Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 36 (2026) 1-43 https://doi.org/10.56947/amcs.v36.950

HORIZONTAL GRADIENT FLOWS ON THE HEISENBERG GROUP

ISHTAQ AHMAD[1] * AND NEYAZ AHMAD SHEIKH[2]

Abstract. This paper studies horizontal gradient flows on the Heisenberg group, an important example of a sub-Riemannian manifold. The analysis focuses on horizontal differential operators and their role in describing evolution processes governed by the intrinsic geometry of the group. The work highlights the structure of the horizontal gradient and related operators, providing insight into gradient-driven dynamics and partial differential equations in non-commutative geometric settings.

Keywords. Heisenberg group, horizontal gradient flow, sub-Riemannian geometry, Carnot groups, sub-Laplacian, geometric analysis.
2020 Mathematics Subject Classification. 53C17, 35R03, 22E25, 35K55.

1. Introduction

The analysis of differential equations and geometric structures on non–commutative Lie groups has become a fundamental topic in modern harmonic analysis and geometric analysis. Among such groups, the Heisenberg group $\mathbb{H}^n$ plays a central role due to its rich algebraic structure and its deep connections with sub–Riemannian geometry, quantum mechanics, and time–frequency analysis. As the simplest example of a stratified nilpotent Lie group, the Heisenberg group provides a natural framework for studying hypoelliptic operators, anisotropic diffusion, and non–Euclidean geometric flows [11, 19, 18, 22, 9, 13].

One of the distinguishing features of the Heisenberg group is the presence of a horizontal distribution generated by left–invariant vector fields that form the first layer of the corresponding Lie algebra. Differential operators constructed from these vector fields capture the intrinsic geometry of the space and lead to the notion of horizontal calculus. In particular, the horizontal gradient operator plays a crucial role in the formulation of variational problems and evolution equations in the sub–Riemannian setting. This operator replaces the classical Euclidean gradient and reflects the anisotropic structure of the underlying manifold.

The study of horizontal differential operators naturally leads to the investigation of the sub–Laplacian, which is the fundamental hypoelliptic operator on

Date: Received: Aug 17, 2026; Revised: Sep 15, 2026; Accepted: Sep 19, 2026.
* Corresponding author

the Heisenberg group. The sub–Laplacian arises in several contexts, including diffusion processes, control theory, and geometric analysis. Its spectral properties, regularity theory, and associated heat kernel estimates have been extensively investigated in the literature [10, 16, 24]. These developments have significantly advanced the understanding of sub–elliptic partial differential equations and their long–time behavior on stratified Lie groups.

In recent years, there has been growing interest in studying evolution equations driven by horizontal differential operators. Such equations appear naturally in models of diffusion and energy dissipation in non–Euclidean geometries. Horizontal gradient flows, in particular, describe the evolution of functions under the action of gradient–type operators adapted to the sub–Riemannian structure. These flows are closely related to energy minimization problems and play an important role in understanding the stability and asymptotic behavior of solutions to sub–elliptic partial differential equations.

The theory of gradient flows in Euclidean spaces is well established and has deep connections with nonlinear diffusion equations, entropy methods, and variational principles. However, extending these ideas to the Heisenberg group requires substantial modifications due to the non–commutative nature of the group and the presence of anisotropic dilations. The horizontal gradient operator introduces additional analytical challenges, particularly in the study of regularity properties and stability estimates for solutions of the associated evolution equations.

Another important motivation for studying analytic problems on the Heisenberg group comes from harmonic analysis and representation theory. The Heisenberg group admits a rich family of unitary representations that form the foundation of modern time–frequency analysis and quantum harmonic analysis [23, 19, 20, 21]. Representation–theoretic approaches have proven particularly effective in analyzing integral transforms and operator structures arising in signal processing and mathematical physics. In this direction, recent work has developed representation–theoretic frameworks for generalized canonical transforms and their analytic properties in quantum harmonic analysis [1].

Furthermore, the interplay between harmonic analysis and functional spaces has stimulated significant research in wavelet theory, frame analysis, and multiresolution techniques on non–Euclidean domains. Although these developments originated in signal processing and approximation theory, they have provided useful analytical tools for studying operators and function spaces on generalized structures. For example, wavelet frame constructions and scaling function characterizations on non–Archimedean fields have been investigated in [2, 3], demonstrating the adaptability of harmonic analysis techniques in non–classical analytic settings.

Motivated by these developments, the purpose of this paper is to investigate horizontal gradient flows on the Heisenberg group and to analyze the evolution of associated energy functionals under sub–elliptic dynamics. Our approach emphasizes the intrinsic geometry of the horizontal distribution and the analytical properties of the associated differential operators. In particular, we establish structural relations between the horizontal gradient operator and the sub–Laplacian and derive several analytical results concerning stability, energy decay, and qualitative properties of the resulting flow equations.

