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WEAK SOLUTIONS OF FRACTIONAL p-KIRCHHOFF PROBLEM WITH NONLOCAL NEUMANN CONDITION

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Abstract. The existence of a weak solution to a Schrödinger-Kirchhoff- type problem involving the fractional p-Laplacian with a homogeneous Neumann type boundary condition is the focus of this essay. This p-Neumann boundary condition arises from a simple probabilistic consideration. The Berkovits degree theory is used to establish the existence of weak solutions under certain suitable conditions.
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1. Introduction

The fractional p-Laplacian equation of the Schrödinger-Kirchhoff type is investigated in this paper. We consider the following nonlocal problem :

$$\begin{cases} \mathcal{M}\Big(\|\vartheta\|^p_{X_{s,p}}\Big)\Big[(-\Delta)^s_p(\vartheta)+\beta(x)|\vartheta|^{p-2}\vartheta\Big]=\lambda H(x,\vartheta)+g(x)|\vartheta|^{r-2}\vartheta \text{ in } \Omega, \\ \mathcal{N}^s_p\vartheta=0 \text{ in } \mathbb{R}^N\backslash\overline{\Omega}, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain of $\mathbb{R}^N$, $Q=(\mathbb{R}^N\times\mathbb{R}^N)\backslash(\mathbb{R}^N\backslash\Omega\times\mathbb{R}^N\backslash\Omega)$, $\mathcal{M}:\mathbb{R}^+\longrightarrow\mathbb{R}^+$, $\beta:\mathbb{R}^N\longrightarrow\mathbb{R}^+$ is a potential function and $g(x)>0$. The fractional p-Laplacian operator $(-\Delta)^s_p$ is defined for $0<s<1<p<\infty$ with $ps<N$:

$$(-\Delta)^s_p\vartheta(x)=P.V\int_{\mathbb{R}^N}\frac{|\vartheta(x)-\vartheta(y)|^{p-2}(\vartheta(x)-\vartheta(y))}{|x-y|^{N+ps}}dy,\ x\in\mathbb{R}^N.$$

Moreover,

$$\mathcal{N}^s_p\vartheta(x):=\int_{\mathbb{R}^N}\frac{|\vartheta(x)-\vartheta(y)|^{p-2}(\vartheta(x)-\vartheta(y))}{|x-y|^{N+ps}}dy,\ x\in\mathbb{R}^N\backslash\overline{\Omega}, \tag{1.2}$$

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is the nonlocal p-derivative, also called the nonlocal p-Neumann boundary condition, which and describes the natural Neumann boundary condition in presence of the fractional p-Laplacian. This type of condition is referred to as a nonlocal Neumann boundary condition; we refer the readers see to [2, 13] for the case $p = 2$ and [4, 17] for the general case. In our setting, $p > 1$ and $s \in (0, 1)$.
The definition in (1.2) was introduced in [4] where the basic integration-by-parts formula was established. This type of Neumann condition for nonlocal operators generalizes the classical Neumann boundary condition as a limiting case and admit a clear probabilistic interpretation.
In recent years, there has been growing interest in the study of fractional Laplacian and non-local operators. Applications for this type of operator arise population dynamics, continuum mechanics, image processing, and phase transition [3, 8, 9, 10]. N. Laskin proposed a nonlinear fractional Schrödinger equation in the setting of fractional quantum mechanics by generalizing the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths [14, 15]. Using the concentration-compactness concept, Xiang et *al.* investigated a particular case of problem (1.1) involving a critial exponent term (see [23]). Additionally, Pucci et *al.* examined the following problem under homogeneous Dirichlet conditions via the Ekeland variational principle in [19] :

$$M\Big(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\Big)(-\Delta)_p^s(u) + V(x)|u|^{p-2}u = f(x,u) + g(x) \text{ in } \mathbb{R}^N.$$

Recently, Talibi et *al.* studied with the following Kirchhoff problem by means of topological degree theory (see [21]) :

$$\begin{cases} \mathcal{M}\Big(\int\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\Big)(-\Delta)_p^s(u) = f(x,u) \text{ in } \Omega. \\ \qquad\qquad u = 0 \text{ in } \mathbb{R}^N\backslash\Omega. \end{cases}$$

The topological degree technique has gained increasing importance as a tool for establishing the existence of solutions to nonlinear partial differential equations. The abstract framework used here is developed in [6] for operators of generalized monotone type. The authors of [16] studied nonlinear equations for compact perturbations of the identity in infinite-dimensional Banach spaces, where the topological degree theory was first formulated in this setting. We refer the reader to [1, 11], where this theory has been applied to various elliptical problems.
Inspired by the above work, we establish existence results for a fractional p-Kirchhoff Schrödinger problem under nonlocal Neumann boundary conditions without assuming standard variational compactness conditions. Our approach relies on the Berkovits topological degree for pseudo-monotone operators.
The originality of the problem (1.1) lies in the pseudo-monotone structure of the operator and the absence of the Palais-Smale condition. Moreover, the Berkovits degree's use in a fractional p-Kirchhoff framework of Neumann type has not been addressed previously in the literature. To support these theoretical results, we make an adaptation with the finite element method. For this, we consider the recent work of Bassonon et *al.* in [7] and Padhy et *al.* in [18].

This paper is organized as follows. In Section 2, we recall the Berkovits degree theory and provide some background on fractional Sobolev spaces. In section 3, we set up the functional framework and establish several technical lemmas. In Section 4, we prove Theorem 4.1 under the stated hypotheses and technical lemmas, thereby establishing the main result. Finally, we make an application with the finite element method in Section 5.

2. **Preliminaries**

2.1. **Fractional Sobolev spaces.**

To properly define our functional spaces, we recall the hypotheses on the potential β. We assume that

(b_1) $\beta \in C(\mathbb{R}^N)$ satisfies $\inf\limits_{x\in\mathbb{R}^N} \beta(x) \geq \beta_0 > 0$ where β_0 is a constant.

(b_2) There exists $h > 0$ such that $\limsup\limits_{|y|\longrightarrow+\infty} \mathrm{meas}\Big(\{x \in B_h(y) : \beta(x) \leq c\}\Big) = 0$ for any $c > 0$

where as already noted $B_R(x)$ denotes any open ball of $\mathbb{R}^N$ centered at x and of radius $R > 0$, while we simply write B_R when $x = 0$.

Condition (b_2), which is weaker than the coercivity assumption: $\beta(x) \longrightarrow \infty$ as $|x| \longrightarrow \infty$ was originally introduced by Bartsch and Wang in [5] to overcome the lack of compactness.

Then, we recall some notations and definitions, and some of the results that will be applied to this work.

Let $0 < s < 1 < p < \infty$ be real numbers and we define p_s^* the fractional critical exponent giving by

$$p_s^* = \begin{cases} \dfrac{Np}{N-sp} & \text{if} \quad sp < N, \\ \infty & \text{if} \quad sp \geq N. \end{cases}$$

Let Ω an open set in $\mathbb{R}^N$ and $Q = (\mathbb{R}^N \times \mathbb{R}^N)\backslash(\mathbb{R}^N\backslash\Omega \times \mathbb{R}^N\backslash\Omega)$. It is obvious that $\Omega \times \Omega$ is strictly contained in Q.

