Annals of Mathematics and Computer Science
ISSN: 2789-7206
Vol 9 (2022) 67-77

SOME CURVATURE IDENTITIES ON HYPERKÄHLER MANIFOLDS

SAMSER ALAM[1*] , AND ARINDAM BHATTACHARYYA[2]

ABSTRACT. The object of the present paper is to study some curvature identities on hyperKähler manifold which is locally symmetric. Also, we study conformal flatness, Bochner flatness and generalised W_2-flatness of a hyperKähler manifold. Finally, we gave some examples of a hyperKähler manifold.

1. INTRODUCTION

A hyperKähler manifold [15] is a Riemannian $4n$-manifold with a family of almost complex structures which act under composition like the multiplication, pure-imaginary, unit quaternions and which are covariantly constant with respect to the Levi-Civita connection. If we only requisite that these almost complex structures exist locally and that the Levi-Civita connection preserves this family generally, then we obtain a quaternionic Kähler structure, at least if $n \geq 2$. Thus hyperKähler manifolds are a special case of quaternionic Kähler manifolds. Although, note that quaternionic Kähler manifold need not be Kähler.

Remember that a Riemannian manifold which has just one such automorphism is called a Kähler manifold. The name "hyperKähler", which established with E. Calabi [8], is a proper description-the metric is Kählerian for several complex structures-even though it does recall Grassmann's "hypercomplex numbers" rather than Hamilton's quaternions. There is, however, an essential difference between Kähler and hyperKähler manifolds. A Kähler metric on a given complex manifold can be modified to another one simply by adding a hermitian form $\partial\bar{\partial}f$ for an arbitrary sufficiently small C^∞ function f. Thus the space of Kähler metrics is infinite dimensional. It is also easy to find examples of Kähler manifolds. Any complex submanifold of CP_n inherits a Kähler metric and so simple writing down algebraic equations for a projective variety gives a vast number of examples.

By contrast, hyperKähler metrics are much more rigid. On a compact manifold, if one such metric exists, then up to isometry there is only a finite dimensional

Date: Received: Aug 19, 2022; Accepted: Oct 28, 2022.
* Corresponding author.
2010 *Mathematics Subject Classification.* 53C26; 53C15.
Key words and phrases. Kähler manifold, hyperKähler manifold, conformally flat Kähler manifold, Bochner flat Kähler manifold, generalised W_2-flat manifold.

space of them. Nor is it easy to find examples. Obviously, we will never find them as quaternionic submanifolds of the quaternionic projective space $\mathbb{H}P_n$ [6].

The idea of a hyperKähler manifold arose first in 1955 with M.Berger's classification of the holonomy groups of Riemannian manifolds. On a hyperKähler manifold, parallel translation preserves I, J and K (since they are covariant constant) and so the holonomy group is contained in both orthogonal group O_{4n} and the group $GL(n, \mathbb{H})$ of quaternionic invertible matrices (i.e., those linear transformations which commute with right multiplication by i, j and k). The maximal such intersection in SP_n, the group of $n \times n$ quaternionic unitary matrices. This group performed in Berger's list.

The group SP_n is also an intersection of U_{2n} and $SP(2n, C)$, the linear transformations of C^{2n} which preserve a non-degenerate skew form. Thus a hyperKähler manifold is naturally a complex manifold with a holomorphic symplectic form. One can see explicitly by taking the three Kähler two-forms, $\omega_1(X, Y) = g(IX, Y)$, $\omega_2(X, Y) = g(JX, Y)$, $\omega_3(X, Y) = g(KX, Y)$ for X, Y $\in TM$, defined for the complex structures I, J and K. As to complex structures I, J and K, the complex form $\omega_+ = \omega_2 + i\omega_3$ is non-degenerate and covariant constant, hence it is closed and holomorphic.