The results presented here contribute to the broader program of understanding evolution equations and geometric flows on stratified Lie groups. In particular, they highlight the deep connections between geometric structures, harmonic analysis, and partial differential equations in non–commutative settings. We expect that the analytical framework developed in this work will be useful for further investigations of sub–elliptic evolution equations and geometric variational problems on the Heisenberg group and related spaces.

Organization of the paper. The remainder of this paper is organized as follows. In Section 2, we review the necessary preliminaries on the Heisenberg group and the associated horizontal differential operators. Section 3 introduces the concept of horizontal gradient flows and discusses their fundamental properties. In Section 4, we present the main analytical results related to these flows and examine their behavior within the sub-Riemannian framework. Finally, Section 5 concludes the paper with remarks and possible directions for future research.

2. Preliminaries

In this section we recall several structural and analytical concepts related to the Heisenberg group that will be required for the formulation of horizontal gradient flows. The analytical framework presented here is based on sub–Riemannian geometry and harmonic analysis on stratified Lie groups. Standard references include [11, 19, 24, 12, 15, 17].

The Heisenberg group serves as the fundamental example of a non–commutative nilpotent Lie group and provides a natural setting for studying hypoelliptic operators and geometric evolution equations.

Definition 2.1. For $n \in \mathbb{N}$, the *n–dimensional Heisenberg group* $\mathbb{H}^n$ is defined as the manifold

$$\mathbb{H}^n = \mathbb{R}^{2n} \times \mathbb{R} \tag{2.1}$$

endowed with the group operation

$$(x, y, t) \circ (x', y', t') = \left(x + x', y + y', t + t' + \frac{1}{2}(x \cdot y' - y \cdot x') \right), \tag{2.2}$$

where $x, y, x', y' \in \mathbb{R}^n$ and $t, t' \in \mathbb{R}$.

The algebraic structure defined above induces a Lie algebra generated by vector fields that determine the horizontal directions of the space.

Definition 2.2. The family of *left–invariant horizontal vector fields* on $\mathbb{H}^n$ is defined by

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_t, \qquad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_t, \quad j = 1, \dots, n. \tag{2.3}$$

The linear span

$$\mathcal{H} = \operatorname{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\} \tag{2.4}$$

is called the horizontal bundle of $\mathbb{H}^n$.

The commutation relations between these vector fields reveal the non–commutative structure of the group.

$$[X_j, Y_k] = \delta_{jk}\partial_t, \qquad [X_j, X_k] = [Y_j, Y_k] = 0, \tag{2.5}$$

where δ_{jk} denotes the Kronecker symbol. These relations imply that the Lie algebra satisfies the Hörmander bracket generating condition, which guarantees hypoellipticity of the associated differential operators [16].

The intrinsic gradient operator on $\mathbb{H}^n$ must be defined with respect to the horizontal directions introduced above.

Definition 2.3. Let $u : \mathbb{H}^n \to \mathbb{R}$ be a smooth function. The *horizontal gradient* is defined by

$$\nabla_H u = \sum_{j=1}^{n} (X_j u) X_j + (Y_j u) Y_j. \tag{2.6}$$

The horizontal gradient plays a central role in variational formulations of sub–elliptic problems and in the construction of gradient flows.

Associated with this operator is a second–order differential operator which governs diffusion processes on the Heisenberg group.

Definition 2.4. The *sub–Laplacian* on $\mathbb{H}^n$ is the differential operator

$$\Delta_H = \sum_{j=1}^{n} (X_j^2 + Y_j^2). \tag{2.7}$$

The operator Δ_H is hypoelliptic and generates the fundamental diffusion semigroup on $\mathbb{H}^n$. Its spectral and analytic properties have been extensively studied in harmonic analysis and geometric PDE theory [10, 19].

An important feature of the Heisenberg group is the presence of anisotropic dilations compatible with the stratified Lie algebra structure.

Definition 2.5. For $\lambda > 0$, the *group dilation* $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ is defined by

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t). \tag{2.8}$$

These dilations induce a natural notion of homogeneity which is essential in scaling arguments and functional inequalities.

Definition 2.6. The *homogeneous dimension* of $\mathbb{H}^n$ is defined as

$$Q = 2n + 2. \tag{2.9}$$

This quantity replaces the Euclidean dimension in several analytic estimates on the Heisenberg group, including Sobolev and Hardy inequalities [6].

We now introduce an intrinsic norm that reflects the geometry of the group.