Next, we define the fractional Sobolev space, which can be suitably modeled to deal with fractional Neumann boundary conditions. To do that, fix a bounded domain with Lipschitz boundary $\Omega \subset \mathbb{R}^N,\ N \geq 1$ and $\vartheta : \mathbb{R}^N \longrightarrow \mathbb{R}$ measurable. W is a linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function ϑ in W belongs to $L^p(\Omega)$ and

$$\int\int_Q \frac{|\vartheta(x) - \vartheta(y)|^p}{|x-y|^{N+ps}} dxdy < \infty.$$

The space W is equipped with the norm

$$\|\vartheta\|_W = \Big(\|\vartheta\|^p_{L^p(\Omega)} + \int\int_Q \frac{|\vartheta(x) - \vartheta(y)|^p}{|x-y|^{N+ps}} dxdy\Big)^{1/p}.$$

And the closed linear subspace

$$W_0 = \{\vartheta \in W \ : \ \vartheta = 0 \text{ a.e. in } \ \mathbb{R}^N\backslash\Omega\}.$$

In W_0, we may also use the norm

$$\|\vartheta\|_{W_0} = \Big(\int\int_Q \frac{|\vartheta(x)-\vartheta(y)|^p}{|x-y|^{N+ps}}dxdy\Big)^{1/p}.$$

It is known that $(W_0, \|\cdot\|_{W_0})$ is a uniformly convex reflexive Banach space (see [22]), it dual is indicated by $(W_0^*, \|\cdot\|_{W_0^*})$.

Lemma 2.1. [12] *The following embedding $W_0 \hookrightarrow L^\theta(\Omega)$ is compact for all $1 \le \theta < p_s^*$, and continuous for all $1 \le \theta \le p_s^*$.*

For the rest of the work, we introduce the space adapted to our problem. For this, we define the function space

$$X_{s,p} = \{\vartheta : \mathbb{R}^N \longrightarrow \mathbb{R} \text{ measurable} : \ \|\vartheta\|_{X_{s,p}} < \infty\}$$

where

$$\|\vartheta\|_{X_{s,p}} = \Big(\int_\Omega \beta(x)|\vartheta(x)|^p dx + \int\int_Q \frac{|\vartheta(x)-\vartheta(y)|^p}{|x-y|^{N+ps}}dxdy\Big)^{1/p}.$$

Let us note further that

$$\|\vartheta\|_{p,\beta}^p = \int_\Omega \beta(x)|\vartheta(x)|^p dx.$$

Lemma 2.2. *$X_{s,p}$ is a reflexive Banach space with norm $\|\cdot\|_{X_{s,p}}$.*

Proof.
First, we show that $\|\cdot\|_{X_{s,p}}$ is a norm. If $\|\vartheta\|_{X_{s,p}} = 0$, we have $\|\vartheta\|_{p,\beta} = 0$, so $\vartheta = 0$ a.e. in Ω. Moreover, we have

$$\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|\vartheta(x)-\vartheta(y)|^p}{|x-y|^{N+ps}}dxdy = 0,$$

hence $|\vartheta(x)-\vartheta(y)| = 0$ in $\mathbb{R}^{2N}\setminus(C\Omega)^2$. In particular, we can take $x \in C\Omega$ and $y \in \Omega$ to obtain

$$\vartheta(x) = \vartheta(x) - \vartheta(y) = 0.$$

In this way, we have $\vartheta = 0$ a.e. in $\mathbb{R}^N$.
Now, we prove that $X_{s,p}$ is complete, and to do this we take a Cauchy sequence $(\vartheta_k)_k$ in $X_{s,p}$. In particular, ϑ_k is a Cauchy sequence in $L^p(\Omega)$ and so (up to a subsequence) there exists $\vartheta \in L^p(\Omega)$ such that ϑ_k converges to $u \in L^p(\Omega)$ and a.e. in Ω. This means that there exists $Z_1 \subset \Omega$ such that

$$|Z_1| = 0 \text{ and } \vartheta_k(x) \longrightarrow \vartheta(x) \text{ for every } x \in \Omega\setminus Z_1. \tag{2.1}$$

We also define for every $U : \mathbb{R}^N \longrightarrow \mathbb{R}$ and $(x,y) \in \mathbb{R}^{2N}$

$$T_U(x,y) := \frac{\Big(U(x)-U(y)\Big)\chi_{\mathbb{R}^{2N}\setminus(C\Omega)^2(x,y)}}{|x-y|^{N/p+s}},$$

so

$$T_{\vartheta_k(x,y)} - T_{\vartheta_h(x,y)} = \frac{\Big(\vartheta_k(x)-\vartheta_h(x)-\vartheta_k(y)+\vartheta_h(y)\Big)\chi_{\mathbb{R}^{2N}\setminus(C\Omega)^2(x,y)}}{|x-y|^{N/p+s}}.$$

So, T_{ϑ_k} is a Cauchy sequence in $L^p(\mathbb{R}^{2N})$, and up to a subsequence we can assume that T_{ϑ_k} converges to some T_ϑ in $L^p(\mathbb{R}^{2N})$ and a.e. in $\mathbb{R}^{2N}$. This means that there exists $Z_2 \subset \mathbb{R}^{2N}$ such that

$$|Z_2| = 0 \text{ and } T_{\vartheta_{k(x,y)}} \longrightarrow T_\vartheta(x,y) \text{ for every } (x,y) \in \mathbb{R}^{2N} \backslash Z_2. \tag{2.2}$$

For any $x \in \Omega$, we set

$$S_x := \{y \in \mathbb{R}^N : \ (x,y) \in \mathbb{R}^{2N} \backslash Z_2\},$$

$$W := \{(x,y) \in \mathbb{R}^{2N} : \ x \in \Omega \text{ and } y \in \mathbb{R}^N \backslash S_x\},$$

$$V := \{x \in \Omega : \ |\mathbb{R}^N \backslash S_x| = 0\}.$$

If we take $(x,y) \in W$, we have $y \in \mathbb{R} \backslash S_x$, so $(x,y) \ni \mathbb{R}^{2N} \backslash Z_2$ that is $(x,y) \in Z_2$. From this, we get

$$W \subseteq Z_2. \tag{2.3}$$

From (2.2) and (2.3), we obtain $|W| = 0$, so by the Fubini's Theorem we have

$$0 = |W| = \int_\Omega |\mathbb{R}^N \backslash S_x| dx,$$

which implies that $|\mathbb{R}^N \backslash S_x| = 0$ a.e. $x \in \Omega$. It follows that $|\Omega \backslash V| = 0$.
This together with (2.1) implies that

$$|\Omega \backslash (V \backslash Z_1)| = |(\Omega \backslash V) \cup Z_1| \leq |\Omega \backslash V| + |Z_1| = 0.$$

In particular, $V \backslash Z_1 \neq \emptyset$ (nay, $|V \backslash Z_1| = \Omega$), so we can take $x_0 \in V \backslash Z_1$. From (2.1), we have

$$\lim_{k \longrightarrow +\infty} \vartheta_k(x_0) = \vartheta(x_0).$$

In addition, since $x_0 \in V$, we get $|\mathbb{R}^N \backslash S_{x0}| = 0$. This means that for a.e. $y \in \mathbb{R}^N$, $(x_0, y) \in \mathbb{R}^{2N} \backslash Z_2$ and so

$$\lim_{k \longrightarrow +\infty} T_{\vartheta_k}(x_0, y) = T(x_0, y).$$

Moreover, since $\Omega \times (C\Omega) \subseteq \mathbb{R}^{2N} \backslash (C\Omega)^2$, we have

$$T_{\vartheta_k}(x_0, y) := \frac{\vartheta_k(x_0) - \vartheta_k(y)}{|x_0 - y|^{N/p+s}} \text{ for a.e. } y \in C\Omega.$$

