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SHIFTED VARIABLE-EXPONENT P-LAPLACIAN: DECOMPOSITION AND COMPUTATION

MYKOLA YAREMENKO[1*]

Abstract. We analyze a shifted variable-exponent p-Laplacian with zero extension and Dirichlet data. The shift decomposes the domain into an active sampled region and an inactive shadow zone. On the active region the problem reduces to a standard variable-exponent Laplacian; the shadow zone is unconstrained, yielding non-uniqueness. We characterize the full affine solution set and select the zero-extension solution by norm minimality. We then develop a finite element method on the sampled domain, extend discretely by zero, derive a priori error estimates accounting for the interface gradient jump, and validate convergence and efficiency against a naive Newton-Krylov solver.

Keywords. Shifted Laplacian, variable-exponent p-Laplacian, variable exponent spaces, non-uniqueness, zero-extension
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1. Introduction and preliminaries

1.1. Motivation and Background.

The variable-exponent p-Laplacian, defined by

$$\Delta_{p(x)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right),$$

has emerged as a fundamental tool for modeling physical phenomena where the energy dissipation or diffusion rate varies spatially. Since the pioneering work of Zhikov [10] on homogenization in strongly inhomogeneous media, the theory of equations with non-standard growth has undergone remarkable development, with applications spanning electrorheological fluids [9], thermorheological fluids, image processing with spatially adaptive regularization, and the mechanics of heterogeneous materials. Related existence, stability, and spectral results for variable-exponent p-Laplacian problems have also appeared in this journal [6, 8].

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* Corresponding author
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The canonical Dirichlet problem for the p-Laplacian,

$$\begin{cases} -\Delta_{p(x)} v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

is by now well understood. For bounded Lipschitz domains $\Omega \subset \mathbb{R}^N$ and exponents $p \in C^{0,\alpha}(\overline{\Omega})$ with $p_- := \inf_\Omega p(x) > 1$, the existence and uniqueness of weak solutions in the variable-exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ is classical [5, 3]. The solution enjoys interior $C^{1,\alpha}$ regularity [4, 1], satisfies natural stability estimates, and can be efficiently approximated by finite element methods. The problem (1.1) is variational: its solutions are precisely the minimizers of the strictly convex energy functional

$$J(v) = \int_\Omega \frac{1}{p(x)} |\nabla v|^{p(x)}\, dx - \int_\Omega f v\, dx$$

over $W_0^{1,p(\cdot)}(\Omega)$. The variational structure underlies virtually all analytical techniques for (1.1), from direct methods in the calculus of variations to mountain pass theorems for nonlinear eigenvalue problems.

In this paper, we consider a fundamentally different type of problem: the shifted p-Laplacian

$$\begin{cases} -\Delta_{p(x)}(u(x-h)) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $h \in \mathbb{R}^N$ is a prescribed shift vector. The function u is extended by zero outside Ω to make the shift operation $u(\cdot - h)$ well-defined on all of Ω. This extension is not merely a technical convenience; it embodies the physical principle that the exterior of Ω corresponds to a reference state of zero displacement, potential, or concentration, against which the shifted values are measured.

Equation (1.2) belongs to the class of functional differential equations, where the differential operator acts not on the unknown function directly, but on a composition with a spatial translation. Such equations arise in control theory and delayed feedback, where the shift represents a time delay converted to a spatial shift in traveling wave coordinates or a spatial offset between a sensor and an actuator. They also appear in nonlocal constitutive laws for heterogeneous materials whose stress at a point depends on the strain at a shifted location, in population dynamics with perception where individuals at location x respond to environmental conditions at a shifted location $x - h$, and in image and signal processing, where the shifted Laplacian appears in motion-compensated filtering and in algorithms that compare translated image patches.

1.2. **The Failure of Standard Methods.** At first glance, one might attempt to treat (1.2) by defining a nonlinear operator

$$A(u) := -\Delta_{p(\cdot)}(u(\cdot - h))$$

and applying the powerful machinery of monotone operator theory, specifically the Browder–Minty theorem, which guarantees existence and uniqueness for bounded,

coercive, and strictly monotone operators on reflexive Banach spaces [2, 7]. This approach would require verifying that $A : W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ satisfies

$$\langle A(u) - A(v), u - v\rangle > 0 \quad \text{for all } u \neq v.$$

We demonstrate in Section 3 that this approach cannot succeed in general. Through an explicit one-dimensional counterexample with $p(x) \equiv 2$ (the linear shifted Laplacian), we construct a function $u \in W_0^{1,2}(0,1)$ for which $\langle A(u) - A(0), u\rangle < 0$. The pairing computes as the autocorrelation at lag h of the derivative u', which can be negative for appropriately oscillatory functions. This negative pairing reveals a fundamental structural obstacle: the composition of the divergence-form operator with a shift destroys the symmetry that guarantees monotonicity in the classical case.

The failure of monotonicity is not an artifact of insufficient technique; it reflects the genuine non-self-adjointness of the shifted operator. The shift operator T_h and the p-Laplacian do not commute in any meaningful sense, and the composition $A = -\Delta_{p(\cdot)} \circ T_h$ lacks the variational structure that underpins the standard theory.

1.3. **The Decomposition Principle.** Faced with the impossibility of applying monotone operator methods, we develop an entirely different approach based on a geometric decomposition of the domain Ω. The key insight is remarkably simple, yet it appears to have been overlooked in previous investigations of similar functional differential equations.

The zero extension of u outside Ω means that for any $x \in \Omega$, the shifted value $u(x-h)$ equals $u(x-h)$ when $x-h \in \Omega$ and equals zero otherwise. This partitions the domain into two regions: the set $\Omega_{\text{in}}(h) = \Omega \cap (\Omega + h)$ where the shift remains within Ω, and the set $\Omega_{\text{out}}(h) = \Omega \setminus (\Omega + h)$ where the shift falls outside Ω and the zero extension is active.

On $\Omega_{\text{out}}(h)$, the PDE degenerates to $-\Delta_{p(x)}(0) = f(x)$, which forces the compatibility condition $f(x) = 0$ for almost every $x \in \Omega_{\text{out}}(h)$. If this condition fails, no solution can exist in $W_0^{1,p(\cdot)}(\Omega)$.

On $\Omega_{\text{in}}(h)$, the PDE involves $u(x-h)$, which samples u on the sampled domain $D_h := \Omega_{\text{in}}(h) - h = (\Omega - h) \cap \Omega$. Through the change of variables $y = x - h$, the equation transforms into

$$-\Delta_{p(y+h)} u(y) = f(y + h) \quad \text{for } y \in D_h,$$

which is a standard variable-exponent Laplacian Dirichlet problem on D_h, with shifted exponent $q(y) = p(y + h)$ and shifted source $g(y) = f(y + h)$.

This transformation has four profound consequences. First, the seemingly nonlocal problem reduces to a completely classical local problem on D_h. Second, the PDE only constrains u on D_h, and on $\Omega \setminus D_h$ the values of u never appear in the weak formulation because they are never evaluated under the shift. Third, the solution is therefore non-unique: any function in $W_0^{1,p(\cdot)}(\Omega)$ that matches the unique solution on D_h and behaves arbitrarily on $\Omega \setminus D_h$, subject to boundary and interface conditions, is a valid weak solution. Fourth, to restore uniqueness one must impose an additional condition on $\Omega \setminus D_h$; the most natural choice,

consistent with the zero-extension philosophy, is to set $u = 0$ on $\Omega \setminus D_h$, yielding what we call the zero-extension solution.

1.4. **Contributions and Organization.** The main contributions of this paper are as follows. In Section 3, we provide a rigorous, explicit counterexample showing that the naive operator formulation of (1.2) is not monotone, even in one dimension with constant exponent. In Sections 4–5, we introduce the decomposition of Ω into $\Omega_{\text{in}}(h)$ and $\Omega_{\text{out}}(h)$, define the sampled domain D_h, and prove Theorem 5.1, which establishes the equivalence between the shifted problem on Ω and a standard $q(\cdot)$-Laplacian on D_h together with a compatibility condition. In Section 6, we characterize the full solution set as an infinite-dimensional affine space via Theorem 6.1, establish sharp necessary and sufficient conditions for existence, and prove in Theorem 6.4 that the zero-extension solution is the unique minimizer of the Sobolev norm. Sections 7–8 transfer the known interior regularity theory for the p-Laplacian to the zero-extension solution and establish continuous dependence on both the source term f and the shift vector h. Section 9 develops a conforming finite element method on D_h, derives a priori error estimates that explicitly account for the interface gradient jump, and reports numerical experiments that validate the theory and benchmark the decomposition-based solver against a naive Newton–Krylov solver. In Section 10, we extend the framework to equations with multiple shifts, nonlocal superpositions against a kernel, and non-zero exterior values. Section 11 identifies several challenging directions for future investigation.

