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FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS INVOLVING p-LAPLACIAN OPERATOR: EXISTENCE THEORY

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Abstract. This work studies the existence, uniqueness and positivity of solutions for fractional integro-differential equation involving a p-Laplacian operator with fractional derivative boundary conditions. The main results are proved by means of some known fixed point theorems such as the nonlinear alternative of Leray-Schauder type, Krasnosel'skii's and Banach contraction principle. At the end a few examples are given to substantiate our results.

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1. Introduction

In mathematical analysis, fractional differential equations ($\mathbb{FDE}$) arise from a variety of applications including in various fields of engineering and science. Recently, very much attention has been given to the linear and nonlinear integro-differential equations ($\mathbb{IDE}$) [1, 7, 10, 14, 18, 19].

The nonlinear $\mathbb{IDE}$s play an important role in many branches of nonlinear functional analysis and their applications in physics, the theory of engineering, mechanics. In 2019 Unhaley and Kendre in [23], established the existence and uniqueness of solutions for iterative $\mathbb{IDE}$ of the form,

$$\begin{cases} \mathcal{D}^{\sigma}\mathfrak{s}(\omega) = \wp(\omega) + \displaystyle\int_0^{\omega} \Bbbk\left(\omega, \varrho\right)\mathfrak{s}\big(\lambda\mathfrak{s}(\varrho)\big)\,\mathrm{d}\varrho, \\ \mathfrak{s}(0) = \mathfrak{a}_0. \end{cases}$$

Hamoud in [8], investigated the uniqueness and stability for the nonlinear Volterra-Fredholm $\mathbb{FIDE}$ involving the standard Caputo derivative ${}^{\mathrm{C}}\mathcal{D}^{\sigma}$, with the boundary

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condition,

$$\begin{cases} {}^{\mathrm{C}}\mathcal{D}^{\sigma}\mathfrak{s}(\omega) = \wp(\omega) + \displaystyle\int_0^{\omega} \mathbb{K}\left(\omega,\varrho\right)\mathfrak{s}\big(\mathfrak{s}(\varrho)\big)\,\mathrm{d}\varrho + \int_0^{\mathfrak{b}} \Bbbk\left(\omega,\varrho\right)\mathfrak{s}\big(\mathfrak{s}(\varrho)\big)\,\mathrm{d}\varrho, \\ \eta_1\mathfrak{s}(0) + \eta_2\mathfrak{s}(T) = \eta_3, \qquad \eta_i \in \mathbb{R}. \end{cases}$$

A p-Laplacian $\mathbb{DE}$ was first introduced in [15] when the author investigated the existence and uniqueness of solutions for the following problem, $\left(L_p\mathfrak{s}'(\omega)\right)' = \wp\big(\omega,\mathfrak{s}(\omega)\big)$, for $\omega \in \mathrm{J} := (0,1)$. Bobisud introduced the $\mathbb{DE}$,

$$\big(L_p\big(\mathfrak{s}'(\omega)\big)\big)' = \wp\big(\omega,\mathfrak{s}(\omega),\mathfrak{s}'(\omega)\big),$$

with an operator $L_p(\omega) = |\omega|^{p-1}\omega$ [3]. Tang *et al.* in [22] studied the existence of solutions for the mixed resonant $\mathbb{FDE}$ with p-Laplacian operator, using the continuation theorem of coincidence degree theory,

$$\begin{cases} {}^{\mathrm{C}}\mathcal{D}^{\sigma}\left(L_p\big(\mathcal{D}^{\sigma_\circ}\mathfrak{s}(\omega)\big)\right) = \wp\big(\omega,\mathfrak{s}(\omega),\mathcal{D}^{\sigma_\circ}\mathfrak{s}(\omega)\big), \quad 0 \le \omega \le T, \\ \omega^{1-\sigma_\circ}\mathfrak{s}(\omega)\big|_{\omega=0} = 0, \qquad \mathcal{D}^{\sigma_\circ}\mathfrak{s}(0) = \mathcal{D}^{\sigma_\circ}\mathfrak{s}(T), \end{cases}$$

Zhang *et al.* in [25] investigated the existence of nontrivial solutions for a class of $\mathbb{FDE}$ with p-Laplacian is formulated by,

$$\begin{cases} \mathcal{D}^{\sigma}\left(L_{p(\omega)}\big(\mathcal{D}^{\sigma_\circ}\left(p(\omega)\mathfrak{s}'(\omega)\right)\big)\right) + \wp\big(\omega,\mathfrak{s}(\omega)\big) = 0, \qquad \omega \in \mathrm{J}, \\ \mathcal{D}^{\sigma_\circ}\big(p(\omega)\mathfrak{s}'(\omega)\big)\big|_{\omega=0} = 0, \\ \eta_1\mathfrak{s}(0) - \eta_2 p(0)\mathfrak{s}'(0) = 0, \quad \eta_3\mathfrak{s}(1) + \eta_4 p(1)\mathfrak{s}'(1) = 0. \end{cases}$$

In [6], Haddouchi studied the existence and positivity of solutions for a p-Laplacian $\mathbb{FDE}$ is expressed by,

$$\begin{cases} \left(L_p\left({}^{\mathrm{C}}\mathcal{D}^{\sigma}\mathfrak{s}(\omega)\right)\right)' + \eta_1(\omega)\wp\big(\omega,\mathfrak{s}(\omega)\big) = 0, \qquad \omega \in \mathrm{J}, \\ {}^{\mathrm{C}}\mathcal{D}^{\sigma}\mathfrak{s}(0) = \mathfrak{s}'(0) = \mathfrak{s}''(0) = 0, \quad \mathfrak{s}(1) + \mathfrak{s}'(1) = \mathfrak{s}'(\zeta). \end{cases}$$

For some other results on $\mathbb{FDE}$s with p-Laplacian operator the reader may refer to [2, 9, 16, 17, 24].

In our article we concentrate on the existence, uniqueness and positivity of solutions for a $\mathbb{FIDE}$ with p-Laplacian operator,

$$\begin{aligned} \left(L_p\left({}^{\mathrm{C}}\mathcal{D}^{\sigma}\mathfrak{s}(\omega)\right)\right)' + \int_0^{\omega} \Bbbk_1\left(\omega,\varrho\right)\wp_1\left(\varrho,\mathfrak{s}(\varrho)\right)\,\mathrm{d}\varrho \\ + \int_0^{1} \Bbbk_2\left(\omega,\varrho\right)\wp_2\left(\varrho,\mathfrak{s}(\varrho)\right)\,\mathrm{d}\varrho = 0, \qquad \omega \in \mathrm{J}, \end{aligned} \tag{1.1}$$

with the three-point boundary condition

$${}^{\mathrm{C}}\mathcal{D}^{\sigma}\mathfrak{s}(0) = \mathfrak{s}'(0) = \mathfrak{s}''(0) = 0, \qquad \mathfrak{s}(1) + \mathfrak{s}'(1) = \mathfrak{s}'(\zeta), \tag{1.2}$$

where $2 < \sigma \le 3$, $\zeta \in \mathrm{J}$, $L_p(\omega) = |\omega|^{p-2}\omega$, the p-Laplacian operator ($p > 1$), $(L_p)^{-1} = L_q$, $p + q = pq$ via assuming the following hypothesis, $\mathrm{E}_\circ := [0,1]$:

H1) $\wp_i \in C\big(\mathrm{E}_\circ \times \mathbb{R}^{\ge 0}, \mathbb{R}^{\ge 0}\big)$ and on $\mathrm{E}_\circ$, $\wp_i(\omega,\cdot) \ne 0$;

H2) $\Bbbk_i \in C\big(\mathrm{E}_\circ^2, \mathbb{R}^{\ge 0}\big)$.

2. Preliminaries

In this section, we recall some important lemmas and definitions related to the fractional calculus which will be used through this work. We denote the Banach space by $\mathcal{B} = C(\mathrm{E}_\circ)$ endowed with the norm $\|\mathfrak{s}\| = \sup\{|\mathfrak{s}(\omega)| : \omega \in \mathrm{E}_\circ\}$. The Riemann-Liouville fractional integral and derivative of order $\sigma > 0$ for a continuous function $\mathfrak{s} : \mathbb{R}^{\geq 0} \to \mathbb{R}$ are given as follows,

$$\mathcal{I}^\sigma \mathfrak{s}(\omega) = \int_0^\omega \tfrac{\mathfrak{s}(\varrho)}{\Gamma(\sigma)}(\omega-\varrho)^{\sigma-1}\,\mathrm{d}\varrho,$$
$${}^{\mathrm{C}}\mathcal{D}^\sigma \mathfrak{s}(\omega) = \int_0^\omega \tfrac{\mathfrak{s}^{(n)}(\varrho)}{\Gamma(n-\sigma)}(\omega-\varrho)^{n-\sigma-1}\,\mathrm{d}\varrho,$$

where $n-1 < \sigma < n$, $n = [\sigma]+1$, and $\Gamma(\cdot)$ is the gamma function.

Lemma 2.1 ([12]). *Let $\sigma > 0$. The $\mathbb{FDE}$, ${}^{\mathrm{C}}\mathcal{D}^\sigma \mathfrak{s}(\omega) = 0$ has the general solution,*

$$\mathfrak{s}(\omega) = \tilde{v}_0 + \tilde{v}_1\omega + \cdots + \tilde{v}_{n-1}\omega^{n-1},$$

where $\tilde{v}_i \in \mathbb{R}$ with $i = \overline{0,\, n-1}$, $n = [\sigma]+1$.