Now, also we define the generalised W_2-curvature tensor $(4n > 8)$ as follows:

$$\overline{W_2}(X, Y)Z = aR(X, Y)Z + \left(b + \frac{c}{4n-7}\right)[g(X, Z)QY - g(Y, Z)QX], \quad (1.1)$$

where $a, b, c \neq 0$. In particular, if $a = 1, b = 0, c = 1$; then it reduces to W_2-curvature tensor. Again if $b = 0$, we call the $\overline{W}_2$ tensor as quasi-W_2 tensor and is denoted by $\widetilde{W}_2$. So a manifold is generalised W_2-flat if $g(\overline{W_2}(X, Y)Z, W) = 0$.

The interest in some curvature identities on hyperKähler manifold is motivated by our study [13] and [15] of hyperKähler manifold and hypercomplex structures in $4n$-dimensional Riemannian manifolds, which is locally symmetric, conformally flat, Bochner flat and generalised W_2-flat.

2. Preliminaries

Let (M, g) be a Riemmanian manifold with I, J, K compatible almost complex structures parallel for the Levi-Civita connection and with $IJ = K = -JI$. Consequently, (a) I, J, K are Integrable, (b) $\omega_1 = g(I.,.)$ etc. are symplectic forms. Let $\mathbb{H} = \mathbb{R}^4$ with basis $\{1, i, j, k\}$, $i^2 = -1 = j^2 = k^2$, quaternion division algebra. In $\mathbb{H}^n$, $Iq = -qi$ holds, with standard inner product. We also know $SP_1 = SU_2 = \{ai + bj + ck : a^2 + b^2 + c^2 = 1\}$ acts on the right. $SP_n = \{A \in M_n(\mathbb{H}) \mid \overline{A}^T A = 1_n\}$ is centraliser in SO_{4n} of SP_1.
A hyperKähler manifold is a Rieminnian $4n$-manifold with holonomy in SP_n.

Now, we have the following propositions:

Proposition 2.1. [14] *A hyperKähler manifold M is a complex manifold with a holomorphic symplectic form. Conversely , any compact Kähler manifold* [10] *with a holomorphic symplectic form is hyperKähler.*

Proposition 2.2. [14] *A hyperKähler manifold is a C^∞ Riemannian manifold together with three covariantly constant orthogonal endomorphisms I, J and K of the tangent bundle which satisfy the quaternionic relations $I^2 = J^2 = K^2 = IJK = -1$.*

Note that, I, J and K give each tangent space the structure of a quaternionic vector space, so the dimension of a hyperKähler manifold is divisible by 4. Since I, J and K are covariantly constant, a parallel transport commutes with the quaternionic multiplication and so the holonomy group is contained in $O_{4n} \bigcap GL_n(\mathbb{H}) \cong SP_n$, the group of quaternionic unitary $n \times n$ matrices. In particular, since $SP_n \subseteq SU_{2n}$ every hyperKähler manifold is Calabi-Yau [9]. Assume that M is an almost hypercomplex manifold. Define the Nijenhuis tensor N of I, J and K by

$$
\begin{aligned}
&N_I(X,Y) = [IX, IY] - I[IX, Y] - I[X, IY] - [X, Y],\\
&N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \textit{ and}\\
&N_K(X,Y) = [KX, KY] - K[KX, Y] - K[X, KY] - [X, Y],
\end{aligned}
$$

for all vector fields X, Y. M is said to be hypercomplex if $N_I = N_J = N_K = 0$. Suppose that g is a pseudo Riemannian metric on M satisfying the condition

$$
\begin{aligned}
&g(IX, Y) + g(X, IY) = 0,\\
&g(JX, Y) + g(X, JY) = 0,\\
&g(KX, Y) + g(X, KY) = 0, \qquad (2.1)
\end{aligned}
$$

for $X, Y \in \chi(M)$. Define the 2-forms $\widetilde{I}(X,Y) = g(X, IY)$, $\widetilde{J}(X,Y) = g(X, JY)$, $\widetilde{K}(X,Y) = g(X, KY)$ for all $X, Y \in \chi(M)$. M is said to be hyperKähler manifold if it is a hypercomplex and the respective 2-forms are closed and $\nabla I = \nabla J = \nabla K = 0$, where ∇ is the Levi-Civita connection on M are equivalent.