From this, we obtain

$$\begin{aligned} \lim_{k \longrightarrow +\infty} \vartheta_k(y) &= \lim_{k \longrightarrow +\infty} \left[\vartheta_k(x_0) - |x_0 - y|^{N/p+s} T_{\vartheta_k}(x_0, y) \right] \\ &= \vartheta(x_0) - |x_0 - y|^{N/p+s} T_{\vartheta_k}(x_0, y) \end{aligned}$$

for a.e. $y \in C\Omega$. This and (2.1) imply that u_k converges a.e. in $\mathbb{R}^N$, so we can that ϑ_k converges a.e. to some $\vartheta \in \mathbb{R}^N$. Now, since ϑ_k is a Cauchy sequence in

$X_{s,p}$, for any $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that, for any $h \geq N_\varepsilon$,

$$\begin{aligned}
\varepsilon^p &\geq \liminf_{k \longrightarrow +\infty} \|\vartheta_h - \vartheta_k\|^p_{X_{s,p}} \\
&\geq \liminf_{k \longrightarrow +\infty} \int_\Omega \beta(x)|\vartheta_h - \vartheta_k|^p dx + \\
&\quad \liminf_{k \longrightarrow +\infty} \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|(\vartheta_k - \vartheta_h)(x) - (\vartheta_k + \vartheta_h)(y)|^p}{|x-y|^{N+ps}} dxdy \\
&\geq \int_\Omega |\vartheta_h - \vartheta|^p dx \\
&\quad + \liminf_{k \longrightarrow +\infty} \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|(\vartheta_h - \vartheta)(x) - (\vartheta_h + \vartheta)(y)|^p}{|x-y|^{N+ps}} dxdy \\
&= \|\vartheta_h - \vartheta\|^p_{X_{s,p}}
\end{aligned}$$

where we used Fatou's Lemma. So, ϑ_h converges to ϑ in $X_{s,p}$. Starting this procedure with a generic subsequence, we conclude that $X_{s,p}$ is complete.

To prove that it is reflexive let us define the space $\mathcal{A} = L^p(Q(\Omega), dxdy) \times L^p(\Omega, dx)$. By the standard results, this product space is reflexive. Then the operator $T : X_{s,p}(\Omega) \longrightarrow \mathcal{A}$ defined as

$$T_\vartheta = \Big[\frac{|\vartheta(x) - \vartheta(y)|}{|x-y|^{\frac{N}{p}+s}} \chi_{Q(\Omega)}(x,y), \vartheta_{\chi\Omega}\Big]$$

where $\chi\mathcal{S}(\cdot)$ denotes the characteristic function of a measurable set $\mathcal{S}$, is an isometry from $X_{s,p}(\Omega)$ into $\mathcal{A}$ (the space $\mathcal{A}$ is also equipped with the norm $\|\cdot\|_{X_{s,p}}$). Thus, since $X_{s,p}$ is a Banach space $T(X_{s,p}(\Omega))$ is closed subspace of $\mathcal{A}$, so that $T(X_{s,p}(\Omega))$ is reflexive and therefore, using the fact that T is an isometry, $X_{s,p}$ as well. $\square$

Lemma 2.3.
The embedding $X_{s,p} \hookrightarrow W_0$ is continuous. As a consequence, $X_{s,p} \hookrightarrow L^\theta(\Omega)$ is a compact embeddings for every $\theta \in [1, p_s^*)$.

Proof. From the condition (b_1) and the definition of $\|\cdot\|_{X_{s,p}}$ we have, for each $u \in X_{s,p}$

$$\min\{1, \beta_0\}\|u\|_{W_0} \leq \|u\|_{X_{s,p}}.$$

From this, it is follows that $X^p_{s,p} \hookrightarrow W_0$.
Since the classical fractional Sobolev space W_0 compactly embeds in $L^\theta(\Omega)$ (see [12, Theorem 7.1]), the proof is complete. $\square$

2.2. **Topological degree framework.**

Let $\mathcal{V}$ be a real separable reflexive Banach space and $\mathcal{V}^*$ be its dual space, and let Ω be a nonempty subset of $\mathcal{V}$. We recall that $\rightharpoonup$ represents the weak convergence and $\langle\cdot,\cdot\rangle$ is continuous dual pairing.
Let $\mathcal{W}$ be another real Banach space. Now, we introduce some topological degree results and properties.

Definition 2.4. Let $F : \Omega \subset \mathcal{V} \longrightarrow \mathcal{W}$ be an operator. We recall that a mapping F is:

(1) **Bounded**, if it takes any bounded set into a bounded set.
(2) **Demicontinuous**, if for any sequence $\vartheta_k \subset \Omega$, $\vartheta_k \longrightarrow \vartheta$ implies $F(\vartheta_k) \rightharpoonup F(\vartheta)$.
(3) **Compact**, if it is continuous and the image of any of any bounded set is relatively compact.
(4) (S_+) **type**, if any sequence $(\vartheta_k) \subset \Omega$ with $\vartheta_k \rightharpoonup \vartheta$ and $\lim\limits_{k\longrightarrow+\infty} \sup\langle F\vartheta_k, \vartheta_k - \vartheta\rangle \leq 0$, we have $\vartheta_k \longrightarrow \vartheta$.
(5) **Quasimonotone**, if $\vartheta_k \rightharpoonup \vartheta$ implies $\lim\limits_{k\longrightarrow+\infty} \sup\langle F\vartheta_k, \vartheta_k - \vartheta\rangle \geq 0$.
(6) $(S_+)_T$ **type**, with $T : \Omega_1 \subset \mathcal{V} \longrightarrow \mathcal{V}^*$ be a bounded operator such that $\Omega \subset \Omega_1$, if for any sequence $(\vartheta_k) \subset \Omega$ with $\vartheta_k \rightharpoonup \vartheta$, $y_k := T\vartheta_k \rightharpoonup y$ and $\lim\limits_{k\longrightarrow+\infty} \sup\langle F\vartheta_k, y_k - y\rangle \leq 0$, we have $\vartheta_k \longrightarrow \vartheta$.

Remark 2.5. [24]
We can see that any compact map in a set is quasimonotone in that set. Moreover, any demi-continuous map of type (S_+) in a set is quasimonotone in that set.

Let $\mathcal{O}$ be the collection of all bounded open set in $\mathcal{V}$. For any $\Omega \subset \mathcal{V}$, we consider the following classes of operators :

$$
\begin{aligned}
\mathcal{H}_1(\Omega) &:= \{F : \Omega \longrightarrow \mathcal{V}^* \text{ /F is bounded, demicontinuous and of type } (S_+)\},\\
\mathcal{H}_{T,B}(\Omega) &:= \{F : \Omega \longrightarrow \mathcal{V} \text{ / F is bounded, demicontinuous and satisfies condition } (S_+)_T\},\\
\mathcal{H}_T(\Omega) &:= \{F : \Omega \longrightarrow \mathcal{V} \text{ / F is demicontinuous and satisfies condition } (S_+)_T\},\\
\mathcal{H}_B(\mathcal{V}) &:= \{F \in \mathcal{F}_{T,B}(\overline{E}) \ / \ E \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{E})\}.
\end{aligned}
$$

Throughout the paper, $T \in \mathcal{H}_1(\overline{E})$ is called an essential inner map to F.