From Lemma 2.1, we deduce

$$\mathcal{I}^\sigma\, {}^{\mathrm{C}}\mathcal{D}^\sigma \mathfrak{s}(\omega) = \mathfrak{s}(\omega) + \tilde{v}_0 + \tilde{v}_1\omega + \cdots + \tilde{v}_{n-1}\omega^{n-1},$$

where $\omega_i \in \mathbb{R}$ with $i = \overline{0,\, n-1}$, $n = [\sigma]+1$.

Lemma 2.2 ([20, 12]). *If $\mathfrak{s} \in L^1(\mathrm{E}_\circ)$ and $\sigma > \sigma_\circ > 0$, then*

i) ${}^{\mathrm{C}}\mathcal{D}^{\sigma_\circ}\, {}^{\mathrm{C}}\mathcal{D}^\sigma \mathfrak{s}(\omega) = \mathcal{I}^{\sigma-\sigma_\circ}\mathfrak{s}(\omega)$, *holds almost everywhere on $\mathrm{E}_\circ$ and it is valid at each point $\omega \in \mathrm{E}_\circ$ if $\mathfrak{s} \in \mathcal{B}$, ${}^{\mathrm{C}}\mathcal{D}^{\sigma_0}\mathcal{I}^{\sigma_0}\mathfrak{s}(\omega) = \mathfrak{s}(\omega)$.*

ii) ${}^{\mathrm{C}}\mathcal{D}^{\sigma_\circ}\omega^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\sigma_\circ)}\omega^{\lambda-\sigma_\circ-1}$, $\lambda > [\sigma_\circ]$ *and* ${}^{\mathrm{C}}\mathcal{D}^{\sigma_\circ}\omega^{\lambda-1} = 0$, $\lambda < [\sigma_\circ]$.

For $m, n > 0$, the Euler beta function is formulated by,

$$\mathfrak{B}(m,n) = \int_{\mathrm{E}_\circ} \omega^{m-1}\,(1-\omega)^{n-1}\,\mathrm{d}\omega.$$

Proposition 2.3. *Assume that $p \in \mathbb{Z}^+$ a, $m, n > 0$ and $\sigma > 0$. Then*

$$\mathfrak{B}(m,n) = \mathfrak{B}(n,m), \qquad \mathfrak{B}(m+1,n) = \mathfrak{B}(m,n)\,\tfrac{m}{m+n},$$
$$\mathfrak{B}(m,n) = \tfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \qquad \mathfrak{B}(m,q) = \mathfrak{B}(m,n+1) + \mathfrak{B}(m+1,n),$$

and

$$\Gamma(\sigma+1) = \sigma\Gamma(\sigma), \quad \Gamma(p+1) = p!, \quad \Gamma\left(p+\tfrac{1}{2}\right) = \tfrac{\sqrt{\pi}(2p)!}{2^{2n}p!}.$$

In particular, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma(1) = 1$ and $\mathfrak{B}(1,n) = \frac{1}{n}$.

A convex and closed subset $C \neq \varnothing$ of a real Banach space $\mathcal{B}$ is a cone whenever i) $\omega \in \mathfrak{C}$, $-\omega \in \mathfrak{C}$ imply $\omega = 0$; ii) $\omega \in \mathfrak{C}$, $\lambda \geq 0$ imply $\lambda\omega \in \mathfrak{C}$.

Theorem 2.4 ([13]). *Consider a cone C of Banach space $\mathcal{B}$, and open subsets O_1, $O_2 \subseteq \mathcal{B}$ with $0 \in O_1$, $\bar{O}_1 \subset O_2$. Let $\beth : C \cap (\bar{O}_2 \setminus O_1) \to C$ be a completely continuous operator as follows:*

i) $\|\beth\omega\| \geq \|\omega\|$, $\omega \in C \cap \partial O_1$, *and* $\|\beth\omega\| \leq \|\omega\|$, $\omega \in C \cap \partial O_2$, *or,*

ii) $\|\beth\omega\| \leq \|\omega\|$, $\omega \in C \cap \partial O_1$, *and* $\|\beth\omega\| \geq \|\omega\|$, $\omega \in C \cap \partial O_2$.

Then, $\beth$ admits a fixed point in $C \cap \left(\bar{O}_2 \setminus O_1\right)$.

Theorem 2.5. *For a closed subset $\varnothing \neq W \subset \mathcal{B}$, any contraction mapping $\beth : W \to W$ admit a unique fixed point.*

Theorem 2.6 ([5])**.** *Consider a convex and closed subset $W \subset \mathcal{B}$, a relatively open subset $0 \in O \subset W$ and suppose that $\beth : \bar{O} \to W$ is a compact map. Then either,*

i) *there is a point $y \in \partial O$ and $0 < \varkappa < 1$ with $y = \varkappa \beth(y)$; or,*
ii) *$\beth$ has a fixed point in $\bar{O}$.*

Lemma 2.7 ([6])**.** *Suppose that L_p be a p-Laplacian operator.*

i) *For $p > 2$, $|\omega_1|, |\omega_1| \leq M$, then*
$$|L_p(\omega_1) - L_p(\omega_2)| \leq (p-1) M^{p-2} |\omega_1 - \omega_2|.$$
ii) *For $1 < p \leq 2$, $\omega_1, \omega_2 > 0$, and $|\omega_1|, |\omega_2| \geq m > 0$, then*
$$|L_p(\omega_1) - L_p(\omega_2)| \leq (p-1) m^{p-2} |\omega_1 - \omega_2|.$$

3. Main results

Let $\hbar \in C\left(\mathrm{E}_\circ\right)$, $2 < \sigma \leq 3$ and consider the following $\mathbb{FDE}$,

$$\begin{cases} \left(L_p\left({}^{\mathrm{C}}\mathcal{D}^{\sigma}\mathfrak{s}(\omega)\right)\right)' + \hbar(\omega) = 0, & \omega \in \mathrm{J}, \\ {}^{\mathrm{C}}\mathcal{D}^{\sigma}\mathfrak{s}(0) = \mathfrak{s}'(0) = \mathfrak{s}''(0) = 0, \quad \mathfrak{s}(1) + \mathfrak{s}'(1) = \mathfrak{s}'(\zeta). \end{cases} \tag{3.1}$$

Regarding the proofs of following Lemmas 3.1, 3.3, 3.2, and 3.4, we refer the reader to [6].

Lemma 3.1 ([6])**.** *The $\mathbb{FDE}$ (3.1) admit a unique solution formulated by,*
$$\mathfrak{s}(\omega) = \int_{\mathrm{E}_\circ} \Bbbk(\omega, \varrho)\, L_q\left(\int_0^{\varrho} \hbar(\xi)\, \mathrm{d}\xi\right) \mathrm{d}\varrho,$$
in which $\Bbbk(\omega, \varrho) = \mathrm{g}_1(\omega, \varrho) + \mathrm{g}_2(\omega, \varrho)$, and for $\omega, \varrho \in \mathrm{E}_\circ$,
$$\mathrm{g}_1(\omega, \varrho) = \begin{cases} \frac{(1-\varrho)^{\sigma-1} - (\omega-\varrho)^{\sigma-1}}{\Gamma(\sigma)}, & \varrho \leq \omega, \\ \frac{(1-\varrho)^{\sigma-1}}{\Gamma(\sigma)}, & \omega \leq \varrho, \end{cases}$$
$$\mathrm{g}_2(\omega, \varrho) = \begin{cases} \frac{(1-\varrho)^{\sigma-2} - (\omega-\varrho)^{\sigma-2}}{\Gamma(\sigma-1)}, & \varrho \leq \omega, \\ \frac{(1-\varrho)^{\sigma-2}}{\Gamma(\sigma-1)}, & \omega \leq \varrho, \end{cases} \tag{3.2}$$

Lemma 3.2 ([6])**.** *For $0 < \varkappa < 1$ and for any $(\omega, \varrho) \in \mathrm{E}_\circ^2$, $\Bbbk(\omega, \varrho)$ satisfies the properties,*

i) $\Bbbk(\omega, \varrho) \geq 0$;
ii) $\left(1 - \zeta^{\sigma-2}\right)\left(1 - \omega^{\sigma-1}\right)\Phi(\varrho) \leq \Bbbk(\omega, \varrho) \leq \Phi(\varrho)$;
iii) $\left(1 - \zeta^{\sigma-2}\right)\left(1 - \varkappa^{\alpha-1}\right)\Phi(\varrho) \leq \Bbbk(\omega, \varrho) \leq \Phi(\varrho)$, *for $(\omega, \varrho) \in [0, \varkappa] \times \mathrm{E}_\circ$;*

where $\Phi(\omega) = \frac{1}{\Gamma(\sigma)}(\sigma - \omega)(1-\omega)^{\sigma-2}$ for $\omega \in [0, \varkappa]$.

Lemma 3.3 ([6])**.** *The functions $\mathrm{g}_i(\omega, \varrho)$, $i = 1, 2$, are continuous on $\mathrm{E}_\circ^2$ and for every $(\omega, \varrho) \in \mathrm{E}_\circ^2$ verify,*

i) $\mathrm{g}_2(\omega,\varrho)\geq 0,\ \mathrm{g}_1(\omega,\varrho)\geq 0$;
ii) $(1-\omega^{\sigma-2})\,\mathrm{g}_2(\varrho,\varrho)\leq \mathrm{g}_2(\omega,\varrho)\leq \mathrm{g}_2(\varrho,\varrho)$;
iii) $(1-\omega^{\sigma-1})\,\mathrm{g}_1(\varrho,\varrho)\leq \mathrm{g}_1(\omega,\varrho)\leq \mathrm{g}_1(\varrho,\varrho)$.