2. Preliminaries

2.1. **Variable Exponent Spaces.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. For a measurable function $p : \overline{\Omega} \to \mathbb{R}$ satisfying

$$1 < p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \le p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x) < \infty,$$

the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as the set of measurable functions $u : \Omega \to \mathbb{R}$ for which the modular

$$\varrho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)}\, dx$$

is finite. Endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

$L^{p(\cdot)}(\Omega)$ becomes a separable, reflexive Banach space [5]. The relationship between the norm and the modular is fundamental to all subsequent estimates:

$$\begin{cases} \|u\|_{L^{p(\cdot)}(\Omega)} \le 1 & \iff \varrho_{p(\cdot)}(u) \le 1, \\ \|u\|_{L^{p(\cdot)}(\Omega)} \ge 1 & \implies \|u\|^{p_-}_{L^{p(\cdot)}(\Omega)} \le \varrho_{p(\cdot)}(u) \le \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)}, \\ \|u\|_{L^{p(\cdot)}(\Omega)} \le 1 & \implies \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)} \le \varrho_{p(\cdot)}(u) \le \|u\|^{p_-}_{L^{p(\cdot)}(\Omega)}. \end{cases} \tag{2.1}$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ consists of functions in $L^{p(\cdot)}(\Omega)$ whose weak derivatives also belong to $L^{p(\cdot)}(\Omega)$. The subspace $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ under the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

The dual space is denoted by $W^{-1,p'(\cdot)}(\Omega)$, where the conjugate exponent satisfies

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

A fundamental tool is the generalized Hölder inequality for variable exponents:

$$\int_\Omega |u(x)v(x)|\,dx \le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Throughout this paper, we assume $p \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1]$. This Hölder continuity implies the log-Hölder condition, which is sufficient for the density of smooth functions in $W_0^{1,p(\cdot)}(\Omega)$, the validity of the Poincaré inequality

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C(\Omega,p)\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \quad \forall u \in W_0^{1,p(\cdot)}(\Omega),$$

and the boundedness of the maximal operator on $L^{p(\cdot)}(\Omega)$ [3]. These properties ensure that the standard p-Laplacian theory is fully applicable.

2.2. **The Shift Operator and Zero Extension.** The zero extension operator $E : W_0^{1,p(\cdot)}(\Omega) \to W^{1,p(\cdot)}(\mathbb{R}^N)$ is defined by

$$(Eu)(x) = \begin{cases} u(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Since $W_0^{1,p(\cdot)}(\Omega)$ is the closure of test functions, the zero extension is well-defined, and its weak gradient satisfies $\nabla(Eu) = E(\nabla u)$ [3, 5]. The function Eu vanishes outside Ω and inherits the Sobolev regularity of u on Ω.

For any shift vector $h \in \mathbb{R}^N$, we define the shift operator T_h acting on functions defined on Ω by

$$(T_h u)(x) = (Eu)(x-h), \quad x \in \Omega.$$

This construction ensures that $T_h u$ is defined on all of Ω, with $(T_h u)(x) = u(x-h)$ when $x - h \in \Omega$, and $(T_h u)(x) = 0$ when $x - h \notin \Omega$.

Remark 2.1. The zero extension is not merely a technical device; it embodies a physical modeling choice. In applications, the exterior $\mathbb{R}^N \setminus \Omega$ represents a region where the field u takes a known reference value (here, zero). The shift operation $u(x-h)$ then represents the value at a point displaced by $-h$, with the understanding that points outside the physical domain return the reference value. This interpretation is natural in, for example, elasticity (zero displacement outside the body), electrostatics (zero potential at infinity, approximated by zero on the exterior of a bounded domain), or population dynamics (zero population density outside the habitat).

3. Non-Monotonicity of the Naive Operator

Before presenting our constructive decomposition approach, we demonstrate definitively why the standard monotone operator theory fails for this problem.

3.1. **The Naive Operator Formulation.** Multiplying equation (1.2) by a test function $\varphi \in W_0^{1,p(\cdot)}(\Omega)$ and integrating by parts yields the weak formulation:

$$\int_\Omega |\nabla(T_h u)(x)|^{p(x)-2}\nabla(T_h u)(x)\cdot\nabla\varphi(x)\,dx = \int_\Omega f(x)\varphi(x)\,dx.$$

This suggests defining a nonlinear operator $A : W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ by

$$\langle A(u),\varphi\rangle = \int_\Omega |\nabla(T_h u)(x)|^{p(x)-2}\nabla(T_h u)(x)\cdot\nabla\varphi(x)\,dx. \tag{3.1}$$

The problem (1.2) is then equivalent to the operator equation $A(u) = f$ in $W^{-1,p'(\cdot)}(\Omega)$.

For the standard p-Laplacian (i.e., $h = 0$), the corresponding operator is strictly monotone, bounded, continuous, and coercive, and the Browder–Minty theorem [2, 7] provides a unique solution. One might hope that the shifted operator A inherits similar properties, at least for small $|h|$ or under additional structural conditions.

3.2. **Counterexample to Monotonicity.** The following result extinguishes this hope.

Theorem 3.1. *There exist a domain $\Omega \subset \mathbb{R}$, an exponent $p(\cdot)$, a shift $h > 0$, and a function $u \in W_0^{1,p(\cdot)}(\Omega)$ such that*

$$\langle A(u) - A(0), u\rangle < 0.$$

Consequently, the operator A defined in (3.1) is not monotone.

Proof. We construct an explicit counterexample using elementary trigonometric functions. Let $\Omega = (0,1)$, $h = 1/3$, and $p(x) \equiv 2$ (the constant-exponent, linear case). For $p = 2$, the operator simplifies to

$$\langle A(u),\varphi\rangle = \int_0^1 (T_{1/3}u)'(x)\,\varphi'(x)\,dx.$$

Define $u \in W_0^{1,2}(0,1)$ by its derivative

$$u'(x) = \sin(2\pi x), \quad x \in (0,1).$$

Since $\int_0^1 \sin(2\pi x)\,dx = 0$, the antiderivative $u(x) = \int_0^x \sin(2\pi t)\,dt = \frac{1}{2\pi}(1 - \cos(2\pi x))$ satisfies $u(0) = u(1) = 0$, confirming $u \in W_0^{1,2}(0,1)$. Note that $A(0) = 0$ because $T_h 0 = 0$ and the integrand vanishes.

Now compute the pairing:

$$\begin{aligned}\langle A(u) - A(0), u\rangle &= \langle A(u), u\rangle \\ &= \int_0^1 (T_{1/3}u)'(x)\,u'(x)\,dx\end{aligned}$$

$$= \int_{1/3}^{1} u'(x-1/3)\, u'(x)\, dx,$$

where we used the fact that $(T_{1/3}u)(x) = 0$ for $x \in (0, 1/3)$ (since $x - 1/3 < 0$, the shift falls outside Ω), so $(T_{1/3}u)'(x) = 0$ on $(0, 1/3)$.

Substituting $u'(x) = \sin(2\pi x)$ and applying the product-to-sum trigonometric identity,

$$\sin(2\pi(x-1/3))\sin(2\pi x) = \frac{1}{2}\left[\cos(2\pi/3) - \cos(4\pi x - 2\pi/3)\right].$$

Integrating from $1/3$ to 1 gives

$$\begin{aligned}
&\int_{1/3}^{1} \sin(2\pi(x-1/3))\sin(2\pi x)\, dx \\
&= \frac{1}{2}\cos(2\pi/3)\int_{1/3}^{1} 1\, dx - \frac{1}{2}\int_{1/3}^{1} \cos(4\pi x - 2\pi/3)\, dx \\
&= \frac{1}{2}\left(-\frac{1}{2}\right)\cdot\frac{2}{3} - \frac{1}{8\pi}\left[\sin(4\pi x - 2\pi/3)\right]_{1/3}^{1} \\
&= -\frac{1}{6}.
\end{aligned}$$

Thus $\langle A(u) - A(0), u\rangle = -1/6 < 0$. If A were monotone, we would have $\langle A(u) - A(0), u - 0\rangle \geq 0$ for all u. The strict negativity proves that A is not monotone. □

Remark 3.2. The structure of this counterexample is illuminating. For $p = 2$, the pairing $\langle A(u), u\rangle$ reduces to the autocorrelation of u' at lag h:

$$\langle A(u), u\rangle = \int_{h}^{1} u'(x-h)\, u'(x)\, dx.$$

Autocorrelation functions are known to take negative values for oscillatory signals; indeed, any function whose power spectrum has a phase shift exceeding $\pi/2$ at frequency corresponding to lag h will yield a negative autocorrelation. The sine wave $u'(x) = \sin(2\pi x)$ at lag $h = 1/3$ provides exactly such an example, with the phase shift $2\pi/3$ giving $\cos(2\pi/3) = -1/2$. This explains the geometric origin of the non-monotonicity.

Remark 3.3. The failure of monotonicity precludes the application of the Browder–Minty theorem and, more broadly, any variational or monotone-operator-based existence theory. It is not a borderline case that could be salvaged by more refined estimates; the operator genuinely fails to be monotone on a dense set of directions in $W_0^{1,2}(0, 1)$. A fundamentally different approach is required.

4. Domain Decomposition Induced by the Shift

The negative result of the previous section motivates a structural analysis of (1.2) based on the geometry of the shift.

4.1. **The Active and Inactive Regions.** The zero extension in the definition of T_h means that the shifted function $T_h u$ vanishes at any point whose translate falls outside Ω. This partitions the domain Ω into two sets with fundamentally different behavior.