3. Main results of Some Curvature Identities on hyperKähler Manifolds

We are investigated some properties of curvature tensors and Ricci tensors of hyperKähler manifold. Let M be a hyperKähler manifold and R denotes the curvature tensor of M.

Theorem 3.1. *The curvature tensor R satisfies*

$$
\begin{aligned}
&(i)R(X,Y)IZ = IR(X,Y)Z,\\
&(ii)R(IX,IY)Z = R(X,Y)Z,\\
&(iii)R(IX,Y)Z + R(X,IY)Z = 0,\\
&(iv)\widetilde{R}(IX,IY,IZ,IW) = \widetilde{R}(X,Y,Z,W),\\
&(v)\widetilde{R}(IX,Y,IZ,W) = \widetilde{R}(X,IY,Z,IW),\\
&(vi)\widetilde{R}(X,Y,IZ,JW) = -\widetilde{R}(IX,IY,Z,IJW),\\
&(vii)\widetilde{R}(IX,IY,JZ,JW) = \widetilde{R}(X,Y,IJZ,IJW),
\end{aligned}
$$

where $\widetilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Proof. (i) Since I is parallel, i.e., $(\nabla_X I)(Y) = 0$, we get

$$\nabla_X I(Y) = I(\nabla_X Y).$$

Now,

$$
\begin{aligned}
R(X,Y)I(Z) &= \nabla_X\nabla_Y I(Z) - \nabla_Y\nabla_X I(Z) - \nabla_{[X,Y]}I(Z)\\
&= \nabla_X I(\nabla_Y Z) - \nabla_Y I(\nabla_X Z) - I(\nabla_{[X,Y](Z)})\\
&= I(\nabla_X\nabla_Y Z) - I(\nabla_Y\nabla_X Z) - I(\nabla_{[X,Y]}Z)\\
&= I(R(X,Y)Z).
\end{aligned}
$$

(ii) Since $g(R(X,Y)V,U) = g(R(U,V)Y,X)$, we have

$$
\begin{aligned}
g(R(IX,IY)V,U) &= g(R(U,V)IY,IX)\\
&= g(I(R)(U,V)Y,IX)\\
&= -g(R(U,V)Y,I^2(X)),\ [\textit{since}\ g(IX,Y) = -g(X,IY)]\\
&= g(R(U,V)Y,X),\ [\textit{since}\ I^2 = J^2 = K^2 = -1\\
&\qquad\qquad \textit{and}\ IJ = -K = JI]\\
&= g(R(X,Y)V,U).
\end{aligned}
$$

Hence, $R(IX,IY)V = R(X,Y)V$.

(iii) Putting $X = IX$ in (ii) we obtain (iii).

(iv) Now,

$$
\begin{aligned}
g(R(IX,IY)IZ,IW) &= -g(I(R)(IX,IY)IZ,W),\\
&\qquad\qquad [\textit{since}\ g(IX,Y) = -g(X,IY)]\\
&= -g(R(IX,IY)Z,W)\\
&= g(R(X,Y)Z,W),\\
&\qquad [\textit{using}\ g(R(IX,IY)V,U) = g(R(X,Y)V,U)]
\end{aligned}
$$

Therefore, $\widetilde{R}(IX,IY,IZ,IW) = \widetilde{R}(X,Y,Z,W)$.

(v) Setting $Y = IY$, $W = IW$ in (iv) we get (v).

(vi) Putting $X = IX$, $Y = IY$, $W = KW$, where $IJ = K = -JI$ in equation (iv) then we obtain

$$\widetilde{R}(X, Y, IZ, JW) = -\widetilde{R}(IX, IY, Z, IJW).$$

(vii) Again putting $Z = KZ$, $W = KW$, where $IJ = K = -JI$ in equation (iv) then we have $\widetilde{R}(IX, IY, JZ, JW) = \widetilde{R}(X, Y, IJZ, IJW)$. □

Remark 3.2. Accordingly, Theorem 3.1 holds for operators J, K. Since $I^2 = J^2 = K^2 = -1$ and $IJ = -K = JI$, so the above curvature identities also hold for the operators IJ and JI.