Lemma 3.4 ([6]). *The unique solution $\mathbb{h}\in C\big(\mathrm{E}_\circ,\mathbb{R}^{\geq 0}\big)$ of $\mathbb{FDE}$ (3.1) is nonnegative and satisfies,* $\min\big\{\mathbb{h}(\omega)\ :\ \omega\in[0,\varkappa]\big\}\geq\gamma\,\|\mathbb{h}\|$, *where* $\gamma=(1-\zeta^{\sigma-2})(1-\varkappa^{\sigma-1})$ *and* $0<\varkappa<1$ *be fixed.*

By considering

$$\Lambda_1=\left[\int_{\mathrm{E}_\circ}\Phi(\varrho)\,\mathrm{d}\varrho\right]^{-1},\qquad \Lambda_2=\left[\gamma\int_0^{\rho}\Phi(\varrho)\,L_q(\varrho)\,\mathrm{d}\varrho\right]^{-1},$$

we find that $0<\Lambda_1<\Lambda_2$ and further,

$$\Lambda_2^{-1}=\gamma\int_0^{\varkappa}\Phi(\varrho)\,L_q(\varrho)\,\mathrm{d}\varrho<\int_0^{\varkappa}\Phi(\varrho)\,L_q(1)\,\mathrm{d}\varrho\leq\int_{\mathrm{E}_\circ}\Phi(\varrho)\,\mathrm{d}\varrho=\Lambda_1^{-1}.$$

Let $\varkappa\in\mathrm{J}$ be fixed. We define the cone $\mathfrak{C}$ such that

$$\mathfrak{C}=\Big\{\mathfrak{s}\in\mathcal{B}\ :\ \mathbb{h}(\omega)\geq 0,\ \omega\in\mathrm{E}_\circ,\ \min_{\omega\in[0,\varkappa]}\mathbb{h}(\omega)\geq\gamma\,\|\mathbb{h}\|\Big\},$$

and the mapping $\beth:\mathfrak{C}\to\mathcal{B}$ as follows

$$\begin{aligned}(\beth\mathbb{h})(\omega)=\int_{\mathrm{E}_\circ}\Bbbk(\omega,\varrho)\,L_q\bigg(\int_0^{\varrho}\bigg[\int_0^{\xi}\Bbbk_1(\xi,u)\,\wp_1(u,\mathbb{h}(u))\,\mathrm{d}u\\+\int_{\mathrm{E}_\circ}\Bbbk_2(\xi,u)\,\wp_2(u,\mathbb{h}(u))\,\mathrm{d}u\bigg]\mathrm{d}\xi\bigg)\,\mathrm{d}\varrho,\end{aligned}\tag{3.3}$$

where $\Bbbk(\omega,\varrho)$ is defined by (3.2). From Lemma 3.1, the fixed points of the mapping $\beth$ in $\mathfrak{C}$ are the nonnegative solutions of $\mathbb{FIDE}$ (1.1). A function $\mathfrak{s}(\omega)$ is called a positive solution of $\mathbb{FIDE}$ (1.1)-(1.2) if $\mathfrak{s}\mathcal{B}$ and for any $\omega\in\mathrm{J}$, $\mathfrak{s}(\omega)>0$.

Lemma 3.5. *The mapping $\beth$ is completely continuous and satisfies $\beth(\mathfrak{C})\subset\mathfrak{C}$.*

Proof. In view of assumption (H1) and Lemma 3.4, we deduce that $\beth(\mathfrak{C})\subset\mathfrak{C}$. Using Lebesgue's dominated convergence theorem and the condition of nonnegativeness and continuity of $\Bbbk(\omega,\varrho)$ and $\wp_i(\omega,\mathbb{h}(\omega))$, $i=1,2$, it follows that $\beth:\mathfrak{C}\to\mathfrak{C}$ is continuous. For an arbitrary bounded set $W\subset\mathfrak{C}$, there exists $M>0$ as follows $W\subset\big\{\mathbb{h}\in\mathfrak{C}\ :\ \|\mathbb{h}\|<M\big\}$. From Lemmas 3.1 and 3.2 and for any $\mathbb{h}\in W$, we obtain,

$$\begin{aligned}(\beth\mathbb{h})(\omega)&=\int_{\mathrm{E}_\circ}\Bbbk(\omega,\varrho)\,L_q\bigg(\int_0^{\varrho}\bigg[\int_0^{\xi}\Bbbk_1(\xi,u)\,\wp_1(u,\mathbb{h}(u))\,\mathrm{d}u\\&\quad+\int_{\mathrm{E}_\circ}\Bbbk_2(\xi,u)\,\wp_2(u,\mathbb{h}(u))\,\mathrm{d}u\bigg]\mathrm{d}\xi\bigg)\,\mathrm{d}\varrho\\&\leq\int_{\mathrm{E}_\circ}\Bbbk(\omega,\varrho)\,L_q\bigg(\int_{\mathrm{E}_\circ}\big[\beta_{\Bbbk_1}L_{\wp_1}+\beta_{\Bbbk_2}L_{\wp_2}\big]\,\mathrm{d}\xi\bigg)\,\mathrm{d}\varrho\\&\leq\big[\beta_{\Bbbk_1}L_{\wp_1}+\beta_{\Bbbk_2}L_{\wp_2}\big]^{q-1}\int_{\mathrm{E}_\circ}\Phi(\varrho)\,\mathrm{d}\varrho,\end{aligned}$$

where for $i=1,2$, $L_{\wp_i}=\max\left\{\wp_i(\omega,\mathbb{h}) : \omega\in \mathrm{E}_\circ, \mathbb{h}\in W\right\}$ and

$$\beta_{\mathbb{k}_i}=\max_{\omega\in \mathrm{E}_\circ}\int_0^{\omega}\mathbb{k}_i(\omega,\varrho)\,\mathrm{d}\varrho.$$

Thus,

$$\|\beth\mathbb{h}\|\leq\left[\beta_{\mathbb{k}_1}L_{\wp_1}+\beta_{\mathbb{k}_2}L_{\wp_2}\right]^{q-1}\int_{\mathrm{E}_\circ}\Phi(\varrho)\,\mathrm{d}\varrho.$$

Thus, $\beth(W)$ is uniformly bounded. Now, for $\mathfrak{s}\in W$, $\omega,\grave{\omega}\in \mathrm{E}_\circ$ with $\omega<\grave{\omega}$, by Lemmas 3.1 and 3.3, we get,

$$\begin{aligned}
\left|(\beth\mathfrak{s})(\omega)-(\beth\mathfrak{s})(\grave{\omega})\right| &= \Bigg|\int_{\mathrm{E}_\circ}\mathbb{k}(\omega,\varrho)L_q\Bigg(\int_0^{\varrho}\Bigg[\int_0^{\xi}\mathbb{k}_1(\xi,u)\wp_1\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u\\
&\quad+\int_{\mathrm{E}_\circ}\mathbb{k}_2(\xi,u)\wp_2\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u\Bigg]\mathrm{d}\xi\Bigg)\mathrm{d}\varrho\\
&\quad-\int_{\mathrm{E}_\circ}\mathbb{k}(\grave{\omega},\varrho)L_q\Bigg(\int_0^{\varrho}\Bigg[\int_0^{\xi}\mathbb{k}_1(\xi,u)\wp_1\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u\\
&\quad+\int_{\mathrm{E}_\circ}\mathbb{k}_2(\xi,u)\wp_2\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u\Bigg]\mathrm{d}\xi\Bigg)\mathrm{d}\varrho\Bigg|\\
&=\Bigg|\int_{\mathrm{E}_\circ}\left[\mathbb{k}(\omega,\varrho)-\mathbb{k}(\grave{\omega},\varrho)\right]\\
&\quad\times L_q\Bigg(\int_0^{s}\Bigg[\int_0^{\xi}\mathbb{k}_1(\xi,u)\wp_1\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u\\
&\quad+\int_{\mathrm{E}_\circ}\mathbb{k}_2(\xi,u)\wp_2\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u\Bigg]\mathrm{d}\xi\Bigg)\mathrm{d}\varrho\Bigg|\\
&\leq\int_{\mathrm{E}_\circ}\left|\mathrm{g}_1(\omega,\varrho)-\mathrm{g}_1(\grave{\omega},\varrho)\right|L_q\Bigg(\int_{\mathrm{E}_\circ}\left[\beta_{\mathbb{k}_1}L_{\wp_1}+\beta_{\mathbb{k}_2}L_{\wp_2}\right]\mathrm{d}\xi\Bigg)\mathrm{d}\varrho\\
&\leq\left[\beta_{\mathbb{k}_1}L_{\wp_1}+\beta_{\mathbb{k}_2}L_{\wp_2}\right]^{q-1}\int_{\mathrm{E}_\circ}\left|\mathrm{g}_1(\omega,\varrho)-\mathcal{G}(\grave{\omega},\varrho)\right|\mathrm{d}\varrho. \qquad (3.4)
\end{aligned}$$

As $\grave{\omega}\to\omega$ the right-side of Ineq. (3.4) tends to zero from the continuity of g_1, which means that $\beth(W)$ is equicontinuous. By Arzelà-Ascoli theorem the mapping we conclude that the mapping $\beth:\mathfrak{C}\to\mathfrak{C}$ is completely continuous. □

Theorem 3.6. *Suppose that* (H1) *and* (H2) *hold. If there exist constants* $\varkappa_i>0$, $i=1,2$, $\varkappa_1<\varkappa_2$, M_1, $N_1\in(0,\Lambda_1]$, *and* M_2, $N_2\in[\Lambda_2,\infty)$, *with and* $M_2\varkappa_1<M_1\varkappa_2$, $N_2\varkappa_1<N_1\varkappa_2$, *such that for* $\omega\in \mathrm{E}_\circ$, $\wp_i$ *satisfies*

i) $\wp_1(\omega,\mathfrak{s})\leq\frac{L_p(M_1\varkappa_2)}{2\beta_{\mathbb{k}_1}}$, $\wp_2(\omega,\mathfrak{s})\leq\frac{L_p(N_1\varkappa_2)}{2\beta_{\mathbb{k}_2}}$ *for each* $\mathfrak{s}\in[0,\varkappa_2]$;

ii) $\wp_1(\omega,\mathfrak{s})\geq\frac{L_p(M_2\varkappa_1)}{2\beta_{\mathbb{k}_1}}$, $\wp_2(\omega,\mathfrak{s})\geq\frac{L_p(N_2\varkappa_1)}{2\beta_{\mathbb{k}_2}}$ *for every* $\mathfrak{s}\in[\gamma\varkappa_1,\varkappa_1]$.