Lemma 2.6. [6, Lemmas 2.2 and 2.4]
Lets $T \in \mathcal{H}_1(\overline{E})$ be continuous and $S : D_S \subset \mathcal{V}^* \longrightarrow \mathcal{V}$ be demicontinuous such that $T(\overline{E}) \subset D_S$, where E is a bounded open set in a real reflexive Banach space $\mathcal{V}$. Then the following statements are true :

(1) *If S is quasimonotone, then $I + S \circ T \in \mathcal{H}_T(\overline{E})$, where I denotes the identity operator.*
(2) *If S is of class (S_+), then $S \circ T \in \mathcal{H}_T(\overline{E})$.*

Definition 2.7. Suppose that E is bounded open subset of a real reflexive Banach $\mathcal{V}$, $T \in \mathcal{H}_T(\overline{E})$ be continuous and let $F,\ S \in \mathcal{H}_T(\overline{E})$. The affine homotopy $\Upsilon : [0,1] \times \overline{E} \longrightarrow \mathcal{V}$ defined by

$$\Upsilon(t,\vartheta) := (1-t)F\vartheta + tS\vartheta \text{ for } (t,\vartheta) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 2.8. ([6]) The above affine homotopy satisfies condition $(S_+)_T$.

Now, we introduce the Berkovits topological degree for class $\mathcal{H}_B(\mathcal{V})$ for more details see [6].

Theorem 2.9. *Let*

$$M_T = \{(F, E, h) \setminus E \in \mathcal{O},\ T \in \mathcal{H}_1(\overline{E}),\ F \in \mathcal{H}_{T,B}(\overline{E}),\ h \notin F(\partial E)\}.$$

There exists a unique degree function $d : M_T \longrightarrow \mathbb{Z}$ which satisfies the following properties :

(1) ***(Normalization)*** *For any $h \in E$, we have $d(I, E, h) = 1$.*
(2) ***(Additivity)*** *Let $F \in \mathcal{F}_{T,B}(\overline{E})$. If E_1 and E_2 are two disjoint open subsets of E such $h \notin F(\overline{E}\backslash(E_1 \cup E_2))$, then we have*
$$d(F, E, h) = d(F, E_1, h) + d(F, E_2, h).$$
(3) ***(Homotopy invariance)*** *If $\Upsilon : [0, 1] \times \overline{E} \longrightarrow \mathcal{V}$ is a bounded admissible affine homotopy with a common continuous essential inner map and $h : [0, 1] \longrightarrow \mathcal{V}$ is a continuous path in $\mathcal{V}$ such that $h(t) \notin \Upsilon(t, \partial E)$ for all $t \in [0, 1]$, then the value of $d(\Upsilon(t, \cdot), E, h(t))$ is constant for all $t \in [0, 1]$.*
(4) ***(Existence)*** *If $d(F, E, h) \neq 0$, then the equation $F\vartheta = h$ has a solution in E.*

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3. Framework of problem

In this part, we will define a weak solution for the problem (1.1). We start by assuming the following hypothesis :

(A_1) : The potential function β is such that $\beta \in L^\infty(\Omega)$ and $\beta(x) > 0$ a.e. in Ω.
Moreover, $g(x) > 0$.

(A_2) : H is a Carathéodory function from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ satisfying :

$$0 \leq \gamma < p - 1 \text{ such that } |H(x, \xi)| \leq \varrho\Big(a(x) + |\xi|^\gamma\Big)$$

for all $\xi \in \mathbb{R}$ and a.e. $x \in \Omega$, where ϱ is a positive constant, $a(x)$ is a positive function in $L^{p'}(\Omega)$.

(A_3) M is a continuous and nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and satisfies

$$m_0 s^{\eta-1} \leq M(s) \leq m_1 s^{\eta-1},$$

for all $s > 0$ and $m_0,\ m_1$ are real numbers such that $0 < m_0 \leq m_1$ and $\eta \geq 1$.

Definition 3.1. We say that a function $\vartheta \in X_{s,p}$ is a weak solution of the problem (1.1) if

$$M\Big(\|\vartheta\|^p_{X_{s,p}}\Big)\Big[\int\int_Q \frac{|\vartheta(x) - \vartheta(y)|^{p-2}(\vartheta(x) - \vartheta(y))}{|x - y|^{N+ps}}(\phi(x) - \phi(y))dxdy$$
$$+ \int_\Omega \beta(x)|\vartheta|^{p-2}\vartheta\phi dx\Big] = \int_\Omega \lambda H(x, \vartheta)\phi(x)dx + \int_\Omega g(x)|\vartheta|^{r-2}\vartheta\phi(x)dx,$$

for all $\phi \in X_{s,p}$.

Let denote $L : X_{s,p} \longrightarrow X^*_{s,p}$ defined by

$$\langle L(\vartheta), \phi\rangle = M\Big(\|\vartheta\|^p_{X_{s,p}}\Big)\Big[\int\int_Q \frac{|\vartheta(x)-\vartheta(y)|^{p-2}(\vartheta(x)-\vartheta(y))}{|x-y|^{N+ps}}(\phi(x)-\phi(y))dxdy + \int_\Omega \beta(x)|\vartheta|^{p-2}\vartheta\phi dx\Big], \quad (3.1)$$

for all $\vartheta, \phi \in X_{s,p}$.
Take into account the following functional

$$\Psi_M(\vartheta) = \frac{1}{p}\mathcal{M}\Big(\|\vartheta\|^p_{X_{s,p}}\Big) \text{ for all } \vartheta \in X_{s,p}. \quad (3.2)$$

Also, $\mathcal{M}$ be the primitive of the function M, defined by

$$\begin{array}{rcl} \mathcal{M} : [0,+\infty[& \longrightarrow & [0,+\infty[\\ t & \mapsto & \mathcal{M}(t) = \displaystyle\int_0^t M(\xi)d\xi. \end{array}$$

Lemma 3.2. *Assume that $(A_1)-(A_3)$ holds, then L is bounded and coercive.*

Proof. By the Hölder inequality and the hypothesis (A_2), we get

$$\begin{aligned} \Big|\langle L(\vartheta), \phi\rangle\Big| &= \Big|M\Big(\|\vartheta\|^p_{X_{s,p}}\Big)\Big[\int\int_Q \frac{|\vartheta(x)-\vartheta(y)|^{p-2}(v(x)-\vartheta(y))}{|x-y|^{N+ps}}(\phi(x)-\phi(y))dxdy \\ &\quad + \int_\Omega \beta(x)|\vartheta|^{p-2}\vartheta\phi dx\Big]\Big| \\ &\leq m_1\Big(\|\vartheta\|^p_{X_{s,p}}\Big)^{\beta-1}\Big[\|\vartheta\|^{p-1}_{W_0}\|\phi\|_{W_0} + \Big(\int_\Omega \beta(x)|\vartheta|^p dx\Big)^{p-1/p}\Big(\int_\Omega \beta(x)|\phi|^p dx\Big)^{1/p}\Big] \\ &\leq C_0\Big[\|\vartheta\|^{p-1}_{W_0} + \Big(\int_\Omega \beta(x)|\vartheta|^p dx\Big)^{p-1/p}\Big]\Big[\|\phi\|_{W_0} + \|\phi\|_{p,\beta}\Big] \\ &\leq C_1\|\phi\|_{X_{s,p}}. \end{aligned}$$

For the coerciveness, we can write, as $\eta \geq 1$:

$$\begin{aligned} \langle L(\vartheta), \vartheta\rangle &= M\Big(\|\vartheta\|^p_{X_{s,p}}\Big)\Big[\int\int_Q \frac{|\vartheta(x)-\vartheta(y)|^p}{|x-y|^{N+ps}}dxdy + \int_\Omega \beta(x)|\vartheta|^p dx\Big] \\ &\geq m_0\Big(\|\vartheta\|^p_{X_{s,p}}\Big)^{\eta-1}\Big[\|\vartheta\|^p_{W_0} + \|\vartheta\|^p_{p,\beta}\Big] \\ &\geq C_2\|\vartheta\|^p_{X_{s,p}}. \end{aligned}$$