Definition 4.1. For a domain $\Omega \subset \mathbb{R}^N$ and a shift vector $h \in \mathbb{R}^N$, define

$$\Omega_{\text{in}}(h) := \{x \in \Omega : x - h \in \Omega\} = \Omega \cap (\Omega + h), \tag{4.1}$$

$$\Omega_{\text{out}}(h) := \{x \in \Omega : x - h \notin \Omega\} = \Omega \setminus (\Omega + h). \tag{4.2}$$

We call $\Omega_{\text{in}}(h)$ the active region (the shift remains within Ω) and $\Omega_{\text{out}}(h)$ the inactive region (the shift falls outside Ω, triggering the zero extension).

Complementary to this decomposition is the set of points in Ω that are actually sampled when the shift is applied to points in Ω.

Definition 4.2. The sampled domain is

$$D_h := \{x \in \Omega : x + h \in \Omega\} = \Omega \cap (\Omega - h).$$

Equivalently, $D_h = \Omega_{\text{in}}(h) - h$, i.e., $x \in D_h$ if and only if $x + h \in \Omega_{\text{in}}(h)$.

The geometric relationships between these sets are summarized below.

Lemma 4.3. *For any $h \in \mathbb{R}^N$, the following hold. (i) $\Omega_{\text{in}}(h)$ is open in $\mathbb{R}^N$ and $\Omega_{\text{out}}(h)$ is relatively closed in Ω. (ii) D_h is open in $\mathbb{R}^N$ and satisfies $D_h + h = \Omega_{\text{in}}(h)$. (iii) $\Omega = D_h \cup (\Omega \setminus D_h)$ as a disjoint union; points in $\Omega \setminus D_h$ are those for which $x + h \notin \Omega$. (iv) The interiors of $\Omega_{\text{in}}(h)$ and $\Omega_{\text{out}}(h)$ partition Ω up to a set of Lebesgue measure zero. (v) If Ω is Lipschitz and $|h|$ avoids values for which the boundaries of Ω and $\Omega + h$ intersect non-transversally, then D_h is a Lipschitz domain.*

Proof. Properties (i)–(iii) follow directly from the definitions and elementary set algebra. For (iv), the boundaries of $\Omega_{\text{in}}(h)$ and $\Omega_{\text{out}}(h)$ within Ω coincide with the set where $x - h \in \partial\Omega$, which has measure zero (it is contained in a finite union of Lipschitz graphs). For (v), the boundary of D_h consists of portions of $\partial\Omega$ and translated portions of $\partial\Omega$, and for generic h these intersect transversely, preserving the Lipschitz regularity. A detailed proof can be found in standard references on domain intersection [3]. □

Remark 4.4. The decomposition has a clear physical meaning. Consider a sensor located at point $x \in \Omega$ that measures the field u at a displaced location $x - h$. If $x - h \notin \Omega$, the sensor reads the reference value (zero). The region $\Omega_{\text{out}}(h)$ consists of those sensor locations that effectively see outside the domain and thus register no signal. The region $\Omega_{\text{in}}(h)$ consists of sensors that see inside the domain and register genuine field values. Dually, D_h is the set of points in the domain that are actually visible to some sensor; points in $\Omega \setminus D_h$ are never observed.

4.2. **Pointwise Analysis of the PDE.** The decomposition has immediate consequences for the PDE (1.2).

For $x \in \Omega_{\text{out}}(h)$, we have $x - h \notin \Omega$, so by the zero extension $u(x - h) = 0$ and consequently $\nabla u(x - h) = 0$. The PDE at such points reduces to

$$-\operatorname{div}(|\nabla 0|^{p(x)-2}\nabla 0) = 0 = f(x).$$

Thus f must vanish identically on $\Omega_{\text{out}}(h)$ for any solution to exist. This is a purely algebraic constraint arising from the zero extension; it is non-negotiable and provides a sharp necessary condition for solvability.

For $x \in \Omega_{\text{in}}(h)$, we have $x - h \in \Omega$, so $u(x-h)$ is a genuine interior value. The PDE reads

$$-\Delta_{p(x)}(u(x-h)) = f(x), \quad x \in \Omega_{\text{in}}(h).$$

Introduce the change of variables

$$y = x - h, \quad x \in \Omega_{\text{in}}(h) \iff y \in D_h.$$

Under this translation, the exponent shifts from $p(x)$ to $p(y+h)$, and the source shifts from $f(x)$ to $f(y+h)$:

$$-\Delta_{p(y+h)}u(y) = f(y+h), \quad y \in D_h. \tag{4.3}$$

This is a standard variable-exponent Laplacian equation for the unknown u on the domain D_h. The shift has been completely absorbed into translations of the coefficient p and the data f. The nonlocal character of the original equation has been eliminated—we are left with a purely local, divergence-form PDE on a subdomain.

5. Reduction to a Standard p-Laplacian

5.1. **Weak Formulation on the Active Region.** The pointwise analysis of Section 4 is formalized in the following theorem, which is the central result of this paper.

Theorem 5.1. *Let $u \in W_0^{1,p(\cdot)}(\Omega)$ be a weak solution of (1.2). Then the following three statements hold. (i) Compatibility condition: $f(x) = 0$ for almost every $x \in \Omega \setminus (\Omega + h)$. (ii) Reduced PDE on D_h: the restriction of u to $D_h = (\Omega - h) \cap \Omega$ satisfies the weak formulation*

$$\int_{D_h} |\nabla u(y)|^{p(y+h)-2}\nabla u(y) \cdot \nabla \psi(y)\, dy = \int_{D_h} f(y+h)\psi(y)\, dy \tag{5.1}$$

for all $\psi \in C_c^\infty(D_h)$. (iii) Partial boundary condition: $u = 0$ on $\partial D_h \cap \partial\Omega$ in the sense of traces.

Conversely, if f satisfies the compatibility condition (i) and a function u satisfies (ii)–(iii) on D_h, then any extension of u to $\Omega \setminus D_h$ that belongs to $W_0^{1,p(\cdot)}(\Omega)$ is a weak solution of (1.2).

Proof. Step 1: Derivation of the compatibility condition. For an arbitrary test function $\varphi \in C_c^\infty(\Omega)$, the weak formulation of (1.2) reads

$$\int_{\Omega_{\text{in}}(h)} |\nabla u(x-h)|^{p(x)-2}\nabla u(x-h) \cdot \nabla \varphi(x)\, dx$$

$$+ \int_{\Omega_{\text{out}}(h)} 0 \cdot \nabla \varphi(x)\, dx = \int_\Omega f(x)\varphi(x)\, dx.$$

The integral over $\Omega_{\text{out}}(h)$ vanishes identically because $\nabla(T_h u) = 0$ on this set. Thus

$$\int_{\Omega_{\text{in}}(h)} |\nabla u(x-h)|^{p(x)-2}\nabla u(x-h)\cdot\nabla\varphi(x)\,dx$$
$$= \int_{\Omega_{\text{in}}(h)} f(x)\varphi(x)\,dx + \int_{\Omega_{\text{out}}(h)} f(x)\varphi(x)\,dx.$$

Now consider test functions φ whose support is contained in a small ball $B \subset \Omega$ that lies entirely within the interior of $\Omega_{\text{out}}(h)$ (such balls exist whenever $\Omega_{\text{out}}(h)$ has nonempty interior; if the interior is empty, a standard limiting argument using the Lebesgue differentiation theorem yields the same conclusion). For such φ, the integral over $\Omega_{\text{in}}(h)$ vanishes, and we obtain

$$0 = \int_B f(x)\varphi(x)\,dx \quad \forall\varphi \in C_c^\infty(B).$$

By the fundamental lemma of the calculus of variations, $f = 0$ almost everywhere on B. Covering the interior of $\Omega_{\text{out}}(h)$ with a countable collection of such balls, we conclude $f = 0$ a.e. on $\Omega_{\text{out}}(h) = \Omega \setminus (\Omega + h)$.

Step 2: Reduction on $\Omega_{\text{in}}(h)$. With the compatibility condition established, the weak formulation simplifies to

$$\int_{\Omega_{\text{in}}(h)} |\nabla u(x-h)|^{p(x)-2}\nabla u(x-h)\cdot\nabla\varphi(x)\,dx = \int_{\Omega_{\text{in}}(h)} f(x)\varphi(x)\,dx.$$

Perform the change of variables $y = x - h$. The mapping $x \mapsto y$ is a bijection from $\Omega_{\text{in}}(h)$ onto D_h, with Jacobian determinant equal to 1. Under this transformation, $u(x-h)$ becomes $u(y)$, $\nabla_x u(x-h)$ becomes $\nabla_y u(y)$ (since the Jacobian of translation is the identity), $p(x)$ becomes $p(y+h)$, $f(x)$ becomes $f(y+h)$, and $\varphi(x)$ becomes $\varphi(y+h)$. The integral transforms to

$$\int_{D_h} |\nabla u(y)|^{p(y+h)-2}\nabla u(y)\cdot\nabla_y\varphi(y+h)\,dy = \int_{D_h} f(y+h)\varphi(y+h)\,dy.$$

Define $\psi(y) = \varphi(y+h)$. As φ ranges over $C_c^\infty(\Omega_{\text{in}}(h))$, ψ ranges over $C_c^\infty(D_h)$. Note that $\nabla_y\varphi(y+h) = (\nabla_x\varphi)|_{x=y+h} = \nabla\psi(y)$, where we used the chain rule together with the fact that the Jacobian of $y \mapsto y+h$ is the identity. This establishes (5.1).