Let S be the Ricci tensor of M, i.e.,

$$S(Y, Z) = trace\{X \to R(X, Y, Z)\} = \sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, Y, Z, e_i),$$

where $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis for M and $\epsilon_i = g(e_i, e_i) = 1$.

Theorem 3.3. *The Ricci tensor of a hyperKähler manifold satisfies*

$$(i) S(IX, IY) = S(X, Y),$$
$$(ii) S(IX, Y) + S(X, IY) = 0.$$

Proof.

$$\begin{aligned} S(IX, IY) &= trace\{Z \to R(Z, IX)IY\} \\ &= trace\{IZ \to R(IZ, IX)IY\} \\ &= trace\{IZ \to R(Z, X)IY\}, \ [by\ (ii)\ of\ Theorem\ 3.1] \\ &= trace\{IZ \to IR(Z, X)Y\}, \ [since IR = RI] \\ &= trace\{Z \to R(Z, X)Y\} \\ &= S(X, Y), \end{aligned}$$

which proves (i).

Now setting $X = IY$ in (i) we obtain (ii). □

Remark 3.4. In parallel, Theorem 3.3 holds for the operators J, K. Since $I^2 = J^2 = K^2 = -1$ and $IJ = -K = JI$, so the above Ricci tensor of a hyperKähler manifold also satisfy for the operators IJ and JI.

Theorem 3.5. *For a hyperKähler manifold of dimension* $4n$ *the following relation holds, i.e.,* $\sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, I(e_i), X, I(Y)) = 0$.

Proof. We have

$$
\begin{aligned}
S(X,Y) &= \sum_{i=1}^{4n} \epsilon_i g(R(e_i,X)Y,e_i) \\
&= -\sum_{i=1}^{4n} \epsilon_i g(R(I(e_i),I(X))Y,e_i) \\
&= -\sum_{i=1}^{4n} \epsilon_i g(R(e_i,Y)I(X),I(e_i)) \\
&= -\sum_{i=1}^{4n} \epsilon_i g(R(Y,e_i)I(e_i),I(X)) \\
&= \sum_{i=1}^{4n} \epsilon_i g(R(Y,e_i)I(X),I(e_i))
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{4n} \epsilon_i g(R(I(X),e_i)Y,I(e_i)) + \sum_{i=1}^{4n} \epsilon_i g(R(Y,I(X))e_i,I(e_i)), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [using\ Bianchi's\ identities] \\
&= -\sum_{i=1}^{4n} \epsilon_i g(R(e_i,I(X))Y,I(e_i)) - \sum_{i=1}^{4n} \epsilon_i g(R(I(X),Y)e_i,I(e_i)) \\
&= \sum_{i=1}^{4n} \epsilon_i g(I(R)(e_i,I(X))Y,e_i) + \sum_{i=1}^{4n} \epsilon_i g(I(R)(I(X),Y)e_i,e_i) \\
&= \sum_{i=1}^{4n} \epsilon_i g(R(e_i,I(X))I(Y),e_i) + \sum_{i=1}^{4n} \epsilon_i g(R(I(X),Y)I(e_i),e_i) \\
&= S(IX,IY) - \sum_{i=1}^{4n} \epsilon_i g(R(Y,I(X))I(e_i),e_i) \\
&= S(X,Y) + \sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i,I(e_i),X,I(Y)).
\end{aligned}
$$

So this implies, $\sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i,I(e_i),X,I(Y)) = 0$. □

Remark 3.6. Comparably, Theorem 3.5 holds for the operators J, K. Since $I^2 = J^2 = K^2 = -1$ and $IJ = -K = JI$, so the above global form of curvature tensors of a hyperKähler manifold also satisfy for the operators IJ and JI, i.e.

$$(i)\sum_{i=1}^{4n}\epsilon_i\widetilde{R}(e_i,J(e_i),X,J(Y))=0.$$

$$(ii)\sum_{i=1}^{4n}\epsilon_i\widetilde{R}(e_i,IJ(e_i),X,IJ(Y))=0.$$

$$(iii)\sum_{i=1}^{4n}\epsilon_i\widetilde{R}(e_i,JI(e_i),X,JI(Y))=0.$$

3.1. **Conformal flatness of hyperKähler manifold.** Since any pseudo-Riemannian as well as Riemannian manifold of dimension 3 is conformally flat, we are focused in dimension $4n \geq 4$. Utilizing the identities from the previous section, we prove the following theorem.