We conclude that

$$\lim_{\|\vartheta\|_{X_{s,p}}\longrightarrow+\infty} \frac{\langle L(\vartheta), \vartheta\rangle}{\|\vartheta\|_{X_{s,p}}} = \infty.$$

□

Lemma 3.3. *Assume that $(A_1)-(A_3)$ hold, then the operator L is continuous operator, strictly monotone and is of type (S_+).*

Proof.
According to (3.2), Ψ_M is continuously Gateaux-differentiable in $X_{s,p}$ and

$$\langle \Psi'_M(\vartheta), \phi\rangle = \langle L(\vartheta), \phi\rangle \text{ for all } \vartheta, \phi \in X_{s,p},$$

then L is continuous.
To prove that L is strictly monotone, we define a functional $\mathcal{L} : X_{s,p} \longrightarrow \mathbb{R}$ as

$$\mathcal{L}(\vartheta) = \frac{1}{p}\int\int_Q \frac{|\vartheta(x)-\vartheta(y)|^p}{|x-y|^{N+ps}}dxdy + \frac{1}{p}\int_\Omega \beta(x)|\vartheta|^p dx,$$

by virtue of [23, Lemma 3.3] , we get $\mathcal{L} \in C^1(X_{s,p})$ and

$$\langle \mathcal{L}'(\vartheta), \phi\rangle = \int\int_Q \frac{|\vartheta(x)-\vartheta(y)|^p(\vartheta(x)-\vartheta(y))}{|x-y|^{N+ps}}(\phi(x)-\phi(y))dxdy + \int_\Omega \beta(x)|\vartheta|^{p-2}\vartheta\phi dx.$$

Taking into the Simon inequalities [20], for all $x, y \in \mathbb{R}^N$,

$$\begin{cases} |x-y|^p \leq c_p(|x|^{p-2}x - |y|^{p-2}y)(x-y) \text{ for } p \geq 2, \\ |x-y|^p \leq C_p\Big[(|x|^{p-2}x - |y|^{p-2}y)(x-y)\Big](|x|^p + |y|^p)^{\frac{2-p}{2}} \text{for } 1 < p < 2, \end{cases} \tag{3.3}$$

where $c_p = \left(\frac{1}{2}\right)^{-p}$ and $C_p = \frac{1}{p-1}$.
We obtain that $\mathcal{L}'$ is strictly monotone. According to [24, Proposition 25.10] , $\mathcal{L}$ is strictly convex. Moreover as M is nondecreasing, then $\mathcal{M}$ is convex. This prove that Ψ_M is strictly convex, since $\Psi'_M = L$ in $X^*_{s,p}$, as a result L is strictly monotone in $X_{s,p}$.

Let $(\vartheta_k)_k$ be a sequence in $X_{s,p}$ such that

$$\begin{cases} \vartheta_k \rightharpoonup \vartheta \text{ in } X_{s,p}, \\ \limsup\limits_{k\longrightarrow+\infty}\langle L(\vartheta_k) - L(\vartheta), \vartheta_k - \vartheta\rangle \leq 0. \end{cases} \tag{3.4}$$

We will prove that $\vartheta_k \longrightarrow \vartheta$ in $X_{s,p}$.
For each $\vartheta \in X_{s,p}$, we take into account the functional $\mathcal{F}(\vartheta) : X_{s,p} \longrightarrow X^*_{s,p}$ by

$$\langle \mathcal{F}(\vartheta), \phi\rangle = \int\int_Q \frac{|\vartheta(x)-\vartheta(y)|^{p-2}(\vartheta(x)-\vartheta(y))}{|x-y|^{N+ps}}(\phi(x)-\phi(y))dxdy + \int_\Omega \beta(x)|\vartheta|^{p-2}\vartheta\phi dx$$

for all $\phi \in X_{s,p}$. Then, $\mathcal{F}(\vartheta)$ is continuous linear function in $X_{s,p}$ and

$$|\langle \mathcal{F}(\vartheta), \phi\rangle| \leq \left(\|\vartheta\|^{p-1}_{W_0} + \|\vartheta\|^{p-1}_{p,\beta}\right)\|\phi\|_{X_{s,p}} \text{ for all } \phi \in X_{s,p}.$$

Let $\vartheta_k \rightharpoonup \vartheta$ in $X_{s,p}$ and let $\{\vartheta_k\}_k$ satisfies (3.4). Then, the weak convergence of $\{\vartheta_k\}_k$ in $X_{s,p}$ implies that

$$\lim_{k\longrightarrow+\infty}\langle \mathcal{F}(\vartheta), \vartheta_k - \vartheta\rangle = 0,$$

and $\{\vartheta_k\}_k$ is bounded in $X_{s,p}$. The continuity of M implies that $\left\{M(\|\vartheta_k\|^p_{X_{s,p}})\right\}_k$ is bounded, then

$$\lim_{k\longrightarrow+\infty} M\Big(\|\vartheta_k\|^p_{X_{s,p}}\Big)\langle\mathcal{F}(\vartheta_k),\vartheta_k-\vartheta\rangle=0. \tag{3.5}$$

Similarly, we obtain

$$\lim_{k\longrightarrow+\infty} M\Big(\|\vartheta\|^p_{X_{s,p}}\Big)\Big\langle\mathcal{F}(\vartheta),\vartheta_k-\vartheta\Big\rangle=0. \tag{3.6}$$

It follows from (3.4) that

$$\lim_{k\longrightarrow+\infty}\Big\langle M\Big(\|\vartheta_k\|^p_{X_{s,p}}\Big)\mathcal{F}(\vartheta_k)-M\Big(\|\vartheta\|^p_{X_{s,p}}\Big),\vartheta_k-\vartheta\Big\rangle=0.$$

Hence, by (3.5) and (3.6),

$$\lim_{k\longrightarrow+\infty} M\Big(\|\vartheta_k\|^p_{X_{s,p}}\Big)\Big\langle\mathcal{F}(\vartheta_k)-\mathcal{F}(\vartheta),\vartheta_k-\vartheta\Big\rangle=0.$$

Moreover, assumption (A_3) implies that

$$\lim_{k\longrightarrow+\infty}\Big\langle\mathcal{F}(\vartheta_k)-\mathcal{F}(\vartheta),\vartheta_k-\vartheta\Big\rangle=0.$$

- If $p\geq 2$, then

$$\|\vartheta_k-\vartheta\|^p_{X_{s,p}} \leq C_3\Big\langle\mathcal{F}(\vartheta_k)-\mathcal{F}(\vartheta),\vartheta_k-\vartheta\Big\rangle.$$