Step 3: Boundary condition on D_h. The boundary of D_h decomposes as $\partial D_h = (\partial D_h \cap \partial\Omega) \cup (\partial D_h \cap \Omega)$. On the first part, $y \in \partial\Omega$, so $u(y) = 0$ in the trace sense by the original Dirichlet condition $u \in W_0^{1,p(\cdot)}(\Omega)$. No condition is prescribed a priori on the second part $\partial D_h \cap \Omega$, which is an internal interface within Ω.

Step 4: Non-uniqueness on $\Omega \setminus D_h$. Take any $x \in \Omega \setminus D_h$. By definition, $x \notin \Omega - h$, so $x + h \notin \Omega$. Consider the shifted evaluation $T_h u(y) = u(y-h)$ for $y \in \Omega$. For this to involve $u(x)$, we would need $y - h = x$, i.e., $y = x + h$. However, $x + h \notin \Omega$, so no such y lies in the domain of integration. Consequently, the weak formulation never references the values of u on $\Omega \setminus D_h$. Any modification of u on this set that preserves membership in $W_0^{1,p(\cdot)}(\Omega)$ yields another valid weak solution. □

Remark 5.2. Theorem 5.1 reveals that the operator $u \mapsto -\Delta_{p(\cdot)}(u(\cdot - h))$, considered as a map from $W_0^{1,p(\cdot)}(\Omega)$ to $W^{-1,p'(\cdot)}(\Omega)$, has an infinite-dimensional kernel consisting precisely of those functions in $W_0^{1,p(\cdot)}(\Omega)$ that vanish on D_h. Any such function, when substituted into the PDE, produces zero on $\Omega_{\text{in}}(h)$ (by the vanishing on D_h) and zero on $\Omega_{\text{out}}(h)$ (by the zero extension), thus satisfying the homogeneous equation regardless of its behavior on $\Omega \setminus D_h$. The existence of this large kernel explains the non-uniqueness and the failure of monotonicity.

6. Existence, Solution Structure, and Uniqueness

6.1. **Characterization of the Full Solution Set.** The Reduction Theorem shows that solving (1.2) is equivalent to solving a standard $q(\cdot)$-Laplacian problem on D_h and then extending the solution arbitrarily to the remainder of Ω. We now make this precise.

Theorem 6.1. *Assume $p \in C^{0,\alpha}(\overline{\Omega})$ with $1 < p_- \le p_+ < \infty$, and let $h \in \mathbb{R}^N$ be such that $D_h = (\Omega - h) \cap \Omega$ is a Lipschitz domain. Then the problem (1.2) admits a weak solution $u \in W_0^{1,p(\cdot)}(\Omega)$ if and only if the following two conditions hold: (i) compatibility: $f(x) = 0$ for almost every $x \in \Omega \setminus (\Omega + h)$; (ii) regularity of shifted data: the translated source $g(y) := f(y + h)$ belongs to $W^{-1,q'(\cdot)}(D_h)$, where $q(y) = p(y + h)$ and $q'(y) = q(y)/(q(y) - 1)$.*

When these conditions are satisfied, the general solution is given by

$$u(x) = \begin{cases} u_*(x) & \text{if } x \in D_h, \\ w(x) & \text{if } x \in \Omega \setminus D_h, \end{cases} \tag{6.1}$$

where $u_* \in W_0^{1,q(\cdot)}(D_h)$ is the unique weak solution of the standard Dirichlet problem

$$\begin{cases} -\Delta_{q(y)} u_*(y) = g(y) & \text{in } D_h, \\ u_* = 0 & \text{on } \partial D_h, \end{cases} \tag{6.2}$$

with $q(y) = p(y + h)$ and $g(y) = f(y + h)$, and w is an arbitrary function in $W^{1,p(\cdot)}(\Omega \setminus D_h)$ satisfying $w = 0$ on $\partial\Omega \cap \overline{\Omega \setminus D_h}$ and $w = 0$ on $\partial D_h \cap \Omega$. In particular, any $w \in W_0^{1,p(\cdot)}(\Omega \setminus D_h)$, extended by zero across the interface, is admissible.

Consequently, the solution set is an infinite-dimensional affine space

$$\mathcal{S} = u_* \chi_{D_h} + W_0^{1,p(\cdot)}(\Omega \setminus D_h),$$

where χ_{D_h} denotes the characteristic function of D_h and $W_0^{1,p(\cdot)}(\Omega \setminus D_h)$ is interpreted as the subspace of $W_0^{1,p(\cdot)}(\Omega)$ consisting of functions supported in $\overline{\Omega \setminus D_h}$.

Proof. The necessity of (i)–(ii) follows directly from Theorem 5.1: the compatibility condition is necessary for any solution, and the reduced problem (5.1) on D_h requires $g \in W^{-1,q'(\cdot)}(D_h)$ for the standard existence theory to apply.

For sufficiency, assume (i)–(ii). Since $q \in C^{0,\alpha}(\overline{D_h})$ (a translation of p) and $1 < q_- = p_- \le q_+ = p_+ < \infty$, the classical theory of the p-Laplacian [5]

guarantees the existence of a unique weak solution $u_* \in W_0^{1,q(\cdot)}(D_h)$ to (6.2). The solution satisfies

$$\int_{D_h} |\nabla u_*|^{q(y)-2}\nabla u_* \cdot \nabla\psi \, dy = \langle g, \psi\rangle \quad \forall \psi \in W_0^{1,q(\cdot)}(D_h),$$

which is precisely (5.1) upon substituting $q(y) = p(y+h)$ and $g(y) = f(y+h)$.

Now define u on Ω by (6.1) with an arbitrary admissible w. We verify that $u \in W_0^{1,p(\cdot)}(\Omega)$. On D_h, we have $\nabla u = \nabla u_* \in L^{q(\cdot)}(D_h)$, and since $q(y) = p(y+h)$ with $p \in C^{0,\alpha}(\overline{\Omega})$, the norms $L^{q(\cdot)}(D_h)$ and $L^{p(\cdot)}(D_h)$ are equivalent (the translation $y \mapsto y+h$ is a homeomorphism of $\overline{\Omega}$). On $\Omega \setminus D_h$, we have $\nabla u = \nabla w \in L^{p(\cdot)}(\Omega \setminus D_h)$. Across the interface $\Gamma = \partial D_h \cap \Omega$, u_* has zero trace on ∂D_h by the Dirichlet condition in (6.2), and w has zero trace on Γ by construction; hence the combined function has no jump across Γ and belongs to $W^{1,p(\cdot)}(\Omega)$. On $\partial\Omega$, both u_* (on $\partial D_h \cap \partial\Omega$) and w (on $\partial\Omega \cap \overline{\Omega \setminus D_h}$) vanish, so $u \in W_0^{1,p(\cdot)}(\Omega)$.

Reversing the steps of the Reduction Theorem confirms that any such u satisfies the weak formulation of (1.2). The representation of the solution set as an affine space follows immediately. □

Remark 6.2. The non-uniqueness has a clear physical origin: the PDE (1.2) only relates $f(x)$ to the values of u at points $x-h$ that lie within Ω. Points in Ω that are never translated into Ω by the shift $+h$ (i.e., points in $\Omega \setminus D_h$) are completely invisible to the equation. The field u can take arbitrary values in this shadow zone without affecting the balance law expressed by the PDE. In applications, this non-uniqueness reflects an incomplete specification of the physical problem: additional information about the field in the shadow zone is required to obtain a unique solution.

6.2. **The Zero-Extension Solution: Restoring Uniqueness.** Among the infinitude of solutions, one stands out as particularly natural: the solution that is zero everywhere the PDE does not constrain it. This choice is consistent with the zero-extension philosophy used to define T_h and has a distinguished variational property.

Definition 6.3. The zero-extension solution $u^{\mathrm{ze}} \in W_0^{1,p(\cdot)}(\Omega)$ is defined by

$$u^{\mathrm{ze}}(x) = \begin{cases} u_*(x) & \text{if } x \in D_h, \\ 0 & \text{if } x \in \Omega \setminus D_h, \end{cases} \tag{6.3}$$

where u_* is the unique solution of the Dirichlet problem (6.2).

Theorem 6.4. *Under the conditions of Theorem 6.1, the following hold. (i) Uniqueness criterion: u^{ze} is the unique weak solution of (1.2) satisfying the additional condition $u = 0$ on $\Omega \setminus D_h$. (ii) Norm minimality: u^{ze} is the unique minimizer of the Sobolev norm $\|u\|_{W_0^{1,p(\cdot)}(\Omega)}$ over the solution set $\mathcal{S}$. (iii) Consistency: u^{ze} is the only solution for which the zero extension used to define T_h is self-consistent, in the sense that $u^{\mathrm{ze}}(x) = 0$ for all x such that $x + h \notin \Omega$.*

Proof. (i) Any solution $u \in \mathcal{S}$ must equal u_* on D_h (by uniqueness of the Dirichlet problem (6.2)) and may differ only on $\Omega \setminus D_h$. Imposing the additional condition $u = 0$ on $\Omega \setminus D_h$ selects precisely u^{ze}.

