<2$

$$\|f\|_{L^{p(\cdot)}(B')}\leq C'\|j^{\alpha(j)}\|f\mathbf{1}_{j,2j}\|_{p(\cdot)}\|_{L^q((2,\infty);\frac{dj}{j})},$$

where $B'=B(0,R)\setminus B(0,2+\theta)$ *and* $C'=C(\rho,R)$.

3. Main results

We now state and prove the main results. Let the sublinear operator T satisfy the following size condition

$$|Tf(x)|\leq C\int_{\mathbb{R}^n}\frac{|f(y)|dy}{|x-y|^n} \tag{3.1}$$

where $x\notin supp\ f$.

3.1. Non-Homogeneous continual Herz-Morrey space(when $\mu=0$).

Theorem 3.1. *Suppose that* $p\in\mathcal{P}^{\log}_\infty(\mathbb{R}^n)$, $q\in\mathcal{P}^{\log}_\infty(\mu,\infty)$ *with* $1<p_-<p_+<\infty$, $1<q_-\leq q_+<\infty$. *Let* $\mu>0$ *and* $0<\gamma<\delta<\infty$. *Let* $\alpha\in M_\infty(\mathbb{R}_{\mu+})$ *and satisfy the condition*

$$-\frac{n}{p_\infty}<\alpha_\infty<\frac{n}{p'_\infty}. \tag{3.2}$$

Then the sublinear operator T *satisfying the condition* (3.1) *is bounded on continual Herz-Morrey spaces with variable exponents.*

Proof. Let $\mu=2$, $\gamma=1$, and it is sufficient to prove that

$$\|T(g)\|_{L^{p(\cdot)}(B(0,2+\theta))}+\mathcal{N}^{p,q,\alpha}_{1,2}(Tg)\leq\|g\|_{L^{p(\cdot)}(B(0,2+\theta))}+\mathcal{N}^{p,q,\alpha}_{\lambda',\delta'}(g) \tag{3.3}$$

with $\lambda'<1$, $\delta'>2$. Define

$$\mathcal{N}^{p,q,\alpha}_{\lambda,\delta}(g):=\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\|t^{\alpha_\infty}\|g\mathbf{1}_{\mathcal{R}\lambda t,\delta t}\|_{L^{p(\cdot)}}\|_{L^{q(\cdot)}((2,k_0),dt/t)}. \tag{3.4}$$

We first estimate the term $\|Tg\|_{L^{p(\cdot)}(B(0,2+\theta))}$, where we choose $\theta\in(0,1)$, and estimate for Tg follows from the boundedness from the $\left(L^{p(\cdot)}\to L^{p(\cdot)}\right)$ of Tg, so we have

$$\begin{aligned}\|Tg\|_{L^{p(\cdot)}(B(0,2+\theta))} &\leq c\|g\|_{L^{p(\cdot)}(B(0,2+\theta))}\\ &\leq c\|g\|_{HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,2;(\gamma',\delta')}(\mathbb{R}^n)}.\end{aligned}$$

For estimating the term $\mathcal{N}(Tg)^{p,q,\alpha}_{1,2}$:

Let $f_0(x) = g(x)\mathbf{1}_{B(0,\frac{1}{2})}(x)$, $f_t(x) = g(x)\mathbf{1}_{B(0,\gamma' t)\setminus B(0,\frac{1}{2}))}(x)$, $g_t(x) = g(x)\mathbf{1}_{B(\delta' t)\setminus B(0,\gamma' t)}$, $h_t(x) = g(x)\mathbf{1}_{\mathbb{R}^n\setminus B(0,\delta' t)}$, then the function $g(x)$ will be splitted as

$$g(x) = f_0(x) + f_t(x) + g_t(x) + h_t(x).$$

Now we have pointwise inequality,

$$|Tg(x)| \leq |Tf_0(x)| + |Tf_t(x)| + |Tg_t(x)| + |Th_t(x)|\,.$$

Now we will find the estimate for $Tf_0(x)$: Let $y \in B(0,\frac{1}{2})$ with $x \in R_{t,2t}$ and $t > 2$ we get

$$\frac{t}{2} \leq t - \frac{1}{2} < |x| - |y| \leq |x - y|,$$

which implies

$$\begin{aligned}|T(f_0)(x) &\leq C\int_{B(0,\frac{1}{2})} |x-y|^{-n}|g(y)|dy\\ &\leq Ct^{-n}\int_{B(0,\frac{1}{2})} |g(y)|dy\\ &\leq Ct^{-n}\|\mathbf{1}_{B(0,\frac{1}{2})}\|_{p'(\cdot)}\|f_0\|_{p(\cdot)}.\end{aligned}$$