Theorem 3.7. *Let M be a conformally flat hyperKähler manifold. Then*
(i) M is locally flat if dim $M \geq 4$,
(ii) M is locally symmetric and its scalar curvature vanishes identically if dim $M = 4$.

Proof. By the vanishing of conformal curvature tensor, we have

$$\begin{aligned}\widetilde{R}(X,Y,Z,W)=&\frac{1}{2n-2}[-g(X,Z)S(Y,W)-g(Y,W)S(X,Z)+g(X,W)S(Y,Z)\\&+g(Y,Z)S(X,W)]-\frac{r}{(2n-1)(2n-2)}[g(X,Z)g(Y,W)\\&-g(X,W)g(Y,Z)],\end{aligned}\tag{3.1}$$

r being the scalar curvature of M. From the above equation with the help of Theorem 3.3 and Theorem 3.1, we get

$$\sum_{i=1}^{4n}\epsilon_i\widetilde{R}(e_i,I(e_i),Z,I(W))=-\frac{2}{n-1}S(Z,W)-\frac{r}{(2n-1)(n-1)}g(Z,W).$$

Then using the result of Theorem 3.5, we obtain

$$S(Z,W)=-\frac{r}{2(2n-1)}g(Z,W).\tag{3.2}$$

Now setting $Z=W=e_i, 1\leq i\leq n$ and summing over i then from Equation (3.2), we have $(4n-1)r=0$, where $r=\sum_{i=1}^{4n}\epsilon_iS(e_i,e_i)$. Then this implies $r=0$, when $4n\geq 4$. Now putting the value of $r=0$ in the Equation (3.2), we get $S=0$.
So from Equation (3.1) it follows that the manifold is locally flat. Now, also if $4n=4$ then $r=0$, then its scalar curvature vanishes identically. Next we proof the manifold is locally symmetric. Now we assume that conformally flatness implies,

$$(\nabla_XS)(Y,Z)-(\nabla_YS)(X,Z)=\frac{1}{6}[(X_r)g(Y,Z)-(Y_r)g(X,Z)].\tag{3.3}$$

Putting $r = 0$ in the Equation (3.3), we obtain

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z). \tag{3.4}$$

On the other hand, from Theorem 3.3 it follows that

$$(\nabla_X S)(Y, I(Z)) + (\nabla_X S)(Z, I(Y)) = 0. \tag{3.5}$$

Using the equality and the Equation (3.4), we get

$$(\nabla_X S)(Y, I(Z)) + (\nabla_Y S)(Z, I(X)) = 0. \tag{3.6}$$

Applying the result of the Theorem 3.3 on the above equation, we obtain $\nabla S = 0$. Now from the Equation (3.1) and $\nabla S = 0$, implies that $\nabla R = 0$, i.e., the manifold is locally symmetric. □

From Theorem 3.7, we have the following Corollary:

Corollary 3.8. A conformally flat hyperKähler manifold of dimension $4n$ is an Einstein manifold.

3.2. **Bochner flatness and generalised W_2-flatness of hyperKähler manifold.** The notion of Bochner curvature tensor [2] is defined by:

$$\begin{aligned} B(X,Y)Z =& R(X,Y)Z - \frac{1}{2n+4}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ & - S(X,Z)Y + g(IY,Z)QIX - g(IX,Z)QIY + S(IY,Z)IX \\ & - S(IX,Z)IY - 2S(IX,Y)IZ - 2g(IX,Y)QIZ] \\ & + \frac{r}{(2n+2)(2n+4)}[g(Y,Z)X - g(X,Z)Y + g(IY,Z)IX \\ & - g(IX,Z)IY - 2g(IX,Y)IZ], \end{aligned} \tag{3.7}$$

where Q is the Ricci operator, defined by $g(QX, Y) = S(X, Y)$ and n is the dimension of the manifold. Moreover a manifold is Bochner flat if $\widetilde{B}(X, Y, Z, U) = g(B(X, Y)Z, U) = 0$.