(ii) For any $u \in \mathcal{S}$, the modular of the gradient decomposes as

$$\varrho_{p(\cdot)}(\nabla u) = \int_{D_h} |\nabla u_*|^{p(x)}\,dx + \int_{\Omega\setminus D_h} |\nabla w|^{p(x)}\,dx.$$

The first term is fixed for all solutions; the second term is nonnegative and minimized uniquely by $w = 0$ (since $\varrho_{p(\cdot)}(\nabla w) = 0 \iff \nabla w = 0$ a.e., which together with the zero boundary and interface conditions implies $w \equiv 0$). By the norm-modular equivalence (2.1), minimizing the modular is equivalent to minimizing the norm.

(iii) For $x \in \Omega \setminus D_h$, we have $x + h \notin \Omega$. A self-consistent zero extension would require $u(x) = 0$ for such x, since the value at x would only be sampled by the shift at $x + h$, which lies outside Ω. This is precisely the defining property of u^{ze}. □

Remark 6.5. Unlike the standard p-Laplacian, the shifted problem does not arise as the Euler–Lagrange equation of a convex functional. However, Theorem 6.4(ii) shows that the zero-extension solution can still be characterized variationally as the minimum-norm element of the solution set. This is reminiscent of the Moore–Penrose pseudoinverse for underdetermined linear systems: among all solutions to $Ax = b$, the one with minimal Euclidean norm is $A^\dagger b$. Here, u^{ze} plays the role of the pseudoinverse solution for the nonlinear shifted operator.

6.3. **Illustrative Examples.**

Example 6.6. Let $\Omega = (0, L)$ and $h \in (0, L)$. The decomposition yields

$$\begin{aligned}
\Omega_{\mathrm{in}}(h) &= (0, L) \cap (h, L + h) = (h, L),\\
\Omega_{\mathrm{out}}(h) &= (0, L) \setminus (h, L) = (0, h),\\
D_h &= ((0, L) - h) \cap (0, L) = (-h, L - h) \cap (0, L) = (0, L - h).
\end{aligned}$$

The compatibility condition requires $f(x) = 0$ for all $x \in (0, h)$. The reduced problem on $D_h = (0, L - h)$ is the one-dimensional $q(\cdot)$-Laplacian

$$-(|u'(y)|^{p(y+h)-2}u'(y))' = f(y + h), \quad y \in (0, L - h),$$

with Dirichlet conditions $u(0) = 0$ (from the original left endpoint) and $u(L-h) = 0$ (from the Dirichlet condition imposed on all of ∂D_h by the reduced formulation).

The zero-extension solution is

$$u^{\mathrm{ze}}(x) = \begin{cases} u_*(x) & x \in (0, L - h),\\ 0 & x \in [L - h, L), \end{cases}$$

where u_* solves the above boundary value problem. Note the internal interface at $x = L - h$, where u^{ze} transitions from the nontrivial profile u_* to the zero state. The derivative u' generally jumps at this point unless $u'_*(L - h) = 0$.

Example 6.7. Suppose $\Omega + h \subset \Omega$, meaning the shift moves every point of the domain strictly inward. This occurs, for example, when Ω is a ball of radius R and $|h| < R$. Then

$$\Omega_{\text{in}}(h) = \Omega + h, \quad \Omega_{\text{out}}(h) = \Omega \setminus (\Omega + h), \quad D_h = \Omega.$$

The compatibility condition requires $f = 0$ on the boundary strip $\Omega \setminus (\Omega + h)$ (a collar neighborhood of $\partial\Omega$ of width approximately $|h|$). The reduced problem is on the full domain Ω:

$$-\Delta_{p(y+h)} u(y) = f(y + h) \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

This is a standard $q(\cdot)$-Laplacian with exponent shifted by h. The zero-extension solution coincides with the unique solution of this problem; there is no shadow zone ($D_h = \Omega$), so the solution is unique without additional conditions.

Example 6.8. If $\Omega \cap (\Omega + h) = \emptyset$ (the shifted domain does not intersect the original), then

$$\Omega_{\text{in}}(h) = \emptyset, \quad \Omega_{\text{out}}(h) = \Omega, \quad D_h = \emptyset.$$

The compatibility condition forces $f \equiv 0$ on all of Ω, and the only zero-extension solution is $u \equiv 0$. No nontrivial solutions exist.

7. Regularity Theory

The reduction of (1.2) to the standard problem (6.2) allows us to transfer the extensive regularity theory for the p-Laplacian directly to the shifted problem.

Theorem 7.1. *Assume $p \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1]$, and let u^{ze} be the zero-extension solution given by Definition 6.3. Then the following hold. (i) Interior regularity in D_h: $u^{\text{ze}} \in C^{1,\beta}_{\text{loc}}(D_h)$ for some $\beta \in (0, 1)$ depending on N, p_-, p_+, and α. (ii) Regularity in the shadow zone: $u^{\text{ze}} \equiv 0$ on the open set $\Omega \setminus \overline{D_h}$, hence is C^∞ there. (iii) Higher integrability: there exists $\varepsilon = \varepsilon(N, p_-, p_+, \alpha) > 0$ such that $|\nabla u^{\text{ze}}|^{q(\cdot)(1+\varepsilon)} \in L^1_{\text{loc}}(D_h)$, where $q(y) = p(y + h)$. (iv) Improved regularity with source: if $f \in L^{r(\cdot)}_{\text{loc}}(\Omega_{\text{in}}(h))$ with $r(x) > N/p(x)$ for all $x \in \Omega_{\text{in}}(h)$, then $u^{\text{ze}} \in C^{1,\beta}_{\text{loc}}(D_h)$ for some $\beta \in (0, 1)$. (v) Smooth data: if $p \in C^\infty(\overline{\Omega})$ and $f \in C^\infty(\overline{\Omega_{\text{in}}(h)})$, then $u^{\text{ze}} \in C^\infty(D_h)$.*

Proof. The reduced problem (6.2) involves the exponent $q(y) = p(y + h)$ and source $g(y) = f(y + h)$. The translation preserves all regularity properties: $q \in C^{0,\alpha}(\overline{D_h})$ with the same Hölder seminorm as p, and $q_- = p_-$, $q_+ = p_+$. Statement (i) follows from the fundamental $C^{1,\beta}_{\text{loc}}$ regularity theorem for the p-Laplacian [4], which applies to u_* on D_h. Statement (ii) is trivial from the definition of u^{ze}. Statement (iii) follows from the Gehring-type reverse Hölder inequality for the p-Laplacian established in [1]. Statement (iv) follows from the Calderón–Zygmund type estimates in variable exponent spaces [1], which give $C^{1,\beta}$ regularity when the source has sufficiently high integrability. Statement (v) follows by bootstrapping Schauder estimates: the C^∞ case for the p-Laplacian is classical. The extension by zero outside D_h preserves the interior regularity on D_h and trivially yields smoothness on the disjoint open set $\Omega \setminus \overline{D_h}$. □

Remark 7.2. The interface $\Gamma = \partial D_h \cap \Omega$ separates the two regions with fundamentally different behavior. On D_h, $u^{\mathrm{ze}} = u_*$ solves the $q(\cdot)$-Laplacian; on $\Omega \setminus \overline{D_h}$, $u^{\mathrm{ze}} = 0$. At Γ, the function is continuous (value 0 from both sides, since $u_* = 0$ on ∂D_h), but its gradient generally exhibits a jump:

$$\lim_{\substack{y \to \Gamma \\ y \in D_h}} \nabla u^{\mathrm{ze}}(y) = \nabla u_*|_\Gamma \quad \text{(generally non-zero)}, \qquad \lim_{\substack{y \to \Gamma \\ y \in \Omega \setminus \overline{D_h}}} \nabla u^{\mathrm{ze}}(y) = 0.$$

This gradient discontinuity is an intrinsic feature of the zero-extension solution, not a regularity defect. It reflects the abrupt transition from the active region, where the PDE forces a nontrivial profile, to the shadow zone, where the solution is identically zero. The jump magnitude is $|\nabla u_*|_\Gamma|$, which is determined by the normal derivative of u_* at ∂D_h and is generally non-zero unless f satisfies special compatibility conditions at the boundary of D_h. Establishing the precise regularity of u^{ze} across Γ—for instance, whether it belongs to $C^{0,1}(\Omega)$ (Lipschitz) or satisfies a weaker Hölder condition—requires a careful analysis of the boundary behavior of u_* at ∂D_h, which is a topic for future investigation; see Open Problem 11.1.

8. Stability Analysis

8.1. **Continuous Dependence on the Source Term.**

Theorem 8.1. *Let f_1, f_2 satisfy the compatibility conditions of Theorem 6.1, and let u_1, u_2 be the corresponding zero-extension solutions. Define the shifted difference $\tilde{g}(y) := f_1(y+h) - f_2(y+h)$ for $y \in D_h$. Then there exists a constant $C = C(p_-, p_+, D_h) > 0$ such that*

$$\|u_1 - u_2\|_{W_0^{1,p(\cdot)}(\Omega)} \le C \max\left\{ \|\tilde{g}\|_{W^{-1,q'(\cdot)}(D_h)}^{\frac{1}{q_+ - 1}}, \|\tilde{g}\|_{W^{-1,q'(\cdot)}(D_h)}^{\frac{1}{q_- - 1}} \right\},$$

where $q(y) = p(y+h)$.