Then, $\mathbb{FIDE}$ (1.1)-(1.2) *admit at least one positive solution* $\mathfrak{s}\in\mathfrak{C}$ *with* $\varkappa_1<\|\mathfrak{s}\|<\varkappa_2$.

Proof. Consider the open set $\Omega_{\varkappa_2} = \{\mathfrak{s} \in \mathcal{B} : \|\mathfrak{s}\| < \varkappa_2\}$. For, $\mathfrak{s} \in \mathfrak{C} \cap \partial\Omega_{\varkappa_2}$, by Lemma 3.2 and assumption (i), we get

$$\begin{aligned}
\left(\beth\mathfrak{s}\right)(\omega) &= \int_{\mathrm{E}_\circ} \Bbbk(\omega,\varrho)\, L_q\Bigg(\int_0^\varrho \Bigg[\int_0^\xi \Bbbk_1(\xi,u)\, \wp_1\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u \\
&\quad + \int_{\mathrm{E}_\circ} \Bbbk(\xi,u)\, \wp_2\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u\Bigg]\mathrm{d}\xi\Bigg)\mathrm{d}\varrho \\
&\leq \int_{\mathrm{E}_\circ} \Bbbk(\omega,\varrho)\, L_q\Bigg(\int_0^\varrho \beta_{\Bbbk_1}\tfrac{L_p(M_1\varkappa_2)}{2\beta_{\Bbbk_1}} + \beta_{\Bbbk_2}\tfrac{L_p(N_1\varkappa_2)}{2\beta_{\Bbbk_2}}\,\mathrm{d}\xi\Bigg)\mathrm{d}\varrho \\
&\leq \int_{\mathrm{E}_\circ} \Bbbk(\omega,\varrho)\, L_q\left(\tfrac{L_p(M_1\varkappa_2)}{2} + \tfrac{L_p(N_1\varkappa_2)}{2}\right)\mathrm{d}\varrho \\
&\leq L_q\left(\tfrac{L_p(\Lambda_1\varkappa_2)}{2} + \tfrac{L_p(\Lambda_1\varkappa_2)}{2}\right)\int_{\mathrm{E}_\circ} \Phi(\varrho)\,\mathrm{d}\varrho \\
&\leq L_q\big(L_p(\Lambda_1\varkappa_2)\big)\int_{\mathrm{E}_\circ} \Phi(\varrho)\,\mathrm{d}\varrho \leq \Lambda_1\Lambda_1^{-1}\varkappa_2 = \varkappa_2,
\end{aligned}$$

which implies that $\|\beth\mathfrak{s}\| \leq \|\mathfrak{s}\|$, for every $\mathfrak{s} \in \mathfrak{C} \cap \partial\Omega_{\varkappa_2}$. Next, consider the open set $\Omega_{\varkappa_1} = \{\mathfrak{s} \in \mathcal{B} : \|\mathfrak{s}\| < \varkappa_1\}$. For any $\mathfrak{s} \in \mathfrak{C} \cap \partial\Omega_{\varkappa_1}$, by using (H1)-(H2), Lemma 3.2 and assumption (ii), for $\omega \in [0,\varkappa]$, we have

$$\begin{aligned}
\left(\beth\mathfrak{s}\right)(\omega) &= \int_{\mathrm{E}_\circ} \Bbbk(\omega,\varrho)\, L_q\Bigg(\int_0^\varrho \Bigg[\int_0^\xi \Bbbk_1(\xi,u)\, \wp_1\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u \\
&\quad + \int_{\mathrm{E}_\circ} \Bbbk_2(\xi,u)\, \wp_2\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u\Bigg]\mathrm{d}\xi\Bigg)\mathrm{d}\varrho \\
&\geq \int_0^{\varkappa} \Bbbk(\omega,\varrho)\, L_q\Bigg(\int_0^{\varrho_1} \beta_{\Bbbk_1}\tfrac{L_p(M_2\varkappa_1)}{2\beta_{\Bbbk_1}} + \beta_{\Bbbk_2}\tfrac{L_p(N_2\varkappa_1)}{2\beta_{\Bbbk_2}}\,\mathrm{d}\xi\Bigg)\mathrm{d}\varrho \\
&\geq \gamma\int_0^{\varkappa} \Phi(\varrho)\, L_q\Bigg(\int_0^\varrho \tfrac{L_p(M_2\varkappa_1)}{2} + \tfrac{L_p(N_2\varkappa_1)}{2}\,\mathrm{d}\xi\Bigg)\mathrm{d}\varrho \\
&\geq \gamma\int_0^{\varkappa} \Phi(\varrho)\, L_q\Bigg(\int_0^\varrho \tfrac{L_p(\Lambda_2\varkappa_1)}{2} + \tfrac{L_p(\Lambda_2\varkappa_1)}{2}\,\mathrm{d}\xi\Bigg)\mathrm{d}\varrho \\
&\geq \gamma\int_0^{\varkappa} \Phi(\varrho)\, L_q\Bigg(L_p(\Lambda_2\varkappa_1)\int_0^\varrho \mathrm{d}\xi\Bigg)\mathrm{d}\varrho \\
&\geq \gamma\int_0^{\varkappa} \Phi(\varrho)\, L_q\big(L_p(\Lambda_2\varkappa_1)\big)\, L_q\Bigg(\int_0^\varrho \mathrm{d}\xi\Bigg)\mathrm{d}\varrho \\
&\geq \gamma\int_0^{\varkappa} \Phi(\varrho)\,(\Lambda_2\varkappa_1)\, L_q\Bigg(\int_0^\varrho \mathrm{d}\xi\Bigg)\mathrm{d}\varrho \geq \varkappa_1\Lambda_2\Lambda_2^{-1} = \varkappa_1,
\end{aligned}$$

which implies that $\|\beth\mathfrak{s}\| \geq \|\mathfrak{s}\|$ for $\mathfrak{s} \in \mathfrak{C} \cap \partial\Omega_{\varkappa_1}$. Hence, thanks to Theorem 2.4-(i), $\beth$ admit at least one fixed point in $\mathfrak{C} \cap \left(\overline{\Omega}_{\varkappa_2} \setminus \Omega_{\varkappa_1}\right)$. Therefore, there exists at least one solution of (1.1)-(1.2) satisfying $\varkappa_1 < \|\mathfrak{s}\| < \varkappa_2$. □

Theorem 3.7. *Suppose that* (H1)-(H2) *hold. If there exist constants* $\varkappa_i$, $i = 1, 2$, $\gamma\varkappa_2 < \varkappa_1 < \varkappa_2$, M_1, $N_1 \in (0, \Lambda_1]$, *and* $M_2, N_2 \in [\Lambda_2, \infty)$, *where* $M_1\varkappa_1 > M_2\varkappa_2$, $N_1\varkappa_1 > N_2\varkappa_2$ *such that* $\wp_i$ *satisfies:*

i) $\wp_1(\omega,\mathfrak{s}) \geq \frac{L_p(M_2\varkappa_2)}{2\beta_{\Bbbk_1}}$, $\wp_2(\omega,\mathfrak{s}) \geq \frac{L_p(N_2\varkappa_2)}{2\beta_{\Bbbk_2}}$ *for* $\mathfrak{s} \in [\gamma\varkappa_2, \varkappa_2]$ *and* $\omega \in [0,\varkappa]$;

ii) $\wp_1(\omega,\mathfrak{s}) \leq \frac{L_p(M_1\varkappa_1)}{2\beta_{\Bbbk_1}}$, $\wp_2(\omega,\mathfrak{s}) \leq \frac{L_p(N_1\varkappa_1)}{2\beta_{\Bbbk_2}}$ *for* $\mathfrak{s} \in [0,\varkappa_1]$ *and* $\omega \in \mathrm{E}_\circ$.

Then, $\mathbb{FIDE}$ (1.1)-(1.2) *admit at least one positive solution* $\mathfrak{s} \in \mathfrak{C}$ *with* $\varkappa_1 < \|\mathfrak{s}\| < \varkappa_2$.