Theorem 3.9. *A Bochner flat hyperKähler manifold is an Einstein manifold.*

Proof. Taking inner product in the Equation (3.7) by W, we get

$$\begin{aligned} g(B(X,Y)Z,W) =& \widetilde{R}(X,Y,Z,W) - \frac{1}{2n+4}[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ & + S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(FY,Z)S(IX,W) \\ & - g(IX,Z)S(IY,W) + S(IY,Z)g(IX,W) - S(IX,Z)g(IY,W) \\ & - 2S(IX,Y)g(IZ,W) - 2g(IX,Y)S(IZ,W)] \\ & + \frac{r}{(2n+2)(2n+4)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ & + g(IY,Z)g(IX,W) - g(IX,Z)g(IY,W) - 2g(IX,Y)g(IZ,W)]. \end{aligned} \tag{3.8}$$

Now as the manifold is Bochner flat then the above equation reduces to

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) =& \frac{1}{2n+4}[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + S(Y,Z)g(X,W) \\ & - S(X,Z)g(Y,W) + g(IY,Z)S(IX,W) - g(IX,Z)S(IY,W) \\ & + S(IY,Z)g(IX,W) - S(IX,Z)g(IY,W) - 2S(IX,Y) \\ & g(IZ,W) - 2g(IX,Y)S(IZ,W)] - \frac{r}{(2n+2)(2n+4)}[g(Y,Z) \\ & g(X,W) - g(X,Z)g(Y,W) + g(IY,Z)g(IX,W) - g(IX,Z) \\ & g(IY,W) - 2g(IX,Y)g(IZ,W)]. \end{aligned} \tag{3.9}$$

Setting $X = e_i, Y = Ie_i, Z = Z$ and $W = IW$ in the above equation and taking summation over i, $1 \le i \le n$ and also using the result of the Theorem 3.5, we obtain

$$S(Z,W) = -\frac{r}{2n+6}g(Z,W). \tag{3.10}$$

Hence the proof. □

From Theorem 3.9 we have the following Corollary:

Corollary 3.10. A Bochner flat hyperKähler manifold is locally flat.

Proof. Taking $Z = W = e_i$ in the above equation and summing over i, $1 \le i \le n$ we obtain

$$r = 0, \ \textit{provided } 3n + 6 \neq 0. \tag{3.11}$$

Then the Equation (3.10) becomes

$$S(Z,W) = 0. \tag{3.12}$$

So the manifold is locally flat. □

Theorem 3.11. A generalised W_2-flat hyperKähler manifold is Ricci flat, provided $a \neq \left(b + \frac{c}{4n-7}\right)$.

Proof. Taking inner product in the Equation (1.1) by W, we get

$$\begin{aligned} g(\overline{W_2}(X,Y)Z,W) =& a\widetilde{R}(X,Y,Z,W) + \left(b + \frac{c}{4n-7}\right)[g(X,Z)S(Y,W) \\ & - g(Y,Z)S(X,W)]. \end{aligned} \tag{3.13}$$

Now as the manifold is $\overline{W}_2$-flat then the above equation reduces to

$$a\widetilde{R}(X,Y,Z,W) + \left(b + \frac{c}{4n-7}\right)[g(X,Z)S(Y,W) - g(Y,Z)S(X,W)] = 0. \tag{3.14}$$

Putting $X = e_i, Y = Ie_i, W = IW$ in the above equation and summing over i, $1 \le i \le 4n$ and operating the result of the Theorem 3.3 we have

$$\left(b + \frac{c}{4n-7}\right)S(Z,W) = 0. \tag{3.15}$$

Then we have $S(Z,W) = 0$, for any $Z, W \in \chi(M)$, being the Lie algebra of vector fields on M. This completes the proof. □

From Theorem 3.11 we have the following Corollary:

Corollary 3.12. A quasi-W_2 flat hyperKähler manifold is Ricci flat, provided $c \neq 0$.