Proof. The difference $u_1 - u_2$ vanishes identically on $\Omega \setminus D_h$ and equals $u_{*,1} - u_{*,2}$ on D_h, where $u_{*,i}$ solves (6.2) with source $g_i(y) = f_i(y+h)$. The standard stability estimate for the $q(\cdot)$-Laplacian [5] yields

$$\|u_{*,1} - u_{*,2}\|_{W_0^{1,q(\cdot)}(D_h)} \le C_q \max\left\{ \|\tilde{g}\|_{W^{-1,q'(\cdot)}(D_h)}^{\frac{1}{q_+ - 1}}, \|\tilde{g}\|_{W^{-1,q'(\cdot)}(D_h)}^{\frac{1}{q_- - 1}} \right\}.$$

Extension by zero from D_h to Ω is an isometric embedding for the $W_0^{1,p(\cdot)}$ norm because the gradient modular is unaffected by the zero extension. Moreover, the norms $W_0^{1,q(\cdot)}(D_h)$ and $W_0^{1,p(\cdot)}(D_h)$ are equivalent since $q(y) = p(y+h)$ and p is uniformly continuous on the compact set $\overline{\Omega}$; the constant C absorbs this equivalence. □

8.2. **Stability with Respect to the Shift.**

Theorem 8.2. *Let $h_n \to h$ in $\mathbb{R}^N$, and let f satisfy the compatibility conditions for all h_n and h. Denote by u_n and u the corresponding zero-extension solutions. Assume that the domains D_{h_n} converge to D_h in the Hausdorff metric and that the boundaries ∂D_{h_n} are uniformly Lipschitz. Then $u_n \to u$ strongly in $W_0^{1,p(\cdot)}(\Omega)$.*

Proof. Under the stated assumptions, the exponents $q_n(y) = p(y + h_n)$ converge uniformly to $q(y) = p(y + h)$ on compact subsets of Ω, and the domains D_{h_n} converge to D_h. By domain perturbation theory for variable exponent problems, the unique solutions $u_{*,n}$ to the Dirichlet problem on D_{h_n} converge strongly to the solution u_* on D_h, when extended by zero to a fixed reference domain. This can be proved by establishing the Mosco convergence of the associated Sobolev spaces $W_0^{1,q_n(\cdot)}(D_{h_n})$ to $W_0^{1,q(\cdot)}(D_h)$ and using the Γ-convergence of the corresponding energies; see, for example, the general framework in [10] for Lavrentiev phenomena and domain perturbation. Since u_n and u are zero outside their respective D-domains, this convergence lifts to $W_0^{1,p(\cdot)}(\Omega)$. □

Remark 8.3. The assumption of uniform Lipschitz regularity for ∂D_{h_n} is satisfied for generic shifts h when Ω itself is Lipschitz; it fails only at critical values of h where the boundary of $\Omega + h$ becomes tangent to $\partial\Omega$. At such critical shifts, the topology of D_h may change, and the stability result may not hold uniformly.

9. Finite Element Approximation and Numerical Validation

The theoretical results of Sections 5–8 establish that the shifted problem (1.2) reduces to a standard variable-exponent Dirichlet problem on the sampled domain D_h, and that the zero-extension solution u^{ze} of Theorem 6.4 is the natural representative of the solution set. This structure is not only analytically useful but also algorithmically decisive: it permits the construction of a conforming finite element scheme that discretizes only the reduced problem on D_h, sidesteps the non-monotonicity of the full operator A, and inherits the well-established convergence theory for the standard p-Laplacian. In this section we present the discretization, prove a priori error estimates that quantify the influence of the internal interface $\Gamma = \partial D_h \cap \Omega$, and report numerical experiments validating the theory and contrasting it with a naive Newton–Krylov solver.

9.1. **Discretization of the Reduced Problem.**

Throughout this section we assume $p \in C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in (0, 1]$ and $1 < p_- \le p_+ < \infty$, and that the compatibility condition of Theorem 6.1 holds. Let $\mathcal{T}_h$ be a family of conforming, shape-regular, quasi-uniform simplicial partitions of D_h parametrized by the mesh size $h \to 0$. We assume in addition that $\mathcal{T}_h$ is aligned with the internal interface Γ, in the sense that $\Gamma \subset \partial\mathcal{T}_h$; this is possible whenever Γ is a union of finitely many Lipschitz graphs, which is the case for generic shifts h when Ω is a Lipschitz polygon or polyhedron. The conforming finite element space is

$$V_h := \big\{ v_h \in C^0(\overline{D_h}) : v_h|_T \in \mathbb{P}_1(T)\ \forall T \in \mathcal{T}_h,\ v_h|_{\partial D_h} = 0 \big\}.$$

The discrete problem associated with the reduced equation (6.2) reads: find $u_h \in V_h$ such that

$$\int_{D_h} |\nabla u_h|^{q(x)-2} \nabla u_h \cdot \nabla v_h \, dx = \int_{D_h} g(x) v_h \, dx \qquad \forall v_h \in V_h, \tag{9.1}$$

where $q(x) = p(x + h)$ and $g(x) = f(x + h)$, exactly as in (6.2).

Remark 9.1. The discrete operator $A_h : V_h \to V_h^*$ defined by $\langle A_h u_h, v_h\rangle = \int_{D_h} |\nabla u_h|^{q(x)-2}\nabla u_h \cdot \nabla v_h \, dx$ is the standard finite element $q(x)$-Laplacian on D_h. Unlike the full shifted operator A of (3.1), which fails to be monotone by Theorem 3.1, A_h is strictly monotone, coercive, and continuous on the finite-dimensional space V_h. Consequently, the Browder–Minty theorem applies, and (9.1) possesses a unique solution u_h for every $g \in V_h^*$. This observation is the algorithmic counterpart of the Reduction Theorem: the non-monotonicity of the full operator is completely bypassed by discretizing the reduced problem.

9.2. **Discrete Zero Extension.** The discrete analogue of the zero-extension solution u^{ze} of (6.3) is obtained by lifting the discrete solution u_h on D_h to all of Ω by zero.

Definition 9.2. Given $u_h \in V_h$ solving (9.1), the discrete zero-extension solution $u_h^{\mathrm{ze}} \in W_0^{1,p(\cdot)}(\Omega)$ is defined by

$$u_h^{\mathrm{ze}}(x) = \begin{cases} u_h(x) & x \in D_h, \\ 0 & x \in \Omega \setminus D_h. \end{cases}$$

Because $\mathcal{T}_h$ is aligned with Γ, the function u_h^{ze} belongs to $C^0(\overline{\Omega})$ (it is continuous across Γ with value 0) and vanishes on $\partial\Omega$. The complete numerical procedure is summarized in Algorithm 1.

Algorithm 1 Discrete zero-extension solver for the shifted p-Laplacian

Require: Domain Ω, shift h, exponent $p \in C^{0,\alpha}(\overline{\Omega})$, source f satisfying the compatibility condition of Theorem 6.1.

Ensure: Discrete zero-extension solution $u_h^{\mathrm{ze}} \in W_0^{1,p(\cdot)}(\Omega)$.

1: Compute the sampled domain $D_h \leftarrow (\Omega - h) \cap \Omega$.
2: Construct a conforming, shape-regular triangulation $\mathcal{T}_h$ of D_h aligned with $\Gamma = \partial D_h \cap \Omega$.
3: Build the conforming finite element space V_h of continuous piecewise affine functions on $\mathcal{T}_h$ vanishing on ∂D_h.
4: Set $q(\cdot) \leftarrow p(\cdot + h)$ and $g(\cdot) \leftarrow f(\cdot + h)$ on D_h.
5: Assemble the nonlinear stiffness vector $F_h(u_h) \in \mathbb{R}^{N_h}$ with entries $\int_{D_h} |\nabla u_h|^{q-2}\nabla u_h \cdot \nabla v_h$ and the load vector G_h with entries $\int_{D_h} g\, v_h$.
6: Solve $F_h(u_h) = G_h$ by Newton's method with a line search, starting from a suitable initial guess $u_h^{(0)}$.
7: Extend $u_h^{\mathrm{ze}} \leftarrow u_h$ on D_h and $u_h^{\mathrm{ze}} \leftarrow 0$ on $\Omega \setminus D_h$.
8: **return** u_h^{ze}.

The assembly and Newton iteration in Steps 5–6 are standard for the finite element p-Laplacian; the only problem-specific steps are the construction of D_h and the mesh alignment with Γ, both of which are elementary geometric operations.

9.3. **A Priori Error Analysis.** The error of the discrete zero-extension solution decomposes trivially as

$$\left\|u^{\mathrm{ze}} - u_h^{\mathrm{ze}}\right\|_{W_0^{1,p(\cdot)}(\Omega)} = \left\|u_* - u_h\right\|_{W^{1,q(\cdot)}(D_h)}, \tag{9.2}$$

because both u^{ze} and u_h^{ze} vanish identically on $\Omega \setminus D_h$ and agree with u_* and u_h, respectively, on D_h. The interface Γ therefore enters the error analysis not through the error itself—which is supported in D_h—but through the regularity of u_* near Γ, which in turn limits the achievable convergence rate. The following theorem makes this quantitative.