$$\begin{aligned}&|T(f_0)(x).\mathbf{1}_{\mathcal{R}t,2t}| \leq C\int_{B(0,\frac{1}{2})} |x-y|^{-n}|g(y)|dy.\mathbf{1}_k(x)\\ &\leq Ct^{-n}\|f_0\|_{p(\cdot)}\|\mathbf{1}_{B(0,\frac{1}{2})}\|_{p'(\cdot)}.\mathbf{1}_{\mathcal{R}t,2t}\\ &\leq Ct^{-n}\|f_0\|_{p(\cdot)}.\mathbf{1}_{\mathcal{R}t,2t}.\end{aligned}$$

$$\begin{aligned}N(Tf_0)_{p,q,\alpha} &\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\|t^{\alpha_\infty - n}\|\mathbf{1}_{\mathcal{R}t,2t}\|_{p(\cdot)}\|_{L^{q(\cdot)}((2,k_0);\frac{dt}{t})}\|f_0\|_{p(\cdot)}\\ &\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\|t^{\alpha_\infty - \frac{n}{p'_\infty}}\|_{L^{q(\cdot)}((2,k_0);\frac{dt}{t})}\|f_0\|_{p(\cdot)}\\ &\leq C\|f_0\|_{p(\cdot)}.\end{aligned}$$

For finding the estimate of $Tf_t(x)$: Let $x \in \mathcal{R}t, 2t$ and considering $(1-\gamma')t < |x-y|$ we get

$$|Tf_t(x)| \leq C\int\limits_{B(0,\gamma' t)\setminus B(0,\frac{1}{2})} |x-y|^{-n}|g(y)|dy.$$

By using Hölder's inequality we obtain,

$$\begin{aligned}
|Tf_t(x)| &\leq \frac{C}{t^n}\int\limits_{\frac{1}{2}}^{t}|g(y)|dy \\
&\leq \frac{C}{t^n}\int\limits_{\frac{1}{2}}^{t}\frac{d\rho}{\rho}\int\limits_{\frac{\rho}{2}<|y|<\rho}|g(y)|dy \\
&\leq \frac{C}{t^n}\int\limits_{\frac{1}{2}}^{t}\frac{d\rho}{\rho}\|g\mathbf{1}_{\mathcal{R}\frac{\rho}{2},\rho}\|_{p(\cdot)}\|\mathbf{1}_{\mathcal{R}\frac{\rho}{2},\rho}\|_{p'(\cdot)} \\
&\leq \frac{C}{t^n}\int\limits_{\frac{1}{2}}^{t}\frac{d\rho}{\rho}\|g\mathbf{1}_{\mathcal{R}\frac{\rho}{2},\rho}\|_{p(\cdot)}\|\mathbf{1}_{\mathcal{R}\frac{\rho}{2},\rho}\|_{p'(\cdot)} \\
&\leq \frac{C}{t^n}\int\limits_{\frac{1}{2}}^{t}\|g\mathbf{1}_{\mathcal{R}\frac{\rho}{2},\rho}\|_{p(\cdot)}\rho^{\frac{n}{p'_\infty}-1}d\rho
\end{aligned}$$

$$\begin{aligned}
|Tf_t(x)\mathbf{1}_{\mathcal{R}t,2t}| &\leq \frac{C}{t^n}\int\limits_{\frac{1}{2}}^{t}d\rho\|g\mathbf{1}_{\mathcal{R}\frac{\rho}{2},\rho}\|_{p(\cdot)}\rho^{\frac{n}{p'_\infty}-1}\mathbf{1}_{\mathcal{R}t,2t} \\
&\leq \frac{C}{t^n}\int\limits_{\frac{1}{2}}^{t}\|g\mathbf{1}_{\mathcal{R}\frac{\rho}{2},\rho}\|_{p(\cdot)}\rho^{\frac{n}{p'_\infty}-1}d\rho
\end{aligned}$$

$$\begin{aligned}
&t^{\alpha_\infty}\|\mathbf{1}_{\mathcal{R}t,2t}Tf_t(x)\|_{p(\cdot)} \\
&\leq Ct^{\alpha_\infty-n}\int\limits_{\frac{1}{2}}^{t}d\rho\|g\mathbf{1}_{\mathcal{R}\frac{\rho}{2},\rho}\|_{p(\cdot)}\rho^{\frac{n}{p'_\infty}-1}\|\mathbf{1}_{\mathcal{R}t,2t}\|_{p(\cdot)} \\
&\leq Ct^{\alpha_\infty-\frac{n}{p'_\infty}}\int\limits_{\frac{1}{2}}^{t}\|g\mathbf{1}_{\mathcal{R}\frac{\rho}{2},\rho}\|_{p(\cdot)}\rho^{\frac{n}{p'_\infty}-1}d\rho \\
&\leq \int\limits_{\frac{1}{2}}^{t}\left(\frac{t}{\rho}\right)^{\alpha_\infty-\frac{n}{p'_\infty}}\omega(\rho)d\rho,
\end{aligned}$$

where $\omega(\rho)=\rho^{\alpha_\infty}\|g\mathbf{1}_{R_{\frac{\rho}{2},\rho}}\|_{p(\cdot)}$. Define $\Gamma(t)$ as

$$\Gamma(t)=\begin{cases}t^{\alpha_\infty-\frac{n}{p'_\infty}}, & t>1,\\ 0, & 0<t<1.\end{cases}\tag{3.5}$$