Definition 2.7. For $\xi = (x, y, t) \in \mathbb{H}^n$, the *Korányi gauge norm* is defined by

$$|\xi|_H = \left((|x|^2 + |y|^2)^2 + 16t^2\right)^{1/4}. \tag{2.10}$$

The Korányi norm is homogeneous with respect to the dilations (2.8) and satisfies the relation

$$|\delta_\lambda(\xi)|_H = \lambda|\xi|_H. \tag{2.11}$$

Finally, we introduce the energy functional that will generate the horizontal gradient flow studied in subsequent sections.

Definition 2.8. Let $u : \mathbb{H}^n \to \mathbb{R}$ be a sufficiently smooth function. The *horizontal Dirichlet energy* is defined by

$$E(u) = \frac{1}{2}\int_{\mathbb{H}^n} |\nabla_H u(\xi)|^2 \, d\xi. \tag{2.12}$$

Energy functionals of the form (2.12) arise naturally in sub–elliptic variational problems and play a fundamental role in the construction of horizontal gradient flows. In particular, the evolution equations studied later in this work can be interpreted as the gradient flow associated with the energy functional (2.12).

The analytical tools introduced in this section will serve as the foundation for the study of horizontal gradient flow equations and their stability properties in the subsequent sections.

3. Gradient Flow Structures on the Heisenberg Group

In this section we introduce the horizontal gradient flow associated with the intrinsic energy functional defined in (2.12). The results presented here describe several structural and dynamical properties of the flow on the Heisenberg group. Throughout this section we consider sufficiently smooth functions $u : \mathbb{H}^n \times (0,\infty) \to \mathbb{R}$ and adopt the notation introduced in the previous section. Related analytical frameworks for sub–elliptic evolution equations may be found in [19, 24].

We first define the intrinsic gradient flow generated by the horizontal Dirichlet energy.

Definition 3.1. Let $E(u)$ be the horizontal energy functional defined in (2.12). The *horizontal gradient flow* associated with E is the evolution equation

$$\partial_t u(\xi,t) = -\nabla_H^* \nabla_H u(\xi,t), \qquad (\xi,t) \in \mathbb{H}^n \times (0,\infty), \tag{3.1}$$

where ∇_H^* denotes the formal adjoint of the horizontal gradient operator.

Using the identity $\nabla_H^* \nabla_H = -\Delta_H$, equation (3.1) can be rewritten in the form

$$\partial_t u(\xi,t) = \Delta_H u(\xi,t). \tag{3.2}$$

The equation (3.2) represents a diffusion process governed by the intrinsic geometry of the Heisenberg group.

We begin by establishing the well–posedness of the horizontal gradient flow associated with the sub–Laplacian on the Heisenberg group. The following theorem guarantees the existence and uniqueness of a weak solution for initial data with finite horizontal energy.

Theorem 3.2 (Existence of Horizontal Gradient Flow). *Let $u_0 \in L^2(\mathbb{H}^n)$ and suppose u_0 possesses finite horizontal energy. Then there exists a unique weak solution*

$$u \in L^2((0,T); H_H^1(\mathbb{H}^n))$$

to the evolution problem

$$\begin{cases} \partial_t u = \Delta_H u, \\ u(\xi, 0) = u_0(\xi), \end{cases} \tag{3.3}$$

for every $T > 0$.

Proof. The proof follows from standard methods for evolution equations generated by sub-elliptic operators on the Heisenberg group.

First, recall that the horizontal Laplacian Δ_H is defined as

$$\Delta_H = \sum_{i=1}^{n} \left(X_i^2 + Y_i^2 \right),$$

where $\{X_i, Y_i\}_{i=1}^n$ denote the left-invariant horizontal vector fields generating the horizontal distribution on $\mathbb{H}^n$. This operator is hypoelliptic and symmetric on $L^2(\mathbb{H}^n)$.

Let $H_H^1(\mathbb{H}^n)$ denote the horizontal Sobolev space equipped with the norm

$$\|u\|_{H_H^1}^2 = \int_{\mathbb{H}^n} \left(|u|^2 + |\nabla_H u|^2 \right) d\xi.$$

The associated horizontal energy functional is given by

$$E(u) = \frac{1}{2} \int_{\mathbb{H}^n} |\nabla_H u|^2 \, d\xi.$$

Since $u_0 \in L^2(\mathbb{H}^n)$ with finite horizontal energy, we have $u_0 \in H_H^1(\mathbb{H}^n)$.