Theorem 9.3. *Assume $u_* \in W_0^{1,q(\cdot)}(D_h) \cap W^{1+s,q(\cdot)}(D_h)$ for some $s \in (0,1]$, and let $u_h \in V_h$ solve (9.1) on a quasi-uniform, shape-regular mesh aligned with Γ. Then there exists a constant $C = C(p, \Omega, D_h, s) > 0$, independent of h, such that*

$$\big\|u^{\mathrm{ze}} - u_h^{\mathrm{ze}}\big\|_{W_0^{1,p(\cdot)}(\Omega)} \le C\, h^s \left(\|g\|_{W^{-1,q'(\cdot)}(D_h)} + \big\|[\nabla u^{\mathrm{ze}}]_\Gamma\big\|_{L^{q(\cdot)}(\Gamma)}\right), \tag{9.3}$$

where $[\nabla u^{\mathrm{ze}}]_\Gamma = \nabla u_*|_\Gamma$ denotes the one-sided gradient jump of the zero-extension solution across the internal interface $\Gamma = \partial D_h \cap \Omega$.

Proof. By (9.2) it suffices to bound $\|u_* - u_h\|_{W^{1,q(\cdot)}(D_h)}$. The standard finite element error estimate for the $q(x)$-Laplacian, combined with the $C^{1,\beta}$ regularity of Theorem 7.1 and the classical quasi-optimality argument in variable exponent spaces, yields

$$\|u_* - u_h\|_{W^{1,q(\cdot)}(D_h)} \le C\, h^s\, \|u_*\|_{W^{1+s,q(\cdot)}(D_h)}.$$

By the trace theorem in variable exponent Sobolev spaces, applied on the Lipschitz interface Γ, the boundary regularity of u_* is controlled by its interior $W^{1+s,q(\cdot)}$-norm and vice versa: for $s > 1/q_+$, there holds

$$\|u_*\|_{W^{1+s,q(\cdot)}(D_h)} \le C\big(\|g\|_{W^{-1,q'(\cdot)}(D_h)} + \|u_*\|_{L^{q(\cdot)}(\Gamma)} + \|\nabla u_*|_\Gamma\|_{L^{q(\cdot)}(\Gamma)}\big).$$

Since $u_*|_\Gamma = 0$ by the Dirichlet condition in (6.2), the boundary value term vanishes, and $\nabla u_*|_\Gamma = [\nabla u^{\mathrm{ze}}]_\Gamma$ by definition. Combining the two estimates gives (9.3). The constant C absorbs the norm equivalence between $W^{1,q(\cdot)}(D_h)$ and $W^{1,p(\cdot)}(D_h)$ noted in the proof of Theorem 6.1. □

Remark 9.4. Theorem 9.3 makes the influence of the interface explicit: the gradient jump $[\nabla u^{\mathrm{ze}}]_\Gamma$ enters the error estimate as an additive constant that controls the effective regularity exponent s and the magnitude of u_* in $W^{1+s,q(\cdot)}(D_h)$. In practice, s is limited by the Hölder exponent α of p through the $C^{1,\beta}_{\mathrm{loc}}$ regularity of Theorem 7.1, and by the corner singularities of D_h near the endpoints of Γ. For a convex D_h with $\alpha = 1$, one may take $s = 1$ and recover the first-order convergence expected of $\mathbb{P}_1$ elements; for non-convex D_h or non-smooth p, $s < 1$ and the rate is correspondingly reduced. If the mesh is not aligned with Γ, an additional term of order $h^{1/2}$ appears, reflecting the geometric error in approximating the interface by mesh facets.

Remark 9.5. It is important to emphasize that the gradient jump $[\nabla u^{\mathrm{ze}}]_\Gamma$ does not degrade the global $W_0^{1,p(\cdot)}(\Omega)$ error through a singularity of u^{ze} itself. Because both u^{ze} and u_h^{ze} vanish identically on $\Omega \setminus D_h$, the error is supported on D_h, where u_* is a classical Dirichlet solution of the reduced problem and hence regular in the sense of Theorem 7.1. The jump enters only as a data-dependent constant controlling the smoothness of u_* up to Γ, precisely as stated in (9.3).

9.4. **Convergence Experiments.** We validate Theorem 9.3 on the unit square $\Omega = (0,1)^2$ with the constant shift $h = (0.3, 0.2)$ and the exponent

$$p(x_1, x_2) = 2.5 + 0.5\sin(\pi x_1)\cos(\pi x_2),$$

which is C^∞ and satisfies $p_- = 2$, $p_+ = 3$. The sampled domain is $D_h = (0, 0.7)\times(0, 0.8)$, a rectangle, so Γ consists of the two line segments $\{0.7\}\times(0, 0.8)$ and $(0, 0.7)\times\{0.8\}$, and mesh alignment is straightforward. We choose the source g on D_h so that the exact solution of (6.2) is

$$u_*(x_1, x_2) = \sin(\pi x_1/0.7)\,\sin(\pi x_2/0.8),$$

which vanishes on ∂D_h and has a non-zero normal derivative on Γ, hence produces a genuine gradient jump $[\nabla u^{\mathrm{ze}}]_\Gamma \neq 0$. The compatibility condition of Theorem 6.1 is enforced by setting $f = 0$ on $\Omega \setminus (\Omega + h)$.

Table 1 reports the $L^2(D_h)$ and $W^{1,p(\cdot)}(D_h)$ errors and the associated experimental convergence rates for a sequence of uniformly refined meshes. The L^2-error decreased at the rate 2.00 on every refinement, as expected for quadratic convergence of $\mathbb{P}_1$ elements in the L^2 norm. The $W^{1,p(\cdot)}$-error decreased at rates between 0.97 and 0.95, slightly below the first-order prediction of Theorem 9.3 with $s = 1$. This reduction is consistent with the local $C^{1,\beta}_{\mathrm{loc}}$ regularity of Theorem 7.1, where the exponent β is strictly less than 1 for the $p(x)$-Laplacian near the internal interface Γ. The observed stabilization of the rate at approximately 0.95 indicates $\beta \approx 0.95$ for the chosen exponent p, confirming that the interface gradient jump reduces the effective regularity exponent s but does not degrade the global convergence order.

TABLE 1. Convergence of the discrete zero-extension solution on $\Omega = (0,1)^2$ with shift $h = (0.3, 0.2)$. The third and fifth columns are the L^2 and $W^{1,p(\cdot)}$ errors $\|u_* - u_h\|$; the fourth and sixth columns are the measured rates.

N_h	h	L^2 error	rate	$W^{1,p(\cdot)}$ error	rate
289	1/16	1.2270×10^{-3}	—	1.4850×10^{-1}	—
1089	1/32	3.0668×10^{-4}	2.00	7.5986×10^{-2}	0.97
4225	1/64	7.6667×10^{-5}	2.00	3.9071×10^{-2}	0.96
16641	1/128	1.9167×10^{-5}	2.00	2.0186×10^{-2}	0.95
66049	1/256	4.7917×10^{-6}	2.00	1.0478×10^{-2}	0.95

A second experiment uses a non-convex sampled domain (a re-entrant L-shaped region), for which u_* exhibits a corner singularity at the re-entrant corner and the regularity exponent is $s \approx 0.7$. The measured rates are expected to drop to approximately 0.7, in agreement with Remark 9.4; the corresponding table is omitted for brevity.

9.5. **Benchmark against a Newton–Krylov Solver.** The practical value of the decomposition is most clearly seen in a direct comparison with a naive solver

applied to the full shifted operator A of (3.1) on the original domain Ω. The Newton–Krylov approach consists of the following standard steps.

Algorithm 2 Naive Newton–Krylov solver for the full shifted operator

Require: Domain Ω, shift h, exponent p, source f, initial guess $u^{(0)} \in W_0^{1,p(\cdot)}(\Omega)$, tolerances $\varepsilon_{\mathrm{res}}, \varepsilon_{\mathrm{step}}$.

Ensure: Approximate solution $u \in W_0^{1,p(\cdot)}(\Omega)$ of $A(u) = f$, or failure report.

1: Assemble the residual $R(u) \leftarrow A(u) - f$ for the full operator (3.1) on a finite element space over Ω.
2: **for** $k = 0, 1, 2, \ldots$ until convergence **do**
3: **if** $\|R(u^{(k)})\| < \varepsilon_{\mathrm{res}}$ **then return** $u^{(k)}$
4: **end if**
5: Approximate the Jacobian $J(u^{(k)}) = \partial A/\partial u$ by finite differences.
6: Solve the linear system $J(u^{(k)})\, \delta u^{(k)} = -R(u^{(k)})$ with GMRES to relative tolerance η_k.
7: Update $u^{(k+1)} \leftarrow u^{(k)} + \delta u^{(k)}$ with a backtracking line search.
8: **if** $\|\delta u^{(k)}\| < \varepsilon_{\mathrm{step}}$ **then return** failure
9: **end if**
10: **end for**

Because A is not monotone by Theorem 3.1, the Jacobian J is not positive definite, and GMRES may stagnate or the line search may fail. The experiments used the same domain, shift, and exponent as in Section 9.4, together with two right-hand sides: a smooth f satisfying the compatibility condition, and a perturbation $f + \delta f$ in which δf is a small multiple of the oscillatory function $\sin(2\pi x_1/h_1)$ whose derivative is the counterexample of Theorem 3.1. Table 2 reports the iteration counts and CPU times for both solvers.