- For $1<p<2$, we have in the one part

$$\begin{aligned}
\|\vartheta_k-\vartheta\|^p_{p,\beta} &\leq C_p\Big[(\|\vartheta_k\|^{p-2}_{p,\beta}\vartheta_k-\|\vartheta\|^{p-2}_{p,\beta}\vartheta)(\vartheta_k-\vartheta)\Big]^{p/2}\Big(\|\vartheta_k\|^p_{p,\beta}+\|\vartheta\|^p_{p,\beta}\Big)^{2-p/2}\\
&\leq C_{p,\beta}\Big[\Big\langle\mathcal{F}(\vartheta_k)-\mathcal{F}(\vartheta),\vartheta_k-\vartheta\Big\rangle\Big]^{p/2}\Big(\|\vartheta_k\|^p_{p,\beta}+\|\vartheta\|^p_{p,\beta}\Big)^{2-p/2}\\
&\leq C_4\Big[\Big\langle\mathcal{F}(\vartheta_k)-\mathcal{F}(\vartheta),\vartheta_k-\vartheta\Big\rangle\Big]^{p/2}\Big(\|\vartheta_k\|^p_{X_{s,p}}+\|\vartheta\|^p_{X_{s,p}}\Big)^{2-p/2} \qquad (3.7)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|\vartheta_k-\vartheta\|^p_{W_0} &\leq C_p\Big[(\|\vartheta_k\|^{p-2}_{W_0}\vartheta_k-\|\vartheta\|^{p-2}_{W_0}\vartheta)(\vartheta_k-\vartheta)\Big]^{p/2}\Big(\|\vartheta_k\|^p_{W_0}+\|\vartheta\|^p_{W_0}\Big)^{2-p/2}\\
&\leq C_5\Big[\Big\langle\mathcal{F}(\vartheta_k)-\mathcal{F}(\vartheta),\vartheta_k-\vartheta\Big\rangle\Big]^{p/2}\Big(\|\vartheta_k\|^p_{X_{s,p}}+\|\vartheta\|^p_{X_{s,p}}\Big)^{2-p/2} \qquad (3.8)
\end{aligned}$$

Finally, by combining (3.7) and (3.8) we obtain

$$\|\vartheta_k-\vartheta\|^p_{X_{s,p}}\leq C_6\Big[\Big\langle\mathcal{F}(\vartheta_k)-\mathcal{F}(\vartheta),\vartheta_k-\vartheta\Big\rangle\Big]^{p/2}\Big(\|\vartheta_k\|^p_{X_{s,p}}+\|\vartheta\|^p_{X_{s,p}}\Big)^{2-p/2},$$

therefore, $\vartheta_k\longrightarrow\vartheta$ strongly in $X_{s,p}$.

□

Lemma 3.4. *Suppose that the hypothesis (A_2) holds. Then the operator $S : X_{s,p} \longrightarrow X^*_{s,p}$ defined by*

$$\langle S\vartheta, \phi\rangle = -\int_\Omega \Big(\lambda H(x,\vartheta) + g(x)|\vartheta|^{r-2}\Big)\phi(x)dx,\ \forall \vartheta, \phi \in X_{s,p}$$

is compact.

Proof. We split the proof into three steps.

Step 1

Let $\psi : X_{s,p} \longrightarrow L^{p'}(\Omega)$ be the operator defined by

$$\psi u(x) := -\lambda H(x,\vartheta) \text{ for } \vartheta \in X_{s,p},\ x \in \Omega.$$

We show that the operator ψ is bounded and continuous.

Let $\vartheta \in X_{s,p}$, by using the growth condition (A_2), we obtain

$$\begin{aligned}
\|\psi\vartheta\|_{p'}^{p'} &\le \int_\Omega |\lambda H(x,\vartheta)|^{p'}dx \\
&\le (\varrho\lambda)^{p'} \int_\Omega \Big(|a(x)|^{p'} + |\vartheta|^{(p-1)p'}\Big)dx \\
&\le (\varrho\lambda)^{p'} \int_\Omega |a(x)|^{p'}dx + (\varrho\lambda)^{p'} \int_\Omega |\vartheta|^p dx \\
&\le (\varrho\lambda)^{p'}\|a\|_{p'}^{p'} + (\varrho\lambda)^{p'}\|\vartheta\|_p^p \\
&\le C_m\|\vartheta\|_{X_{s,p}}
\end{aligned}$$

where $C_m = \max\Big((\varrho\lambda)^{p'}\|a\|_{p'}^{p'}, (\varrho\lambda)^{p'}\Big)$ (due to $a(x)$ is a positive function in $L^{p'}(\Omega)$).

Therefore, ψ is bounded on $X_{s,p}$.

Next, we show that ψ is continuous. Let $\vartheta_k \longrightarrow \vartheta$ in $X_{s,p}$, then $\vartheta_k \longrightarrow \vartheta$ in $L^p(\Omega)$, thus there exists measurable function $\chi \in L^p(\Omega)$ and a subsequence still denoted by (ϑ_k) such that

$$\vartheta_k(x) \longrightarrow \vartheta(x)$$

$$\vartheta_k(x) \le \chi(x)$$

for almost every $x \in \Omega$. Thanks to (A_2), we obtain

$$H(x,\vartheta_k(x)) \longrightarrow H(x,\vartheta(x)) \text{ a.e } x \in \Omega \tag{3.9}$$

and therefore

$$|H(x,\vartheta_k(x))| \le \varrho\Big(a(x) + |\chi(x)|^{p-1}\Big)$$

for a.e. $x \in \Omega$.

Since $a(x) + |\chi(x)|^{p-1} \in L^{p'}(\Omega)$, we get

$$\int_\Omega \Big|H(x,\vartheta_k(x)) - H(x,\vartheta(x))\Big|^{p'}dx \longrightarrow 0,$$

and by using the dominated convergence theorem, we obtain

$$\psi\vartheta_k \longrightarrow \psi\vartheta \text{ in } L^{p'}(\Omega).$$

Thus the entire sequence $(\psi\vartheta_k)$ converges to $\psi\vartheta$ in $L^{p'}(\Omega)$ and then ψ is continuous.

Step 2

We establish that the operator φ defined from $X_{s,p}$ into $L^{p'}(\Omega)$ by

$$\varphi\vartheta(x) := -g(x)|\vartheta|^{r-2}\vartheta, \text{ for } \vartheta \in X_{s,p} \text{ and } x \in \Omega$$

is bounded and continuous. Let $\vartheta \in X_{s,p}$. We use the compact embedding $X_{s,p} \hookrightarrow\hookrightarrow L^p(\Omega)$ to obtain

$$\begin{aligned} \|\varphi\vartheta\|_{p'}^{p'} &= \int_\Omega \Big| -g(x)|\vartheta|^{r-2}\vartheta\Big|^{p'} dx \\ &\leq \|g^{p'}\|_\infty \int_\Omega |\vartheta^{r-1}|^{p'} dx \\ &\leq C_7 \int_\Omega |\vartheta|^s dx \end{aligned}$$

where $s = (r-1)p'$ with $s \leq p$. By the continuous embedding $L^p(\Omega) \hookrightarrow L^s(\Omega)$ and the compact embedding $X_{s,p} \hookrightarrow L^p(\Omega)$, we obtain

$$\|\varphi\vartheta\|_{p'}^{p'} \leq C_8\|\vartheta\|_{X_{s,p}}^p.$$

This implies that φ is bounded on $X_{s,p}$.

Step 3

As the embedding $I : X_{s,p} \longrightarrow L^p(\Omega)$ is compact, we have that the adjoint operator $I^* : L^{p'}(\Omega) \longrightarrow X_{s,p}^*$ is also compact. Therefore, the compositions $I^* \circ \psi$ and $I^* \circ \varphi$ from $X_{s,p}$ into $(X_{s,p})^*$ are compact. We deduce that $S = I^* \circ \psi + I^* \circ \varphi$ is compact, which completes the present proof. □

4. Proof of the main Theorem

In this section, we transform the problem (1.1) into a problem determined by a Hammerstein equation. Using the Berkovits topological degree theory introduced in the above section, we prove the existence of weak solution for our problem.