Next, we consider the weak formulation of problem (3.3). For any test function $\varphi \in H_H^1(\mathbb{H}^n)$, multiplying the equation by φ and integrating over $\mathbb{H}^n$ yields

$$\int_{\mathbb{H}^n} \partial_t u \, \varphi \, d\xi = \int_{\mathbb{H}^n} \Delta_H u \, \varphi \, d\xi.$$

Using horizontal integration by parts, we obtain

$$\int_{\mathbb{H}^n} \Delta_H u \, \varphi \, d\xi = - \int_{\mathbb{H}^n} \nabla_H u \cdot \nabla_H \varphi \, d\xi.$$

Hence the weak formulation becomes

$$\int_{\mathbb{H}^n} \partial_t u \, \varphi \, d\xi + \int_{\mathbb{H}^n} \nabla_H u \cdot \nabla_H \varphi \, d\xi = 0.$$

The existence of a weak solution follows by applying the Galerkin approximation method. One constructs a sequence of finite-dimensional approximations using a basis of $H_H^1(\mathbb{H}^n)$ and derives uniform energy estimates. These estimates ensure boundedness of the approximate solutions in $L^2((0,T); H_H^1(\mathbb{H}^n))$. Using compactness arguments and weak convergence, one obtains a limit function

$$u \in L^2((0,T); H_H^1(\mathbb{H}^n))$$

that satisfies the weak formulation of (3.3).

Finally, uniqueness follows from the energy method. Let u_1 and u_2 be two weak solutions and set $w = u_1 - u_2$. Then w satisfies

$$\partial_t w = \Delta_H w, \qquad w(\xi, 0) = 0.$$

Multiplying the equation by w and integrating over $\mathbb{H}^n$ gives

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{H}^n} |w|^2 \, d\xi = -\int_{\mathbb{H}^n} |\nabla_H w|^2 \, d\xi \leq 0.$$

Since $w(\xi, 0) = 0$, it follows that

$$\int_{\mathbb{H}^n} |w(\xi, t)|^2 \, d\xi = 0$$

for all $t \geq 0$, which implies $w \equiv 0$. Therefore the solution is unique.

This completes the proof. □

The horizontal gradient flow admits an intrinsic energy dissipation mechanism that reflects the geometric structure of the Heisenberg group.

An essential property of horizontal gradient flows is the monotonic decay of the associated energy functional along the evolution. The following result establishes the intrinsic dissipation mechanism governed by the sub–Laplacian on the Heisenberg group.

Proposition 3.3 (Energy Dissipation)**.** *Let $u(\xi, t)$ be a sufficiently smooth solution of (3.2). Then the horizontal energy defined in (2.12) satisfies*

$$\frac{d}{dt}E(u(t)) = -\int_{\mathbb{H}^n} |\Delta_H u(\xi, t)|^2 \, d\xi. \tag{3.4}$$

Proof. Recall that the horizontal energy functional defined in (2.12) is

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{H}^n} |\nabla_H u(\xi, t)|^2 \, d\xi.$$

We differentiate this quantity with respect to time. Assuming that $u(\xi, t)$ is sufficiently smooth, we obtain

$$\frac{d}{dt}E(u(t)) = \int_{\mathbb{H}^n} \nabla_H u(\xi, t) \cdot \nabla_H(\partial_t u(\xi, t)) \, d\xi. \tag{3.5}$$

Next, using horizontal integration by parts on the Heisenberg group, we obtain

$$\int_{\mathbb{H}^n} \nabla_H u \cdot \nabla_H(\partial_t u) \, d\xi = -\int_{\mathbb{H}^n} \Delta_H u \, \partial_t u \, d\xi. \tag{3.6}$$

Substituting (3.6) into (3.5), we get

$$\frac{d}{dt}E(u(t)) = -\int_{\mathbb{H}^n} \Delta_H u \, \partial_t u \, d\xi. \tag{3.7}$$

Since $u(\xi, t)$ satisfies the horizontal gradient flow equation (3.2), we have

$$\partial_t u = \Delta_H u.$$

Substituting this relation into (3.7) yields

$$\frac{d}{dt}E(u(t)) = -\int_{\mathbb{H}^n} (\Delta_H u)^2 \, d\xi.$$

Hence we obtain

$$\frac{d}{dt}E(u(t)) = -\int_{\mathbb{H}^n} |\Delta_H u(\xi,t)|^2 \, d\xi,$$

which is precisely the energy dissipation identity stated in (3.4). This shows that the horizontal energy is non-increasing along the flow.

□

The previous result shows that the energy functional acts as a Lyapunov functional for the horizontal gradient flow.

Corollary 3.4 (Monotonicity of Horizontal Energy)**.** *Let $u(\xi, t)$ be a solution of* (3.2)*. Then the mapping*

$$t \longmapsto E(u(t))$$

is non–increasing for $t > 0$.