Define the operator $\Gamma\omega(t)=\int_0^\infty K(\frac{t}{\rho})\omega(\rho)d\rho$. Therefore

where $\omega(\rho)=\rho^{\alpha_\infty}\|\mathbf{1}_{\mathcal{R}\rho,2\rho}\|_{L^{p(\cdot)}}$. Left hand side of above equation is a Hardy type operator and by applying Lemma (2.2) we get

$$\mathcal{N}^{p,q,\alpha}_{1,2}(Tf_t)\leq\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\|\Gamma\omega\|_{L^{q(\cdot)}((2,k_0);\frac{dt}{t})}\leq\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\|\omega\|_{L^{q(\cdot)}((2,k_0);\frac{dt}{t})}\leq\|g\|_{HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,2;(\gamma',\delta')}(\mathbb{R}^n)}.$$

For finding the estimate of Tg_t: As the operator T is bounded on the space $L^{p(\cdot)}(\mathbb{R}^n)\to L^{p(\cdot)}(\mathbb{R}^n)$ so we obtain

$$\|Tg_t)\mathbf{1}_{\mathcal{R}t,2t}\|_{p(\cdot)}\leq C\|g_t\|_{p(\cdot)}=C\|g\mathbf{1}_{R\gamma' t,\delta' t}\|_{p(\cdot)}$$

which implies

$$\mathcal{N}^{p,q,\alpha}_{1,2}(Tg_t)\leq C\|g\mathbf{1}_{R\gamma' t,\delta' t}\|_{HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,2;(\gamma',\delta')}(\mathbb{R}^n)}.$$

For finding the estimate of $Th_t(x)$: Let $x\in\mathcal{R}t,2t$, by using the fact that $|x-y|\geq\rho/2$ and Hölder's inequality, we get

$$\begin{aligned}|Th_t(x)|&\leq C\int_{|y|>8t}\frac{|g(y)|}{|x-y|^n}dy\\&\leq C\int_{4t}^\infty\rho^{-n}\frac{d\rho}{\rho}\|g\mathbf{1}_{\mathcal{R}\rho,2\rho}\|_{p(\cdot)}\mathbf{1}_{\mathcal{R}\rho,2\rho}\|_{p'(\cdot)}\\&\leq C\int_{4t}^\infty\rho^{-n}\frac{d\rho}{\rho}\|g\mathbf{1}_{\mathcal{R}\rho,2\rho}\|_{p(\cdot)}\rho^{\frac{n}{p'_\infty}}.\end{aligned}$$

$$\begin{aligned}&\|T(h_t).\mathbf{1}_{\mathcal{R}t,2t}\|_{p(\cdot)}\\&\leq C\int_{4t}^\infty\rho^{-n}\frac{d\rho}{\rho}\|g\mathbf{1}_{\mathcal{R}\rho,2\rho}\|_{p(\cdot)}\rho^{\frac{n}{p'_\infty}}\|\mathbf{1}_{\mathcal{R}t,2t}\|_{p(\cdot)}\\&\leq C\int_{4t}^\infty\rho^{\frac{-n}{p_\infty}}\frac{d\rho}{\rho}\|g\mathbf{1}_{\mathcal{R}\rho,2\rho}\|_{p(\cdot)}t^{\frac{n}{p_\infty}}.\end{aligned}$$

$$\begin{aligned}&t^{\alpha_\infty}\|T(h_t).\mathbf{1}_{\mathcal{R}t,2t}\|_{p(\cdot)}\\&\leq C\rho^{\alpha_\infty+\frac{n}{p_\infty}}\int_{4t}^\infty\|g\mathbf{1}_{\mathcal{R}\rho,2\rho}\|_{p(\cdot)}\rho^{\frac{-n}{p_\infty}-1}d\rho\end{aligned}$$

$$\leq \int_t^\infty \left(\frac{t}{\rho}\right)^{\alpha_\infty + \frac{n}{p_\infty}} \omega(\rho) d\rho,$$

where $\omega(\rho) = \rho^{\alpha_\infty} \|g\mathbf{1}_{R_{\rho,2\rho}}\|_{L^{p(\cdot)}} \mathbf{1}_{(2,\infty)}(\rho)$. This yields a Hardy-type inequality, now the inequality $\alpha_\infty + n/p_\infty > 0$ and using Lemma (2.2) implies

$$\sup_{k_0 \in \mathbb{Z}} 2^{-k_0\beta} \|t^{\alpha_\infty} \|\mathbf{1}_{R_{t,2t}} Th_t(x)\|_{p(\cdot)}\|_{L^{q(\cdot)}((2,k_0);\frac{dt}{t})} \leq C\|g\|_{HM^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\beta,2;(\gamma',\delta')}(\mathbb{R}^n)}.$$

Combining the above estimates completes the proof. □

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6. Data availability

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7. Authors contribution

All authors contributed equally.
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