Theorem 3.8. *Let* (H1) *and* (H2) *hold. Moreover suppose that there exists a constant* $\nu > 0$ *as follows:*

$$\nu > \left[\beta_{\Bbbk_1}\wp_1 + \beta_{\Bbbk_2}\wp_2\right]^{q-1} \int_{\mathrm{E}_\circ} \Phi(\varrho)\, \mathrm{d}\varrho, \tag{3.5}$$

where $L_{\wp_i} = \max\left\{\wp_i(\omega,\mathfrak{s}) : (\omega,\mathfrak{s}) \in \mathrm{E}_\circ \times [0,\nu]\right\}$, $i = 1,2$. *Then* $\mathbb{FIDE}$ (1.1)-(1.2) *admit at least one positive solution.*

Proof. Define $\mathcal{U} = \left\{\mathfrak{s} \in \mathfrak{C} : \|\mathfrak{s}\| < \nu\right\}$. From Lemma 3.5, the mapping $\beth : \overline{\mathcal{U}} \to \mathfrak{C}$ is completely continuous. Suppose that there exist $\mathfrak{s} \in \overline{\mathcal{U}}$ and $\lambda \in \mathrm{J}$ as follows $\mathfrak{s} = \lambda\beth\mathfrak{s}$. Hence, we obtain,

$$\begin{aligned}
\left|\mathfrak{s}(\omega)\right| &= \left|\lambda\left(\beth\mathfrak{s}\right)(\omega)\right| \\
&= \left|\lambda \int_{\mathrm{E}_\circ} \Bbbk(\omega,\varrho)\, L_q\Bigg(\int_0^{\varrho}\Bigg[\int_0^{\tau} \Bbbk_1(\xi,u)\,\wp_1\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u \right. \\
&\quad \left. + \int_{\mathrm{E}_\circ} \Bbbk_2(\xi,u)\,\wp_2\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u\Bigg]\,\mathrm{d}\xi\Bigg)\,\mathrm{d}s\right| \\
&\leq \left|\int_{\mathrm{E}_\circ} \Bbbk(\omega,\varrho)\, L_q\Bigg(\int_{\mathrm{E}_\circ} \left(\beta_{\Bbbk_1}L_1 + \beta_{\Bbbk_2}L_2\right)\,\mathrm{d}\tau\Bigg)\,\mathrm{d}s\right| \\
&\leq \left[\beta_{\Bbbk_1}L_1 + \beta_{\Bbbk_2}L_2\right]^{q-1} \int_{\mathrm{E}_\circ} \Phi(\varrho)\,\mathrm{d}\varrho.
\end{aligned}$$

So,

$$\|u\| \leq \left[\beta_{\Bbbk_1}L_1 + \beta_{\Bbbk_2}L_2\right]^{q-1} \int_{\mathrm{E}_\circ} \Phi(\varrho)\,\mathrm{d}\varrho.$$

Then by (3.5), we have that $\|\mathfrak{s}\| < \nu$, we conclude that $\mathfrak{s} \notin \partial\mathcal{U}$. Thus, it follows that there is no $\mathfrak{s} \in \partial\mathcal{U}$ such that $\mathfrak{s} = \lambda\beth\mathfrak{s}$ for some $\lambda \in \mathrm{E}_\circ \setminus \{0,1\}$. Therefore by Theorem 2.6, which means that $\mathbb{FIDE}$ (1.1)-(1.2) admit at least one positive solution. □

Theorem 3.9. *Let* $p > 2$ *and Suppose that* (H1)-(H2) *hold. Further, we Suppose that the two following hypothesis hold.*

C1) *There exists a nonnegatives functions* $\Upsilon_{\wp_i}(\omega) \in \mathcal{B}$ *with* $\wp_i(\omega,\mathfrak{s}) \leq \Upsilon_{\wp_i}(\omega)$, *for all* $(\omega,\mathfrak{s}) \in \mathrm{E}_\circ \times \mathbb{R}^{\geq 0}$ *and* $i = 1,2$;

C2) *There exists two constants* $L_{\wp_i}$, $i = 1,2$ *such that*

$$\left|\wp_i(\omega,\mathfrak{s}) - \wp_i(\omega,\grave{\mathfrak{s}})\right| \leq L_{\wp_i}\left|\mathfrak{s} - \grave{\mathfrak{s}}\right|, \qquad \omega \in \mathrm{E}_\circ, \mathfrak{s}, \grave{\mathfrak{s}} \in \mathbb{R}^{\geq 0}.$$

Then, $\mathbb{FIDE}$ (1.1)-(1.2) *admit a unique solution whenever,*

$$(q-1)\,M^{q-2}\left[\beta_{\Bbbk_1}L_{\wp_1} + \beta_{\Bbbk_2}L_{\wp_2}\right] \int_{\mathrm{E}_\circ} \Phi(\varrho)\,\mathrm{d}\varrho < 1. \tag{3.6}$$

Proof. From (C1), for each $\omega \in \mathrm{E}_{\circ}$, we obtain

$$\int_0^{\omega} \left[\int_0^{\xi} \Bbbk_1(\xi,u)\, \wp_1\big(u, \mathfrak{s}(u)\big)\, \mathrm{d}u + \int_{\mathrm{E}_{\circ}} \Bbbk_2(\xi,u)\, \wp_2\big(u, \mathfrak{s}(u)\big)\, \mathrm{d}u \right] \mathrm{d}\xi$$
$$\leq \int_{\mathrm{E}_{\circ}} \left[\beta_{\Bbbk_1} \Upsilon_{\wp_1}(u) + \beta_{\Bbbk_2} \Upsilon_{\wp_2}(u) \right] \mathrm{d}\xi \leq \beta_{\Bbbk_1} \Upsilon_{\wp_1} + \beta_{\Bbbk_2} \Upsilon_{\wp_2} := M.$$

By Lemma 2.7-(i) and (C2), for each $\mathfrak{s}, \grave{\mathfrak{s}} \in \mathcal{B}$, we get

$$\begin{aligned}
\big|\big(\beth\mathfrak{s}\big)(\omega) - \big(\beth\grave{\mathfrak{s}}\big)(\omega)\big| &= \int_{\mathrm{E}_{\circ}} \Bbbk(\omega,\varrho) \\
&\quad \times L_q\bigg(\int_0^{\varrho} \bigg[\int_0^{\xi} \Bbbk_1(\xi,u)\, \wp_1\big(u,\mathfrak{s}(u)\big)\, \mathrm{d}u \\
&\quad + \int_{\mathrm{E}_{\circ}} \Bbbk_2(\xi,u)\, \wp_2\big(u,\mathfrak{s}(u)\big)\, \mathrm{d}u \bigg] \mathrm{d}\xi \bigg)\, \mathrm{d}\varrho \\
&\quad - \int_{\mathrm{E}_{\circ}} \Bbbk(\omega,\varrho)\, L_q\bigg(\int_0^{\varrho} \bigg[\int_0^{\xi} \Bbbk_1(\xi,u)\, \wp_1\big(u,\grave{\mathfrak{s}}(u)\big)\, \mathrm{d}u \\
&\quad + \int_{\mathrm{E}_{\circ}} \Bbbk_2(\xi,u)\, \wp_2\big(u,\grave{\mathfrak{s}}(u)\big)\, \mathrm{d}u \bigg] \mathrm{d}\tau \bigg)\, \mathrm{d}\varrho \\
&= \int_{\mathrm{E}_{\circ}} \Bbbk(\omega,\varrho) \bigg[L_q\bigg(\int_0^{\varrho} \bigg[\int_0^{\xi} \Bbbk_1(\xi,u)\, \wp_1\big(u,\mathfrak{s}(u)\big)\, \mathrm{d}u \\
&\quad + \int_{\mathrm{E}_{\circ}} \Bbbk_2(\xi,u)\, \wp_2\big(u,\mathfrak{s}(u)\big)\, \mathrm{d}u \bigg] \mathrm{d}\xi \bigg) \\
&\quad - L_q\bigg(\int_0^{\varrho} \bigg[\int_0^{\xi} \Bbbk_1(\xi,u)\, \wp_1\big(u,\grave{\mathfrak{s}}(u)\big)\, \mathrm{d}u \\
&\quad + \int_{\mathrm{E}_{\circ}} \Bbbk_2(\xi,u)\, \wp_2\big(u,\grave{\mathfrak{s}}(u)\big)\, \mathrm{d}u \bigg] \mathrm{d}\xi \bigg) \bigg]\, \mathrm{d}\varrho \\
&\leq (q-1)\, M^{q-2} \int_{\mathrm{E}_{\circ}} \Bbbk(\omega,\varrho) \int_0^{\varrho} \bigg[\int_0^{\xi} \Bbbk_1(\xi,u) \Big[\wp_1\big(u,\mathfrak{s}(u)\big) \\
&\quad - \wp_1\big(u,\grave{\mathfrak{s}}(u)\big) \Big]\, \mathrm{d}u + \int_{\mathrm{E}_{\circ}} \Bbbk_2(\xi,u) \Big[\wp_2\big(u,\mathfrak{s}(u)\big) \\
&\quad - \wp_2\big(u,\grave{\mathfrak{s}}(u)\big) \Big]\, \mathrm{d}u \bigg]\, \mathrm{d}\xi\, \mathrm{d}\varrho \\
&\leq (q-1)\, M^{q-2} \int_{\mathrm{E}_{\circ}} \Bbbk(\omega,\varrho) \\
&\quad \times \int_{\omega,\varrho} \left[\beta_{\Bbbk_1} L_{\wp_1} \|\mathfrak{s} - \grave{\mathfrak{s}}\| + \beta_{\Bbbk_2} L_{\wp_2} \|\mathfrak{s} - \grave{\mathfrak{s}}\| \right] \mathrm{d}\xi\, \mathrm{d}\varrho \\
&\leq (q-1)\, M^{q-2} \left[\beta_{\Bbbk_1} L_{\wp_1} + \beta_{\Bbbk_2} L_{\wp_2} \right] \int_{\mathrm{E}_{\circ}} \Phi(\varrho)\, \mathrm{d}\varrho\, \|\mathfrak{s} - \grave{\mathfrak{s}}\|.
\end{aligned}$$