Proof. Let $u(\xi, t)$ be a solution of (3.2) and recall the horizontal energy functional

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{H}^n} |\nabla_H u(\xi,t)|^2 \, d\xi.$$

Differentiating $E(u(t))$ with respect to t, we obtain

$$\frac{d}{dt}E(u(t)) = \int_{\mathbb{H}^n} \langle \nabla_H u(\xi,t), \nabla_H(\partial_t u(\xi,t))\rangle \, d\xi. \tag{3.8}$$

Using integration by parts on the horizontal gradient (valid since u decays sufficiently at infinity or is smooth and compactly supported), and noting that $\partial_t u = \Delta_H u$ from (3.2), we get

$$\frac{d}{dt}E(u(t)) = -\int_{\mathbb{H}^n} |\Delta_H u(\xi,t)|^2 \, d\xi \le 0. \tag{3.9}$$

Hence, $t \mapsto E(u(t))$ is non–increasing for $t > 0$, which proves the corollary. □

Next we establish a decay estimate which depends on the homogeneous dimension introduced in (2.9).

The horizontal diffusion generated by the sub–Laplacian produces a quantitative decay of energy over time. The next result provides an intrinsic L^2 decay estimate reflecting the sub–Riemannian geometry and homogeneous dimension of the Heisenberg group.

Theorem 3.5 (Intrinsic Energy Decay)**.** *Let $u(\xi, t)$ be the solution of* (3.2) *corresponding to initial data $u_0 \in L^2(\mathbb{H}^n)$. Then there exists a constant $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^2(\mathbb{H}^n)} \le C\, t^{-Q/4} \|u_0\|_{L^1(\mathbb{H}^n)}, \qquad t > 0. \tag{3.10}$$

Proof. Let $u(\xi, t)$ be the solution of the horizontal gradient flow equation (3.2) corresponding to the initial datum $u_0 \in L^2(\mathbb{H}^n)$. The operator Δ_H denotes the horizontal sub-Laplacian associated with the system of left-invariant horizontal vector fields generating the horizontal distribution on the Heisenberg group. It is well known that Δ_H is a hypoelliptic, essentially self-adjoint operator on $L^2(\mathbb{H}^n)$ and generates a strongly continuous contraction semigroup $\{e^{t\Delta_H}\}_{t\geq 0}$. Therefore the solution of the initial value problem (3.2) can be written in the semigroup form

$$u(\xi, t) = e^{t\Delta_H} u_0(\xi), \tag{3.11}$$

for every $t > 0$.

The semigroup generated by the horizontal Laplacian admits an integral representation through the heat kernel associated with Δ_H. Denoting by $p_t(\xi)$ the fundamental solution of the heat equation $\partial_t u = \Delta_H u$, the representation formula for the solution can be written as

$$u(\xi, t) = (p_t * u_0)(\xi) = \int_{\mathbb{H}^n} p_t(\eta^{-1} \circ \xi)\, u_0(\eta)\, d\eta, \tag{3.12}$$

where $\circ$ denotes the group law on $\mathbb{H}^n$ and $d\eta$ denotes the Haar measure on the group. This representation shows that the evolution of the solution is governed by convolution with the intrinsic heat kernel of the sub-Laplacian.

The behavior of the kernel $p_t(\xi)$ is closely related to the sub-Riemannian geometry of the Heisenberg group. In particular, the kernel satisfies a homogeneity property with respect to the non-isotropic group dilations. If δ_r denotes the dilation on $\mathbb{H}^n$ defined by the natural grading of the Lie algebra, then the kernel satisfies the scaling relation

$$p_t(\xi) = t^{-Q/2} p_1(\delta_{t^{-1/2}}\xi), \tag{3.13}$$

where Q denotes the homogeneous dimension of the Heisenberg group. This property reflects the intrinsic scaling of the sub-Laplacian and plays a fundamental role in deriving decay estimates.

Using the scaling relation (3.13) together with the invariance of the Haar measure under the dilations, one obtains information on the L^p norms of the kernel. In particular, computing the L^2 norm of p_t and performing the change of variables induced by the dilation $\delta_{t^{-1/2}}$, one obtains

$$\|p_t\|^2_{L^2(\mathbb{H}^n)} = \int_{\mathbb{H}^n} |p_t(\xi)|^2\, d\xi = t^{-Q} \int_{\mathbb{H}^n} |p_1(\delta_{t^{-1/2}}\xi)|^2\, d\xi. \tag{3.14}$$

Applying the change of variables $\zeta = \delta_{t^{-1/2}}\xi$ and using the homogeneity of the Haar measure under dilations yields

$$\int_{\mathbb{H}^n} |p_1(\delta_{t^{-1/2}}\xi)|^2\, d\xi = t^{Q/2} \int_{\mathbb{H}^n} |p_1(\zeta)|^2\, d\zeta.$$

Substituting this relation into (3.14) gives

$$\|p_t\|^2_{L^2(\mathbb{H}^n)} = t^{-Q/2} \int_{\mathbb{H}^n} |p_1(\zeta)|^2\, d\zeta. \tag{3.15}$$

Taking square roots in (3.15) leads to the estimate

$$\|p_t\|_{L^2(\mathbb{H}^n)} \leq C\, t^{-Q/4}, \tag{3.16}$$

where the constant $C > 0$ depends only on the L^2 norm of the kernel p_1 and therefore does not depend on t.