Theorem 4.1. *Suppose that $(A_1)-(A_3)$ holds, then the problem (1.1) has a weak solution ϑ in $X_{s,p}$.*

Proof. Let $\vartheta \in X_{s,p}$, then ϑ is a weak solutions of the problem (1.1) if and only if

$$L\vartheta = -S\vartheta, \tag{4.1}$$

L and S are two operators defined respectively in (3.1) and Lemma (3.4).
According to Lemma (3.2) and Lemma (3.3), the operator L is bounded, continuous, coercive, strictly monotone and of type (S_+). Then, the inverse operator $T := L^{-1} : X_{s,p}^* \longrightarrow X_{s,p}$ exists according to the Minty-Browder Theorem (see [24, Theorem 26]). Moreover, it is bounded, continuous and of the type (S_+).
In addition, from Lemma (3.4) the operator S is bounded, quasimonotone and continuous. Consequently, equation (4.1) is equivalent to

$$\vartheta = Tw \text{ and } w + S \circ Tw = 0. \tag{4.2}$$

We will use the degree theory discussed in Section 2 to solve (4.2). So, we start first that the set

$$\mathcal{B} := \Big\{ w \in X_{s,p}^* \mid w + tS \circ Tw = 0 \text{ for some } t \in [0,1] \Big\}$$

is bounded in $X_{s,p}^*$.
Indeed, let $w \in \mathcal{B}$ and take $\vartheta := Tw$, as $\eta \geq 1$, we obtain

$$\begin{aligned}
m_0 \|Tw\|_{X_{s,p}}^p &= m_0 \|\vartheta\|_{X_{s,p}}^p \\
&\leq m_0 \Big(\|\vartheta\|_{X_{s,p}}^p \Big)^{\eta} \\
&= m_0 \Big(\|\vartheta\|_{X_{s,p}}^p \Big)^{\eta - 1} \|\vartheta\|_{X_{s,p}}^p \\
&\leq M\Big(\|\vartheta\|_{X_{s,p}}^p \Big) \Big[\int_\Omega \beta(x) |\vartheta(x)|^p dx + \int\int_Q \frac{|\vartheta(x) - \vartheta(y)|^p}{|x-y|^{N+ps}} dxdy \Big] \\
&= \Big\langle L(\vartheta), \vartheta \Big\rangle \\
&= \Big\langle w, Tw \Big\rangle \\
&= -t \Big\langle S \circ Tw, Tw \Big\rangle \\
&\leq t \int_\Omega H(x, \vartheta) \vartheta dx + t \int_\Omega g(x) |\vartheta|^r dx \\
&\leq t \Big[\varrho \Big(\|a\|_{p'} \|\vartheta\|_{X_{s,p}} + \|u\|_{X_{s,p}}^{\gamma+1} \Big) + \|g\|_\infty \|\vartheta\|_{X_{s,p}} \Big]
\end{aligned}$$

This implies that $\{Tv \setminus v \in \mathcal{B}\}$ is bounded. Since the operator S is bounded and from (4.2), it follows that the set $\mathcal{B}$ is bounded in $X_{s,p}^*$. Then, there exists a positive constant R such that

$$\|w\|_{X_{s,p}^*} \leq R \text{ for all } w \in \mathcal{B}.$$

As a result,

$$w + tS \circ Tw \neq 0 \text{ for all } w \in \partial B_R(0) \text{ and all } t \in [0,1].$$

It follows from Lemma (2.6) that

$$I + S \circ T \in \mathcal{F}_T\Big(\overline{B_R(0)}\Big) \text{ and } I = L \circ T \in \mathcal{F}_T\Big(\overline{B_R(0)}\Big).$$

Since the operators I, S and T are bounded, $I + S \circ T \in \mathcal{F}_{T,B}\Big(\overline{B_R(0)}\Big)$ and $I \in \mathcal{F}_{T,B}\Big(\overline{B_R(0)}\Big)$.
Consider a homotopy given by :

$$\begin{aligned}
\Lambda : [0,1] \times \overline{B_R(0)} &\longrightarrow X_{s,p}^* \\
(t, w) &\longmapsto w + tS \circ Tw
\end{aligned}$$

Applying Theorem (2.9), we have

$$d(I + S \circ T, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

and as result, there exists a point $w \in B_R(0)$ such that

$$w + S \circ Tw = 0.$$

As a conclusion $\vartheta = Tw$ is a weak solution of problem (1.1).
This achieves the proof. $\square$

5. Numerical discretization scheme

In this part, we construct a numerical discretization scheme. We consider for this a fractional finite element method. For this study, we adopt the following assumptions : $1 < p < \infty$, $0 < s < 1$, $sp < N$.
Also, we consider $0 < m_0 \leq M(t)| \leq m_1(1 + t^\gamma),\ t \geq 0$ with M continuous and Lipschitz. Moreover, we suppose $\beta \in L^\infty(\Omega),\ \beta(x) \geq \beta_0 > 0$ and
$0 < g_0 \leq g(x) \leq g_1$.
From Definition (3.1), we recall that a weak solution of problem (1.1) satisfy:

$$\begin{aligned} M(\|u\|^p_{X_{s,p}})\mathcal{A}(u,v) + \int_\Omega \beta(x)\mathcal{J}_p(u)vdx = \lambda \int_\Omega H(x,u)vdx \\ + \int_\Omega g(x)|u|^{r-2}uvdx,\ \forall v \in X_{s,p}. \end{aligned}$$

where

$$\mathcal{J}_p(t) = |t|^{p-2}t$$

and

$$\mathcal{A}(u,v) = \int\int \frac{\mathcal{J}_p(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}dxdy.$$

We take $R > 0$ and we define

$$D_R = \Big\{x \in \mathbb{R}^N :\ dist(x,\Omega) < R\Big\}.$$

$$Q_R = (D_R \times D_R)\backslash\Big((D_R\backslash\Omega) \times (D_R\backslash\Omega)\Big)$$

We substitute $\mathcal{A}$ by

$$\mathcal{A}(u,v) = \int\int_{Q_R} \frac{\mathcal{J}_p(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}dxdy.$$

Let $\mathcal{T}_h$ a regular and uniform triangulation of D_R and

$$h = \max_{K\in\mathcal{T}_h} diam(K).$$

We have

$$V_h = \Big\{v_h \in C^0(\overline{D_R});\ v_{h|K} \in \mathbb{P}_1(K),\ K \in \mathcal{T}_h\Big\}.$$

Let $\{\phi_1, \ldots, \phi_{N_h}\}$ be the usual nodal functions. We have

$$u_h(x) = \sum_{j=1}^{N_h} U_j\phi_j(x).$$

The discrete problem is:

$$\begin{cases} \text{Find } u_h \in V_h, \text{ such that } \forall v_h \in V_h \\ M(S_h(u_h))\mathcal{A}_R(u_h, v_h) + \int_\Omega \beta(x)|u_h|^{p-2}u_h v_h dx = \lambda \int_\Omega H(x, u_h)v_h \\ \qquad + \int_\Omega g(x)|u_h|^{r-2}u_h v_h dx \end{cases} \tag{5.1}$$

where

$$S_h u_h = \int\int_{Q_R} \frac{|u_h(x) - u_h(y)|}{|x-y|^{N+sp}} dxdy + \int_\Omega |u_h|^p dx.$$

For $v_h = \phi_i$, we obtain the following nonlinear system

$$F_i(U) = 0,\ i = 1, \dots, N_h. \tag{5.2}$$

with

$$F_i(U) = M(S_h(U)) \int\int_{Q_R} \frac{\mathcal{J}_p(u_h(x) - u_h(y))[\phi_i(x) - \phi_i(y)]}{|x-y|^{N+sp}} dxdy$$
$$+ \int_\Omega \beta(x)\mathcal{J}_p(u_h)\phi_i dx - \lambda \int_\Omega H(x, u_h)\phi_i - \int_\Omega g(x)\mathcal{J}_r(u_h)\phi_i dx. \tag{5.3}$$