Hence

$$\left\|\mathfrak{J}\mathfrak{s}-\mathfrak{J}\grave{\mathfrak{s}}\right\| \leq (q-1)\, M^{q-2}\left[\beta_{\Bbbk_1} L_{\wp_1}+\beta_{\Bbbk_2} L_{\wp_2}\right] \int_{\mathrm{E}_\circ} \Phi(\varrho)\, \mathrm{d}\varrho\, \|\mathfrak{s}-\grave{\mathfrak{s}}\|.$$

From the principle of Banach fixed point theorem, the mapping $\mathfrak{J}$ admit a unique fixed point, which corresponds to the unique solution for $\mathbb{FIDE}$ (1.1)-(1.2). □

Theorem 3.10. *Let $1<p<2$, and suppose that* (H1), (H2) *hold. Further, we assume that there exist two constants $\mu>0$, and $0<\varkappa<\frac{2}{2-q}$ such that*

$$\wp_i(\omega,\mathfrak{s}(\omega)) \geq \tfrac{\mu\omega^{\varkappa-1}}{2\varkappa_{\Bbbk_i}}, \qquad \varkappa_{\Bbbk_i}=\min\int_0^{\omega} \Bbbk_i(\omega,\varrho)\, \mathrm{d}\varrho,\ i=1,2. \tag{3.7}$$

Then $\mathbb{FIDE}$ (1.1)-(1.2) admit a unique solution, whenever

$$\tfrac{(q-1)\mu^{q-2}\left(\beta_{\Bbbk_1}L_{\wp_1}+\beta_{\Bbbk_2}L_{\wp_2}\right)(\varkappa(q-2)+\sigma+1)}{(\varkappa(q-2)+\sigma)\Gamma(\sigma-1)}\mathfrak{B}\big(\sigma-1,\varkappa(q-2)+1\big)<1.$$

Proof. From (3.7), we have

$$\begin{aligned}\int_0^{\xi} \Bbbk_1(\xi,u)\wp_1\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u+\int_{\mathrm{E}_\circ} \Bbbk_2(\xi,u)\,\wp_2\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u\\ \geq \varkappa_{\Bbbk_1}\tfrac{\mu\omega^{\varkappa-1}}{2\varkappa_{\Bbbk_1}}+\varkappa_{\Bbbk_2}\tfrac{\mu\omega^{\varkappa-1}}{2\varkappa_{\Bbbk_2}} \geq \mu\omega^{\varkappa-1}, \qquad (\omega,u)\in \mathrm{E}_\circ\times\mathbb{R}^{\geq 0}.\end{aligned}$$

From (C2) and Lemma 2.7-(ii), for all $\mathfrak{s},\grave{\mathfrak{s}}\in\mathcal{B}$, we get,

$$\begin{aligned}\left|\left(\mathfrak{J}\mathfrak{s}\right)(\omega)-\left(\mathfrak{J}\grave{\mathfrak{s}}\right)(\omega)\right| &= \int_{\mathrm{E}_\circ} \Bbbk(\omega,\varrho)\\ &\quad\times L_q\Bigg(\int_0^{\varrho}\Bigg[\int_0^{\xi}\Bbbk_1(\xi,u)\,\wp_1\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u\\ &\quad+\int_{\mathrm{E}_\circ}\Bbbk_2(\xi,u)\,\wp_2\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u\Bigg]\mathrm{d}\xi\Bigg)\mathrm{d}\varrho\\ &\quad-\int_{\mathrm{E}_\circ}\Bbbk(\omega,\varrho)\,L_q\Bigg(\int_0^{\varrho}\Bigg[\int_0^{\xi}\Bbbk_1(\xi,u)\,\wp_1\big(u,\grave{\mathfrak{s}}(u)\big)\,\mathrm{d}u\\ &\quad+\int_{\mathrm{E}_\circ}\Bbbk_2(\xi,u)\,\wp_2\big(u,\grave{\mathfrak{s}}(u)\big)\,\mathrm{d}u\Bigg]\mathrm{d}\tau\Bigg)\mathrm{d}\varrho\\ &= \int_{\mathrm{E}_\circ}\Bbbk(\omega,\varrho)\Bigg[L_q\Bigg(\int_0^{\varrho}\Bigg[\int_0^{\xi}\Bbbk_1(\xi,u)\,\wp_1\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u\\ &\quad+\int_{\mathrm{E}_\circ}\Bbbk_2(\xi,u)\,\wp_2\big(u,\mathfrak{s}(u)\big)\,\mathrm{d}u\Bigg]\mathrm{d}\tau\Bigg)\\ &\quad-L_q\Bigg(\int_0^{\varrho}\Bigg[\int_0^{\xi}\Bbbk_1(\xi,u)\,\wp_1\big(u,\grave{\mathfrak{s}}(u)\big)\,\mathrm{d}u\\ &\quad+\int_{\mathrm{E}_\circ}\Bbbk_2(\xi,u)\,\wp_2\big(u,\grave{\mathfrak{s}}(u)\big)\,\mathrm{d}u\Bigg]\mathrm{d}\xi\Bigg)\Bigg]\mathrm{d}\varrho\\ &\leq (q-1)\,(\mu s^{\varkappa})^{q-2}\int_{\mathrm{E}_\circ}\Bbbk(\omega,\varrho)\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^{\varrho} \left[\int_0^{\xi} \Bbbk_1(\xi,u) \left[\wp_1\big(u,\mathfrak{s}(u)\big) - \wp_1\big(u,\grave{\mathfrak{s}}(u)\big) \right] \mathrm{d}u \right. \\
&\left. + \int_{\mathrm{E}_\circ} \Bbbk_2(\xi,u) \left[\wp_2\big(u,\mathfrak{s}(u)\big) - \wp_2\big(u,\grave{\mathfrak{s}}(u)\big) \right] \mathrm{d}u \right] \mathrm{d}\xi\, \mathrm{d}\varrho \\
&\leq (q-1)\,(\mu\varrho^{\varkappa})^{q-2} \int_{\mathrm{E}_\circ} \Bbbk(\omega,\varrho) \\
&\times \int_{\mathrm{E}_\circ} \left[\beta_{\Bbbk_1} L_{\wp_1} \|\mathfrak{s}-\grave{\mathfrak{s}}\| + \beta_{\Bbbk_2} L_{\wp_2} \|\mathfrak{s}-\grave{\mathfrak{s}}\| \right] \mathrm{d}\xi\, \mathrm{d}\varrho \\
&\leq (q-1)\,\mu^{q-2} \left[\beta_{\Bbbk_1} L_{\wp_1} + \beta_{\Bbbk_2} L_{\wp_2} \right] \\
&\times \|\mathfrak{s}-\grave{\mathfrak{s}}\| \int_{\mathrm{E}_\circ} \Bbbk(\omega,\varrho)\, \varrho^{\varkappa(q-2)}\, \mathrm{d}\varrho \\
&\leq (q-1)\,\mu^{q-2} \left[\beta_{\Bbbk_1} L_{\wp_1} + \beta_{\Bbbk_2} L_{\wp_2} \right] \\
&\times \|\mathfrak{s}-\grave{\mathfrak{s}}\| \int_{\mathrm{E}_\circ} \left[\mathrm{g}_1(\omega,\varrho) + \mathrm{g}_2(\omega,\varrho) \right] \varrho^{\varkappa(q-2)}\, \mathrm{d}\varrho \\
&= (q-1)\,\mu^{q-2} \left[\beta_{\Bbbk_1} L_{\wp_1} + \beta_{\Bbbk_2} L_{\wp_2} \right] \|\mathfrak{s}-\grave{\mathfrak{s}}\| \\
&\times \left[\frac{\mathfrak{B}(\sigma,\varkappa(q-2)+1)}{\Gamma(\sigma)} + \frac{\mathfrak{B}(\sigma-1,\varkappa(q-2)+1)}{\Gamma(\sigma-1)} \right] \\
&= \frac{(q-1)\mu^{q-2}\left(\beta_{\Bbbk_1} L_{\wp_1} + \beta_{\Bbbk_2} L_{\wp_2}\right)(\varkappa(q-2)+\sigma+1)}{(\varkappa(q-2)+\sigma)\Gamma(\sigma-1)} \\
&\times \mathfrak{B}(\sigma-1,\varkappa(q-2)+1)\,\|\mathfrak{s}-\grave{\mathfrak{s}}\|.
\end{aligned}
$$

Hence, the operator $\beth : \mathcal{B} \to \mathcal{B}$ is a contraction mapping. Using the principle of Banach fixed point theorem, we deduce that $\mathbb{FIDE}$ (1.1)-(1.2) admit a unique solution in $\mathcal{B}$. □

4. Illustrative examples

In the first example, we check our results for three values of derivative order σ.