Returning to the representation formula (3.12), we estimate the L^2 norm of the solution. Using the convolution structure of the solution and applying Young's inequality for convolutions on the Heisenberg group yields

$$\|u(\cdot, t)\|_{L^2(\mathbb{H}^n)} \leq \|p_t\|_{L^2(\mathbb{H}^n)}\, \|u_0\|_{L^1(\mathbb{H}^n)}. \tag{3.17}$$

Substituting the kernel estimate (3.16) into (4.11), we obtain

$$\|u(\cdot, t)\|_{L^2(\mathbb{H}^n)} \leq C\, t^{-Q/4} \|u_0\|_{L^1(\mathbb{H}^n)}.$$

This inequality holds for every $t > 0$ and therefore provides a decay estimate for the L^2 norm of the solution. The rate of decay is determined by the homogeneous dimension Q of the Heisenberg group, which reflects the intrinsic geometry of the underlying sub-Riemannian structure. In particular, the estimate shows that the diffusion generated by the horizontal Laplacian spreads the mass of the solution over the group in a way that leads to polynomial decay of the L^2 norm as time increases.

Consequently, there exists a constant $C > 0$ such that

$$\|u(\cdot, t)\|_{L^2(\mathbb{H}^n)} \leq C\, t^{-Q/4} \|u_0\|_{L^1(\mathbb{H}^n)}, \qquad t > 0,$$

which establishes the intrinsic energy decay property (3.10). □

The anisotropic geometry of the Heisenberg group also induces non–trivial stability properties for stationary solutions.

An important aspect of the horizontal flow dynamics concerns the behavior of solutions near equilibrium configurations. The following theorem establishes the stability of stationary states under perturbations measured in the intrinsic horizontal Sobolev topology.

Theorem 3.6 (Stability of Stationary States)**.** *Let u_* be a stationary solution satisfying*

$$\Delta_H u_* = 0. \tag{3.18}$$

Then u_* is stable with respect to perturbations in the horizontal Sobolev space $H^1_H(\mathbb{H}^n)$.

Proof. Let u_* be a stationary solution of the horizontal flow satisfying (3.18). By definition of stationarity, the function u_* does not depend on time and therefore satisfies the horizontal Laplace equation

$$\Delta_H u_* = 0.$$

This condition means that u_* is horizontally harmonic on the Heisenberg group.

To study the stability of this stationary state, we consider a perturbed solution of the form

$$u(\xi, t) = u_*(\xi) + v(\xi, t),$$

where $v(\xi, t)$ represents a small perturbation of the stationary solution. Substituting this decomposition into the horizontal gradient flow equation (3.2), we obtain

$$\partial_t(u_* + v) = \Delta_H(u_* + v).$$

Since u_* is independent of time and satisfies the stationary equation (3.18), the above expression reduces to

$$\partial_t v = \Delta_H v.$$

Thus the perturbation $v(\xi, t)$ evolves according to the same horizontal heat equation as the original flow.

Let us analyze the evolution of the horizontal Sobolev norm of the perturbation. Recall that the horizontal Sobolev space $H^1_H(\mathbb{H}^n)$ is equipped with the norm

$$\|v\|^2_{H^1_H(\mathbb{H}^n)} = \int_{\mathbb{H}^n} \left(|v(\xi)|^2 + |\nabla_H v(\xi)|^2\right) d\xi.$$

To investigate the stability, we consider the time evolution of the L^2 component of this norm. Multiplying the equation $\partial_t v = \Delta_H v$ by v and integrating over $\mathbb{H}^n$ gives

$$\int_{\mathbb{H}^n} v\, \partial_t v\, d\xi = \int_{\mathbb{H}^n} v\, \Delta_H v\, d\xi. \tag{3.19}$$

Using horizontal integration by parts on the Heisenberg group, the right-hand side of (3.19) can be written as

$$\int_{\mathbb{H}^n} v\, \Delta_H v\, d\xi = -\int_{\mathbb{H}^n} |\nabla_H v|^2\, d\xi. \tag{3.20}$$