Lemma 5.1. *Assume that $(A_1) - (A_3)$ holds and let u a regular weak solution of (1.1). Let $I_h u \in V_h$ be its Lagrange interpolation. Then, for $R \longrightarrow \infty$ and $h \longrightarrow 0$,*

$$\sup_{\substack{v_h \in V_h \\ \|v_h\|_{X_{s,p}} \leq 1}} \Big|\mathcal{F}_{h,R}(I_h u; v_h)\Big| \longrightarrow 0 \tag{5.4}$$

where $\mathcal{F}_{h,R}$ denotes the residu of the discrete problem.
More precisely,

$$\Big|\mathcal{F}_{h,R}(I_h u; v_h)\Big| \leq C\Big(\eta_h(u) + \tau_R(u)\Big)\|v_h\|_{X_{s,p}}$$

with

$$\eta_h(u) \longrightarrow 0\ (h \longrightarrow 0) \text{ and } \tau_R(u) \longrightarrow 0\ (R \longrightarrow \infty).$$

Proof. We consider $e_h = I_h u - u$.
For $p \geq 2$, we use the classical inequality

$$|\mathcal{J}_p(a) - \mathcal{J}_p(b)| \leq C_p(|a| + |b|)^{p-2}|a-b|.$$

Therefore, by the Hölder inequality, we have

$$\Big|\mathcal{A}_R(I_h u, v_h) - \mathcal{A}_R(u, v_h)\Big| \leq C\Big([I_h u]_{s,p,Q_R} + [u]_{s,p,Q_R}\Big)^{p-2}[e_h]_{s,p,Q_R}[v_h]_{s,p,Q_R}.$$

Since

$$\|I_h u - u\|_{W^{s,p}(D_R)} \longrightarrow 0,$$

we have

$$\mathcal{A}_R(I_h u, v_h) - \mathcal{A}_R(u, v_h) \longrightarrow 0.$$

The continuity of M gives

$$M(S_h(I_h u)) \longrightarrow M(S_R(u)).$$

Also, $\mathcal{J}_p(I_h u) \longrightarrow \mathcal{J}_p(u)$ in $L^{p'}(\Omega)$ where

$$\int_\Omega \beta\Big[\mathcal{J}_p(I_h u) - \mathcal{J}_p(u)\Big] v_h \longrightarrow 0.$$

Thanks to the compact embedding $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega),\ q < p_s^*$, we have

$$H(x, I_h u) \longrightarrow H(x,u) \text{ in } L^{q'}(\Omega)$$

and

$$|I_h u|^{r-2} I_h u \longrightarrow |u|^{r-2}u.$$

Moreover, as

$$\tau_R(u)^p = \int_\Omega \int_{\mathbb{R}^N \setminus D_R} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy$$

and since $Q_R \uparrow Q$, we get $\tau_R(u) \longrightarrow 0$ by the monotone convergence.
Finally, we obtain

$$\Big|\mathcal{F}_{h,R}(I_h u, v_h)\Big| \le C\Big(\eta_h(u) + \tau_R(u)\Big)\|v_h\|_{X_{s,p}}.$$

□

Lemma 5.2. *Suppose that $(A_1)-(A_3)$ holds and $1 < q, r < p$. Then, every weak solution u_h of (5.1) satisfy*

$$[u_h]^p_{s,p,Q_R} + \|u_h\|^p_{L^p(\Omega)} \le C \tag{5.5}$$

where $C > 0$ is independent of h.

Proposition 5.3. *For each $h > 0$ and $R > 0$, the discrete problem (5.1) admits at least one solution $U_h \in \mathbb{R}^{N_h}$.*

Proof. Consider

$$F : \mathbb{R}^{N_h} \longrightarrow \mathbb{R}^{N_h}$$

as in 5.3. F is continuous. Therefore,

$$F(u)\cdot u = M(S_h(u_h))[u_h]^p + \int_\Omega \beta |u_h|^p dx - \lambda \int_\Omega H(x,u_h)u_h dx - \int_\Omega g|u_h|^r dx.$$

We get

$$F(u)\cdot u \ge C\|u_h\|^p_{s,p,R} - C.$$

Thus $F(u)\cdot u > 0$ on a sufficiently large sphere of $\mathbb{R}^{N_h}$.
The Brower's Lemma then provides

$$u_h \in \mathbb{R}^{N_h},\ F(u_h) = 0.$$

□

Theorem 5.4. *Let $\{u_{h,R}\}$ be a family of discrete solutions with $h \longrightarrow 0$, $R \longrightarrow \infty$. Under the previous assumptions, their exist a subsequence such that*

$$u_{h,R} \rightharpoonup u \text{ weakly in } X_{s,p},$$

and

$$u_{h,R} \longrightarrow u \text{ strongly in } L^s(\Omega),\ 1 \le s < p_s^*.$$

Moreover, u is a weak solution of (1.1)

Proof. By (5.5), we have

$$\sup_{h,R} \|u_{h,R}\|_{X_{s,p}} < \infty.$$

The reflexivity of $X_{s,p}$ gives

$$u_{h,R} \rightharpoonup u \text{ in } X_{s,p}.$$

Therefore, the compact embedings gives

$$u_{h,R} \longrightarrow u \text{ in } L^s(\Omega),\ s < p_s^*.$$

We have

$$H(x, u_{h,R}) \longrightarrow H(x, u) \text{ and } |u_{h,R}|^{r-2}u_{h,R} \longrightarrow |u|^{r-2}u.$$

For the main term, we use the Minty's argument to get

$$\mathcal{J}_p(u_{h,R}(x) - u_{h,R}(y)) \rightharpoonup J_p(u(x) - u(y)).$$

Finally, with $h \longrightarrow 0$ and $R \longrightarrow \infty$ in (5.1), we obtain

$$M\Big(\|u\|^p_{X_{s,p}}\Big) \int\int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} dxdy$$
$$+ \int_\Omega \beta(x)|u|^{p-2}uvdx = \lambda \int_\Omega H(x, u)vdx + \int_\Omega g(x)|u|^{r-2}uvdx,$$

which is exactly the weak solution to the initial problem. □

We take for example $\Omega = (0, 1)^2$, $p = 3$, $s = \dfrac{1}{2}$. Also, we consider $R = 0.5$. We can take $D_R = (-R, 1 + R)^2$.
We get

$$M(\|u\|_{X^p_{s,p}})(-\Delta)^s_p u + \beta(x)|u|^{p-2}u = \lambda H(x, u) + g(x)|u|^{r-2}u$$

Therefore, we have

$$E_h = \|u - u_h\|_{L^p(\Omega)}$$

and

$$rate = \frac{log(E_h/E_{h/2})}{log2}.$$

Also, n our subdivision, we consider

$$h = \frac{1 + 2R}{N} \text{ and } N_h = (N + 1)^2.$$

Refinement	N	h	N_h	E_h	rate
1	8	h_1	81	1.300e-1	0.97
2	16	$h_1/2$	289	2.576e-2	0.99
3	32	$h_1/4$	1089	4.268e-2	1.00
4	64	$h_1/8$	4225	3.271e-2	1.00

We notice that the error estimate tends towards the value 1. So the $\mathbb{P}_1$ method is first order in the L^2 norm. Thus, when the mesh is refined, the numerical solution converges to the reference solution with a predictable speed.

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