Example 4.1. We consider the following $\mathbb{FIDE}$,

$$
\begin{cases}
\left(L_{4/3}\left({}^{\mathrm{C}}\mathcal{D}^{\sigma}\mathfrak{s}(\omega)\right)\right)' + \displaystyle\int_0^{\omega} \frac{\varrho^4}{3} \cos\left(\mathfrak{s}(\varrho)\right) \frac{2\varrho^6 \sin\left(\varrho^2 \arctan(\mathfrak{s}(\varrho))^3\right)}{1+\varrho^2}\, \mathrm{d}\varrho \\
\quad + \displaystyle\int_{\mathrm{E}_\circ} \frac{2\varrho}{5} \sin\left(\mathfrak{s}(\varrho)\right) + \frac{5}{7} \frac{1}{\exp(1)+\varrho} \left(\exp(1) + \frac{\varrho^4 \mathfrak{s}(\varrho)}{2+\mathfrak{s}(\varrho)} \right) \mathrm{d}\varrho = 0, \\
{}^{\mathrm{C}}\mathcal{D}^{7/3}\mathfrak{s}(0) = \mathfrak{s}'(0) = \mathfrak{s}''(0) = 0, \quad \varphi(1) + \mathfrak{s}'(1) = \mathfrak{s}'\left(\frac{\sqrt{13}}{4}\right),
\end{cases}
\tag{4.1}
$$

for $\omega \in (0,1)$ and three fractional derivative order $\sigma \in \left\{\frac{9}{4}, \frac{7}{3}, \frac{5}{2}\right\}$. Clearly, $p = \frac{4}{3}$, $q = 4$, $\zeta = \frac{\sqrt{13}}{4} \in \mathrm{E}_\circ \setminus \{0,1\}$. We choose

$$
\begin{aligned}
\wp_1(\omega,\mathfrak{s}(\omega)) &= \frac{2\omega^6 \sin\left(\omega^2 \arctan \mathfrak{s}^3\right)}{1+\omega^2}, \\
\wp_2(\omega,\mathfrak{s}(\omega)) &= \frac{1}{\exp(1)+\omega} \left[\exp(1) + \frac{\omega^4 \mathfrak{s}}{2+\mathfrak{s}} \right],
\end{aligned}
$$

and

$$
\Bbbk_1(\omega,\varrho) = \frac{\omega^4}{3} \cos(\varrho), \qquad \Bbbk_2(\omega,\varrho) = \frac{2\omega}{5} \sin(\varrho) + \frac{5}{7},
$$

which it follows $\wp_i \in C\big(\mathrm{E}_\circ \times \mathbb{R}^{\geq 0}, \mathbb{R}^{\geq 0}\big)$. By taking $M = 1$, we get

$$L_{\wp_1} = \max_{\omega\in \mathrm{E}_\circ,\, \|\mathfrak{s}\|<M} |\wp_1(\omega, \mathfrak{s}(\omega))| = \max_{\omega\in \mathrm{E}_\circ,\, \|\mathfrak{s}\|<M} \left|\frac{2\omega^6 \sin\big(\omega^2 \arctan \mathfrak{s}^3\big)}{1+\omega^2}\right| = \sqrt{2},$$

$$L_{\wp_2} = \max_{\omega\in \mathrm{E}_\circ,\, \|\mathfrak{s}\|<M} |\wp_2(\omega, \mathfrak{s}(\omega))| = \max_{\omega\in \mathrm{E}_\circ,\, \|\mathfrak{s}\|<M} \left|\frac{1}{\exp(1)+\omega}\left[\exp(1) + \frac{\omega^4 \mathfrak{s}}{2+\mathfrak{s}}\right]\right| = 1,$$

and

$$\beta_{\mathbb{k}_1} = \max_{\omega\in \mathrm{E}_\circ} \int_0^\omega \mathbb{k}_1(\omega, \varrho)\, \mathrm{d}\varrho = \max_{\omega\in \mathrm{E}_\circ} \int_0^\omega \frac{\omega^4}{3} \cos(\varrho)\, \mathrm{d}\varrho \approx 0.2804,$$
$$\beta_{\mathbb{k}_2} = \max_{\omega\in \mathrm{E}_\circ} \int_0^\omega \mathbb{k}_2(\omega, \varrho)\, \mathrm{d}\varrho = \max_{\omega\in \mathrm{E}_\circ} \int_0^\omega \left[\frac{2\omega}{5} \sin(\varrho) + \frac{5}{7}\right] \mathrm{d}\varrho \approx 0.8981.$$

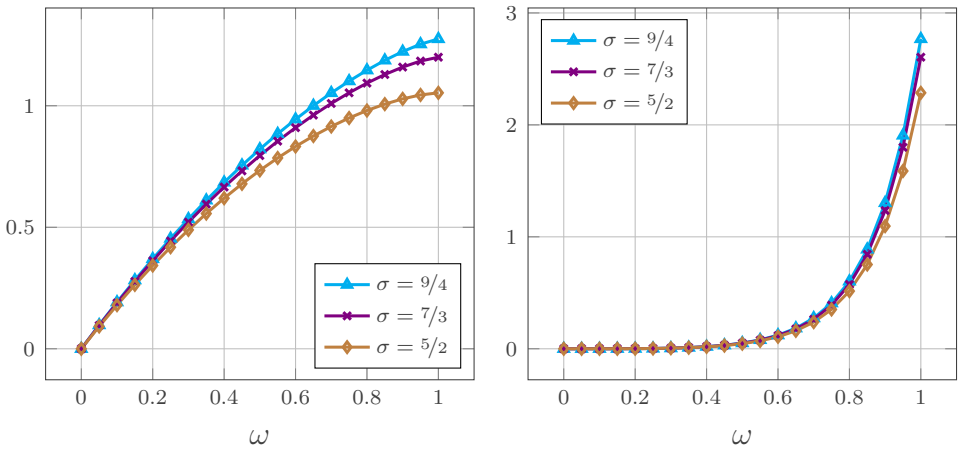
Further,

$$\mathcal{I}_\Phi := \int_{\mathrm{E}_\circ} \Phi(\varrho)\, \mathrm{d}\varrho = \int_{\mathrm{E}_\circ} \frac{(\sigma-\varrho)(1-\varrho)^{\sigma-2}}{\Gamma(\sigma)}\, \mathrm{d}\varrho \simeq \begin{cases} 1.2748, & \sigma = 9/4, \\ 1.1998, & \sigma = 7/3, \\ 1.0531, & \sigma = 5/2, \end{cases}$$

By taking $\nu = 2.8$,

$$\nu > \left\{ \begin{array}{ll} 2.7676, & \sigma = 9/4, \\ 2.6047, & \sigma = 7/3, \\ 2.2863, & \sigma = 5/2, \end{array} \right\} \simeq \big[\beta_{\mathbb{k}_1} L_{\wp_i} + \beta_{\mathbb{k}_2} L_{\wp_2}\big]^{q-1} \int_{\mathrm{E}_\circ} \Phi(\varrho)\, \mathrm{d}\varrho := \Delta.$$

Table 1 shows the numerical results for $\mathcal{I}_\Psi$ and the obtained results for Δ with three values of derivative order β_1 respectively. Further, Figs. 1a and 1b show the curves of $\mathcal{I}_\Psi$ and Δ for $\sigma = \frac{9}{4}, \frac{7}{3}, \frac{5}{2}$, respectively. By means of Theorem 3.8,



(A) $\mathcal{I}_\Psi$ (B) Δ

FIGURE 1. Representation of $\mathcal{I}_\Psi$ and Δ for FIDE (4.1) with three values of derivative order σ in Example 4.1.

FIDE (4.1) has at least one positive solution.

TABLE 1. The obtained results of $\mathcal{I}_\Psi$ and Δ for FIDE (4.1) with three values of derivative order σ in Example 4.1.

ω	$\sigma = 9/4$		$\sigma = 7/3$		$\sigma = 5/2$	
	$\mathcal{I}_\Psi$	Δ	$\mathcal{I}_\Psi$	Δ	$\mathcal{I}_\Psi$	Δ
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.05	0.0976	0.0000	0.0961	0.0000	0.0919	0.0000
0.10	0.1917	0.0001	0.1885	0.0001	0.1796	0.0001
0.15	0.2824	0.0004	0.2772	0.0003	0.2632	0.0003
0.20	0.3697	0.0011	0.3622	0.0011	0.3426	0.0010
0.25	0.4535	0.0027	0.4436	0.0027	0.4180	0.0025
0.30	0.5339	0.0057	0.5212	0.0056	0.4892	0.0053
0.35	0.6109	0.0109	0.5952	0.0106	0.5564	0.0099
0.40	0.6845	0.0191	0.6655	0.0185	0.6196	0.0172
0.45	0.7546	0.0317	0.7321	0.0308	0.6788	0.0285
0.50	0.8213	0.0509	0.7951	0.0493	0.7340	0.0455
0.55	0.8845	0.0795	0.8544	0.0767	0.7852	0.0705
0.60	0.9442	0.1216	0.9099	0.1172	0.8324	0.1072
0.65	1.0003	0.1835	0.9616	0.1764	0.8756	0.1606
0.70	1.0528	0.2744	1.0095	0.2631	0.9147	0.2384
0.75	1.1015	0.4075	1.0534	0.3897	0.9497	0.3514
0.80	1.1463	0.6025	1.0932	0.5745	0.9805	0.5153
0.85	1.1870	0.8875	1.1286	0.8438	1.0068	0.7528
0.90	1.2230	1.3031	1.1592	1.2350	1.0284	1.0957
0.95	1.2536	1.9058	1.1840	1.8001	1.0446	1.5881
1.00	1.2749	2.7677	1.1998	2.6048	1.0532	2.2863

The next example examines Theorem 3.9 for three different values of derivative order σ.