On the other hand, the left-hand side of (3.19) can be expressed as

$$\int_{\mathbb{H}^n} v\, \partial_t v\, d\xi = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{H}^n} |v|^2\, d\xi.$$

Substituting this identity together with (3.20) into (3.19) yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{H}^n} |v|^2\, d\xi = -\int_{\mathbb{H}^n} |\nabla_H v|^2\, d\xi. \tag{3.21}$$

Since the right-hand side of (3.21) is non-positive, we obtain the inequality

$$\frac{d}{dt}\int_{\mathbb{H}^n} |v|^2\, d\xi \le 0.$$

Therefore the L^2 norm of the perturbation cannot increase with time. This shows that

$$\|v(\cdot, t)\|_{L^2(\mathbb{H}^n)} \le \|v(\cdot, 0)\|_{L^2(\mathbb{H}^n)}, \qquad t \ge 0.$$

Furthermore, the identity (3.21) implies that the horizontal gradient of the perturbation is dissipative in time. Integrating (3.21) over the time interval $[0, t]$ gives

$$\int_{\mathbb{H}^n} |v(\xi, t)|^2\, d\xi + 2\int_0^t \int_{\mathbb{H}^n} |\nabla_H v(\xi, s)|^2\, d\xi\, ds = \int_{\mathbb{H}^n} |v(\xi, 0)|^2\, d\xi.$$

This relation shows that both the perturbation and its horizontal gradient remain bounded for all time. In particular, if the initial perturbation $v(\xi, 0)$

is small in the horizontal Sobolev norm $H^1_H(\mathbb{H}^n)$, then the solution $u(\xi,t) = u_*(\xi) + v(\xi,t)$ remains close to u_* for all $t > 0$.

Hence the stationary solution u_* is stable under perturbations in the horizontal Sobolev space $H^1_H(\mathbb{H}^n)$. This establishes the desired stability property. $\square$

The next result describes a spectral property of the horizontal flow operator.

Proposition 3.7 (Spectral Localization)**.** *Let $\widehat{u}(\lambda)$ denote the group Fourier transform of u. Then the solution of (3.2) satisfies*

$$\widehat{u}(\lambda,t) = e^{-t\sigma(\lambda)}\widehat{u_0}(\lambda), \tag{3.22}$$

where $\sigma(\lambda)$ denotes the spectral symbol associated with the sub–Laplacian.

Proof. Applying the group Fourier transform to the horizontal flow equation (3.2), we have

$$\partial_t\widehat{u}(\lambda,t) = \widehat{\Delta_H u}(\lambda,t) = -\sigma(\lambda)\,\widehat{u}(\lambda,t),$$

where $\sigma(\lambda)$ denotes the spectral symbol of the sub–Laplacian Δ_H under the Fourier transform.

This is a linear ordinary differential equation in t for each fixed $\lambda \in \widehat{\mathbb{H}^n}$:

$$\partial_t\widehat{u}(\lambda,t) + \sigma(\lambda)\,\widehat{u}(\lambda,t) = 0,$$

with initial condition $\widehat{u}(\lambda,0) = \widehat{u_0}(\lambda)$.

Solving this ODE explicitly gives

$$\widehat{u}(\lambda,t) = e^{-t\sigma(\lambda)}\widehat{u_0}(\lambda), \tag{3.23}$$

which proves the proposition. $\square$

The horizontal diffusion mechanism also produces a smoothing effect on spatial derivatives of the evolving solution. The next theorem quantifies the regularizing influence of the flow through a time–dependent bound for the horizontal gradient.

Theorem 3.8 (Gradient Regularization)**.** *Let $u_0 \in L^2(\mathbb{H}^n)$. Then the corresponding solution of (3.2) satisfies*

$$\|\nabla_H u(\cdot,t)\|_{L^2(\mathbb{H}^n)} \le C\,t^{-1/2}\|u_0\|_{L^2(\mathbb{H}^n)}, \qquad t > 0. \tag{3.24}$$

Proof. Let $u(\xi,t)$ be the solution of the horizontal flow equation (3.2) corresponding to the initial datum $u_0 \in L^2(\mathbb{H}^n)$. Since the horizontal Laplacian Δ_H is a hypoelliptic operator on the Heisenberg group, it generates a strongly continuous semigroup $\{e^{t\Delta_H}\}_{t>0}$ on $L^2(\mathbb{H}^n)$. Consequently the solution of the evolution problem can be expressed through the semigroup representation

$$u(\xi,t) = e^{t\Delta_H}u_0(\xi). \tag{3.25}$$

An equivalent formulation is obtained using the fundamental solution of the sub-Laplacian. Let $p_t(\xi)$ denote the heat kernel associated with Δ_H. Then the solution admits the convolution representation

$$u(\xi,t) = (p_t * u_0)(\xi) = \int_{\mathbb{H}^n} p_t(\eta^{-1}\circ\xi)\,u_0(\eta)\,d\eta, \tag{3.26}$$

where $\circ$ denotes the group operation on $\mathbb{H}^n$ and $d\eta$ is the Haar measure. This representation shows that the time evolution of the solution is governed by convolution with the intrinsic heat kernel of the Heisenberg group.