The following examples are given in the paper [14]

Example 3.13. A trivial example is $\mathbb{H}^n$. However, in contrast to the Kähler case, $\mathbb{H}P_n$ is not hyperKähler and neither do its generic quaternionic submanifolds.

Example 3.14. In the particular case $n = 1$, then $SP_1 = SU_2$ in SO_4, so a 4-dimensional Riemannian manifold is hyperKähler exactly when it is Kähler and Ricci flat. Specifically, this shows that any compact complex surface M of Kähler type with vanishing first Chern class is either a torus or simply connected and admits a unique complex-symplectic structure, i.e., is a so-called "K3-surface".

Example 3.15. A class of non-compact hyperKähler manifolds of real dimension 4 can be obtained by resolving the singularity of C^2/Γ for $\Gamma \subset SU_2$ a finite subgroup.

Example 3.16. Many examples of non-compact hyperKähler manifolds arise as moduli spaces of solutions to gauge-theoretic equations. The hyperKähler structure is obtained by a hyperKähler reduction from $\mathbb{H}^n$.

These results can be verified in these examples.

Acknowledgement. The authors are grateful to the anonymous reviewer(s) for valuable suggestions and comments to improve the paper.

References

1. Arthur L. Basse, *Einstein Manifolds*, Springer. Berlin. Classics in Mathematics. ISBN 978-3-540-74120-6, (2007)[1987], 399-412.
2. G. Calvaruso, *Nulity index of Bochner-Kähler manifolds* , Note di Mathematica. **29** (2009), 115-122.
3. B. B. Chaturvedi and P. Pandey, *Study of special type of a weakly symmetric Kälher manifold*, Diff. Geom. Dyn. Syst. **17** (2015), 32-37.
4. A. Gray , *Curvature identities for hermitian and almost hermitian manifolds*, Tohoku. Math. Journ. **28** (1976), 601-612.
5. A. Gray, *Some examples of almost Hermitian manifolds*, Illinois J . Math. **10** (1966), 353-366.
6. A. Gray, *A note on manifolds whose holonomy group is a subgroup of SP(n).SP(1)*, Michigan Math. J. **16** (1969) 125-128.
7. N. Hitchin and J. Sawon, *Curvature and characteristic numbers of hyperkähler manifolds*, Duke. Math. J. **106(3)** (2001), 599-615.
8. N. Hitchin, *HyperKähler manifolds, Seminaire Bourbaki, exp. n°, 748*, Asterisque t. **206** (1992), 137-166.
9. N. Hitchin, *Generalized Calabi-Yau manifolds*, Quaterly Mathematical Journal **54** (2003), 281-308.
10. D. Huybrechts, *Compact hyperKähler manifolds*, Basic results Invent. Math. **135(1)**, (1999), 63-113.
11. J. Keller, *About the Calabi-Yau theorem and its applications*, Publi. Mathematical Journal, (2009) 1-26.

12. Curtis.T.Mcmullen, *The moduli space of Riemannian surfaces in Kähler hyperbolic*, Annal Mathematical Journal, **151** (2000), 327-357.
13. Z. Olszak, *On conformally flat ParaKähler manifolds*, Publ.Math.Balkanica, **5** (1991), 302-307.
14. P. Safronov, *HyperKähler Manifolds*, Talk at 2011 Talbot Workshop.
15. Andrew F. Swann, *HyperKähler and quaternionic Kähler geometry* , Aarhus University Mathematische Annalen, **289(1)** (1991), 421-450.
16. Q. Yang, *Vanishing theorem on Compact HyperKähler manifolds*, Hindawi Mathematical Journal, (2014), 1-14.
17. E.Yanhee, J.Park, K. Sekigawa, *A curvature identities on a* 6*-dimensional Riemannian Manifolds and its applications*, Czechoslovak Mathematical Journal **67** (2017), 253-270.

[1] Department of Mathematics, Jadavpur University, Kolkata, India.
Email address: samser.alam@gmail.com

[2] Department of Mathematics, Jadavpur University, Kolkata, India
Email address: bhattachar1968@yahoo.co.in