Example 4.2. We consider the following FIDE

$$\begin{cases} \left(L_3\left({}^{\mathrm{C}}\mathcal{D}^{\sigma}\mathfrak{s}\left(\omega\right)\right)\right)' + \displaystyle\int_0^{\omega}\left[\frac{\sin(\varrho)+\varrho}{3}+\frac{5}{27}\right]\frac{\varrho^4}{11+\varrho}\left(2+\sin\varrho\right)\mathrm{d}\varrho \\ \quad + \displaystyle\int_{\mathrm{E}_\circ}\frac{(\varrho+2)^3}{145}\cos\left(\varrho\right)\left[\frac{\sqrt{2}\cos\varrho+\sin\varrho}{\cos\varrho}\left(\frac{\mathfrak{s}(\varrho)+1}{\mathfrak{s}(\varrho)+2}\right)\right]\mathrm{d}\varrho = 0, \\ {}^{\mathrm{C}}\mathcal{D}^{7/3}\mathfrak{s}(0) = \mathfrak{s}'(0) = \mathfrak{s}''(0) = 0, \quad \mathfrak{s}(1)+\mathfrak{s}'(1) = \mathfrak{s}'\left(\frac{\sqrt{7}}{5}\right), \end{cases} \tag{4.2}$$

for $0 < \omega < 1$ and three fractional derivative order $\sigma \in \left\{\frac{8}{3}, \frac{14}{5}, \frac{32}{11}\right\}$. We get $\eta = \frac{\sqrt{7}}{5} \in (0,1)$, $p = 3$, and $q = \frac{3}{2}$. By choosing,

$$\begin{aligned} \wp_1\left(\omega, \mathfrak{s}(\omega)\right) &= \frac{\omega^4}{11+\omega}\left(2+\sin\mathfrak{s}(\omega)\right), \\ \wp_2\left(\omega, \mathfrak{s}(\omega)\right) &= \frac{\sqrt{2}\cos\omega+\sin\omega}{\cos\omega}\left(\frac{\mathfrak{s}(\omega)+1}{\mathfrak{s}(\omega)+2}\right), \end{aligned}$$

and

$$\Bbbk_1\left(\omega,\varrho\right) = \frac{\sin(\varrho)+\omega}{3}+\frac{5}{27}, \qquad \Bbbk_2\left(\omega,\varrho\right) = \frac{(\omega+2)^3}{145}\cos(\varrho),$$

we see that $\wp_i \in C\big(\mathrm{E}_\circ \times \mathbb{R}^{\geq 0}, \mathbb{R}^{\geq 0}\big)$, $\Upsilon_{\wp_1}(\omega) = \frac{\omega^4}{11+\omega}$, $\Upsilon_{\wp_2}(\omega) = \frac{\sqrt{2}\cos\omega+\sin\omega}{\cos\omega}$ such that $\Upsilon_{\wp_1} = \frac{1}{11}$, $\Upsilon_{\wp_2} = \sqrt{2}+\tan(1)$,

$$\beta_{\Bbbk_1} = \max_{\omega\in\mathrm{E}_\circ}\int_0^{\omega}\Bbbk_1\left(\omega,\varrho\right)\mathrm{d}\varrho = \max_{\omega\in\mathrm{E}_\circ}\int_0^{\omega}\left[\frac{\sin(\varrho)+\omega}{3}+\frac{5}{27}\right]\mathrm{d}\varrho \approx 0.6717,$$

$$\beta_{\mathbb{k}_2} = \max_{\omega\in \mathrm{E}_\circ} \int_0^{\omega} \mathbb{k}_2(\omega,\varrho)\,\mathrm{d}\varrho = \max_{\omega\in \mathrm{E}_\circ} \int_0^{\omega} \frac{(\omega+2)^3}{145}\cos(\varrho)\,\mathrm{d}\varrho \approx 0.1566,$$

and

$$L_{\wp_1} = \max_{\omega\in \mathrm{E}_\circ,\,\|\mathfrak{s}\|<M} |\wp_1(\omega,\mathfrak{s}(\omega))| = \max_{\omega\in \mathrm{E}_\circ,\,\|\mathfrak{s}\|<M} \left|\frac{\omega^4}{11+\omega}(2+\sin\mathfrak{s})\right| = \frac{3}{11},$$

$$L_{\wp_2} = \max_{\omega\in \mathrm{E}_\circ,\,\|\mathfrak{s}\|<M} |\wp_2(\omega,\mathfrak{s}(\omega))| = \max_{\omega\in \mathrm{E}_\circ,\,\|\mathfrak{s}\|<M} \left|\frac{\sqrt{2}\cos\omega+\sin\omega}{\cos\omega}\left(\frac{\mathfrak{s}+1}{\mathfrak{s}+2}\right)\right| = \sqrt{2}+\tan 1.$$

Thus, $M = \beta_{\mathbb{k}_1}\Upsilon_{\wp_1} + \beta_{\mathbb{k}_2}\Upsilon_{\wp_2} = 0.5215$ and we get

$$\Delta := (q-1)\,M^{q-2}\left(\beta_h L_f + \beta_k L_g\right)\int_0^1 \Phi(\varrho)\,\mathrm{d}s$$
$$\simeq \left\{\begin{array}{ll} 0.4105, & \sigma = 8/3, \\ 0.3636, & \sigma = 14/5, \\ 0.3276, & \sigma = 32/11, \end{array}\right\} < 1.$$

Table 2 shows the numerical results of Δ when derivative order $\sigma = \frac{8}{3}, \frac{14}{5}, \frac{32}{11}$ respectively. Further, we plot these data in Fig. 2. By Theorem 3.9, FIDE (4.2) admit one solution.

TABLE 2. The obtained results of Δ for FIDE (4.2) with three values of derivative order σ in Example 4.2.

ω	Δ		
	$\sigma = 8/3$	$\sigma = 14/5$	$\sigma = 32/11$
0.05	0.0077	0.0072	0.0068
0.10	0.0217	0.0203	0.0191
0.15	0.0398	0.0372	0.0349
0.20	0.0610	0.0568	0.0532
0.25	0.0846	0.0786	0.0734
0.30	0.1101	0.1019	0.0949
0.35	0.1370	0.1263	0.1174
0.40	0.1647	0.1514	0.1403
0.45	0.1928	0.1767	0.1633
0.50	0.2210	0.2018	0.1860
0.55	0.2487	0.2263	0.2081
0.60	0.2755	0.2499	0.2291
0.65	0.3010	0.2722	0.2489
0.70	0.3249	0.2927	0.2670
0.75	0.3466	0.3113	0.2832
0.80	0.3659	0.3275	0.2972
0.85	0.3824	0.3412	0.3089
0.90	0.3957	0.3520	0.3180
0.95	0.4053	0.3596	0.3243
1.00	0.4105	0.3636	0.3277

5. Conclusion

Basically, the existence, uniqueness and stability of solution for the nonlinear mixed-type FIDEs with boundary conditions are examined based on some

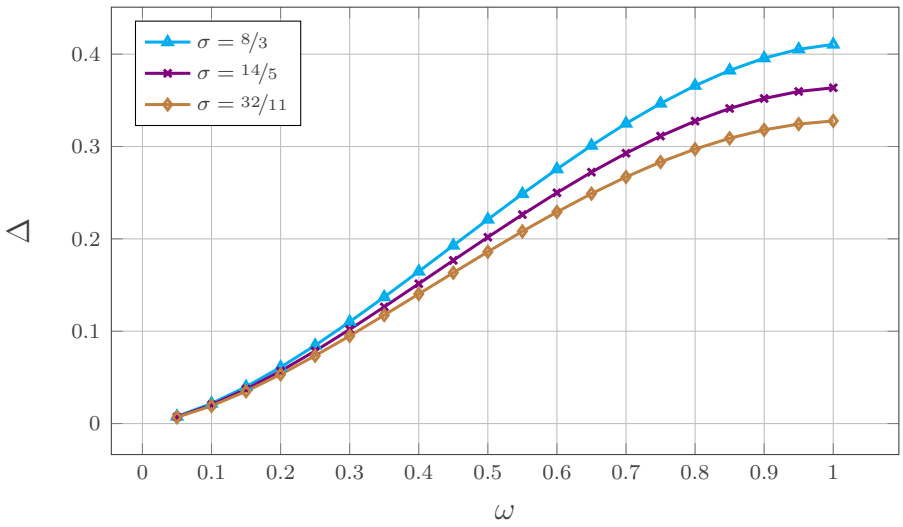


FIGURE 2. Representation of Δ for FIDE (4.2) with three values of derivative order σ in Example 4.2.

potential fixed point theorems such as Schaefers, Schauders, Banach contraction principle and the nonlinear alternative of Leray-Schauder type. Further, the Ulam-Hyers stability of the proposed fractional boundary value problem is studied based on Banach contraction principle. In fact, we have controlled the operators Katugampola and Caputo-Katugampola such that those can be considered as special cases of integral and derivative operators of the RL, Hadamard, Caputo and Caputo-Hadamard type. By branching out from the results obtained in this research, we can discuss the solutions to the FDE and FIE of special state [4, 11, 21].

Abbreviations

- FDE: Differential equation;
- FDE: Fractional differential equation;
- IDE: Integro-differential equation;
- FIDE: Fractional Integro-differential equation;

DECLARATIONS

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Availability of Data and Materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests
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Authors' contributions
CK: Methodology, validation, formal analysis, actualization, investigation, initial draft and was a major contributor in writing the manuscript. **AA**: Methodology, validation, formal analysis, actualization, investigation, review and edit. **MES**: Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft and was a major contributor in writing the manuscript. **FC**: Methodology, validation, formal analysis, actualization, investigation, review and edit. All authors read and approved the final manuscript.

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