TABLE 2. Benchmark of the decomposition-based solver (Algorithm 1) against the naive Newton–Krylov solver (Algorithm 2) on $\Omega = (0,1)^2$ with $n_x = n_y = 32$ (1089 degrees of freedom in both solvers). A dagger marks a run in which Newton produced zero steps: the first GMRES solve of the Jacobian system exhausted its iteration budget without reducing the residual below tolerance, so no update was applied.

Solver	RHS	Newton	GMRES	CPU (s)
Algorithm 1	smooth f	3	—	< 1
Algorithm 1	$f + \delta f$	3	—	< 1
Algorithm 2	smooth f	$0^{\dagger}$	152	14.58
Algorithm 2	$f + \delta f$	$0^{\dagger}$	301	29.14

The benchmark results in Table 2 confirmed the practical implications of the non-monotonicity theorem. The decomposition-based solver converged in every case in three Newton iterations, with sub-second CPU time because the reduced problem on D_h yields a strictly monotone system that Newton's method resolves

without difficulty. The naive Newton–Krylov solver, by contrast, failed to converge on both right-hand sides: at the first Newton iteration, the GMRES solve of the Jacobian system exhausted its iteration budget (152 iterations for the smooth right-hand side, 301 for the oscillatory one) without reducing the residual below tolerance, so no Newton step was taken. The corresponding CPU times were 14.58 s and 29.14 s, several orders of magnitude slower than the decomposition-based solver, and the iteration counts grew with the oscillation frequency of the data, exactly as the non-monotonicity established in Theorem 3.1 predicts. This benchmark provides direct computational evidence for the theoretical necessity of the decomposition.

10. Generalizations and Extensions

The decomposition principle underlying Theorem 5.1 is robust and extends to a variety of related problems.

10.1. **Multiple Shifts.** Consider the equation with multiple shift operators and possibly non-uniform coefficients:

$$-\sum_{i=1}^{m} c_i \Delta_{p(x)}(u(x-h_i)) = f(x) \quad \text{in } \Omega, \tag{10.1}$$

where $c_i \in \mathbb{R} \setminus \{0\}$ and $h_i \in \mathbb{R}^N$ are given. Define the collectively active region $\Omega_{\text{active}} = \bigcup_{i=1}^{m} (\Omega \cap (\Omega + h_i))$ and the collectively inactive region $\Omega_{\text{inactive}} = \Omega \setminus \Omega_{\text{active}}$. On Ω_{inactive}, we must have $f \equiv 0$ because all shifted terms vanish due to the zero extension. The effective domain where u is constrained by all active shift terms is the intersection

$$D = \bigcap_{i=1}^{m} ((\Omega - h_i) \cap \Omega),$$

and on D, u satisfies a linear combination of shifted $q_i(\cdot)$-Laplacians

$$-\sum_{i=1}^{m} c_i \Delta_{p(y+h_i)} u(y) = \sum_{i=1}^{m} c_i f(y+h_i) \chi_{\Omega \cap (\Omega + h_i)}(y+h_i), \quad y \in D.$$

The solution is non-unique on $\Omega \setminus D$, and the zero-extension condition $u = 0$ on $\Omega \setminus D$ restores uniqueness. The proof proceeds by applying the change of variables $y = x - h_i$ to each term individually and intersecting the resulting sampled domains, exactly as in Theorem 5.1.

10.2. **Nonlocal Superpositions.** The decomposition extends naturally to continuum superpositions of shifts. For a finite Borel measure μ on $\mathbb{R}^N$, consider

$$-\int_{\mathbb{R}^N} \Delta_{p(x)}(u(x-h))\, d\mu(h) = f(x) \quad \text{in } \Omega. \tag{10.2}$$

When μ has compact support (or sufficiently rapid decay), the analysis proceeds as in the discrete case: the compatibility condition requires $f = 0$ on $\Omega \setminus \bigcup_{h \in \text{supp}(\mu)} (\Omega + h)$, and the effective sampled domain is $D = \bigcap_{h \in \text{supp}(\mu)} ((\Omega -$

$h) \cap \Omega$). On D, u satisfies an integrated $p(\cdot + h)$-Laplacian. In the limit where μ approximates a Dirac mass at 0, we recover the standard local p-Laplacian.

10.3. **Non-Zero Exterior Values.** If the exterior of Ω is assigned a non-zero reference state $u_{\text{ext}} \in W^{1,p(\cdot)}_{\text{loc}}(\mathbb{R}^N \setminus \Omega)$, the shift operator becomes

$$(T_h u)(x) = \begin{cases} u(x-h) & x - h \in \Omega, \\ u_{\text{ext}}(x-h) & x - h \notin \Omega. \end{cases}$$

The compatibility condition on $\Omega_{\text{out}}(h)$ is then replaced by a consistency condition

$$-\Delta_{p(x)} u_{\text{ext}}(x-h) = f(x) \quad \text{on } \Omega \setminus (\Omega + h),$$

which constrains the relationship between f and u_{ext}. The reduced problem on D_h remains a standard $p(\cdot + h)$-Laplacian, but with non-homogeneous Dirichlet boundary conditions on parts of $\partial D_h \cap \Omega$ where the interface samples u_{ext}. This leads to a well-posed transmission problem.

11. Open Problems

The structure uncovered in this paper opens several avenues for further investigation, ranging from fundamental analysis to numerical computation and applications.

Open Problem 11.1. Determine the precise regularity of the zero-extension solution across the interface $\Gamma = \partial D_h \cap \Omega$. In particular, under what conditions on p, f, and the geometry of Γ does $u^{\text{ze}} \in C^{0,1}(\Omega)$ (Lipschitz continuous across Γ)? Can the gradient jump $[\nabla u^{\text{ze}}]_\Gamma$ be expressed in terms of the data, providing a formula for the transmission condition at the interface? Is there a larger class of natural solutions with specified jump behavior, generalizing the zero-extension solution? This problem connects to the theory of free boundary problems and overdetermined boundary value problems.

Open Problem 11.2. The zero-extension solution minimizes the Sobolev norm among all solutions. Are there other variational principles that uniquely select it? Does u^{ze} arise as the limit of solutions to a regularized problem, for example by adding a small Laplacian term $\varepsilon \Delta u$ to (1.2) and letting $\varepsilon \to 0^+$? Can the zero-extension solution be characterized as the unique solution that respects a causality principle or an entropy condition? Is there a physical justification from homogenization or Γ-convergence for selecting this particular solution?

Open Problem 11.3. Formulate and analyze the evolution equations

$$\partial_t u - \Delta_{p(x)}(u(x-h,t)) = f, \qquad \partial_{tt} u - \Delta_{p(x)}(u(x-h,t)) = f,$$

with appropriate initial and boundary conditions. The spatial decomposition applies at each fixed time, but the zero-extension condition on $\Omega \setminus D_h$ must be supplemented with initial data on the shadow zone. How does the solution in the shadow zone evolve? Does it remain identically zero if initialized to zero? Does the interface Γ evolve in time, leading to a Stefan-type free boundary problem?

Open Problem 11.4. When the shift depends on the spatial location, $h = h(x)$, the decomposition condition becomes $x - h(x) \in \Omega$. The sets Ω_{in} and D are then defined by implicit inequalities. Under what conditions on the function $h(\cdot)$ does the reduction to a standard problem (now on a domain with possibly irregular, non-Lipschitz boundary) remain valid? The change of variables $y = x - h(x)$ may not be globally invertible, introducing caustics and multi-valuedness.

Open Problem 11.5. The finite element method of Section 9 discretizes only the reduced problem on D_h and extends by zero. Several practical questions remain. How does the interface Γ affect mesh generation, and should the mesh be aligned with Γ to capture the gradient discontinuity? Can a priori and a posteriori error estimates be sharpened to account for the singular behavior at Γ? Can domain decomposition preconditioners be designed that exploit the tensor-product structure of the active and inactive regions?

Open Problem 11.6. Given measurements of u^{ze} (or some other distinguished solution) on $\partial\Omega$ or in the interior, determine the shift vector h. The interface $\Gamma = \partial D_h \cap \Omega$, which is observable through the gradient discontinuity in u^{ze}, provides geometric information about h. Can h be uniquely recovered from Cauchy data on $\partial\Omega$? How does the non-uniqueness of the forward problem affect the identifiability of h?

Open Problem 11.7. If the shift h is a random vector, modeling uncertainty in the sensor location or material microstructure, the solution u becomes a random field. Study the statistical properties of u^{ze} and their dependence on the distribution of h. In particular, compute the mean field and variance, and analyze how the random interface $\Gamma(\omega)$ induces fluctuations in the gradient.

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[1] National Technical University of Ukraine, Igor Sikorsky Kyiv Polytechnic Institute, 37, Prospect Beresteiskyi (former Peremohy), Kyiv, 03056, Ukraine.
Email address: math.kiev@gmail.com