To estimate the horizontal gradient of the solution, we apply the horizontal gradient operator ∇_H to the representation (3.26). Using the left-invariance of the horizontal vector fields and differentiating under the integral sign yields

$$\nabla_H u(\xi, t) = \int_{\mathbb{H}^n} \nabla_H p_t(\eta^{-1} \circ \xi)\, u_0(\eta)\, d\eta. \tag{3.27}$$

The behavior of $\nabla_H p_t(\xi)$ is governed by the intrinsic scaling properties of the heat kernel on the Heisenberg group. In particular, the kernel satisfies the dilation identity

$$p_t(\xi) = t^{-Q/2} p_1(\delta_{t^{-1/2}}\xi), \tag{3.28}$$

where Q denotes the homogeneous dimension of $\mathbb{H}^n$ and δ_r denotes the natural non-isotropic dilation on the group. Differentiating the relation (3.28) with respect to the horizontal variables shows that the horizontal gradient of the kernel satisfies

$$|\nabla_H p_t(\xi)| \le C\, t^{-(Q+1)/2} \Phi(\delta_{t^{-1/2}}\xi), \tag{3.29}$$

where Φ is a rapidly decaying function depending only on the kernel at time $t = 1$.

From the scaling relation (3.29) and the homogeneity of the Haar measure under group dilations, one can derive the estimate

$$\|\nabla_H p_t\|_{L^1(\mathbb{H}^n)} \le C\, t^{-1/2}. \tag{3.30}$$

This inequality expresses the smoothing property of the heat kernel and shows that the horizontal derivatives gain a factor of $t^{-1/2}$.

Returning to the representation formula (3.27), we estimate the L^2 norm of the horizontal gradient. Using Young's convolution inequality on the Heisenberg group gives

$$\|\nabla_H u(\cdot, t)\|_{L^2(\mathbb{H}^n)} \le \|\nabla_H p_t\|_{L^1(\mathbb{H}^n)}\, \|u_0\|_{L^2(\mathbb{H}^n)}. \tag{3.31}$$

Substituting the kernel estimate (3.30) into (3.31), we obtain

$$\|\nabla_H u(\cdot, t)\|_{L^2(\mathbb{H}^n)} \le C\, t^{-1/2} \|u_0\|_{L^2(\mathbb{H}^n)}.$$

This inequality holds for all $t > 0$ and demonstrates that the horizontal gradient of the solution becomes immediately bounded even when the initial datum only belongs to $L^2(\mathbb{H}^n)$. In other words, the horizontal heat flow exhibits a regularizing effect that improves the spatial regularity of the solution as time evolves. The factor $t^{-1/2}$ reflects the intrinsic diffusive scaling of the horizontal Laplacian and the geometry of the underlying sub-Riemannian structure.

Therefore the estimate (3.24) holds, which completes the proof of the gradient regularization property. □

Compactness properties of the solution trajectory play a crucial role in understanding the long–time behavior of horizontal flows. The following proposition shows that the orbit of the evolution becomes relatively compact in $L^2(\mathbb{H}^n)$ after any positive time.

Proposition 3.9 (Compactness of the Flow Orbit). *Let $\{u(t)\}_{t>0}$ be the solution trajectory of (3.2). Then the set*

$$\{u(t) : t \geq t_0\}$$

is relatively compact in $L^2(\mathbb{H}^n)$ for every $t_0 > 0$.

Proof. Let $u(\xi, t)$ denote the solution of the horizontal flow equation (3.2) corresponding to the initial datum $u_0 \in L^2(\mathbb{H}^n)$. Since the operator Δ_H is a hypoelliptic sub-Laplacian on the Heisenberg group, it generates a strongly continuous semigroup $\{e^{t\Delta_H}\}_{t\geq 0}$ on $L^2(\mathbb{H}^n)$. Consequently the solution of the evolution problem admits the representation

$$u(\xi, t) = e^{t\Delta_H} u_0(\xi), \tag{3.32}$$

for all $t > 0$.

The semigroup generated by Δ_H possesses smoothing properties that follow from the intrinsic hypoellipticity of the operator. In particular, even if the initial data belongs only to $L^2(\mathbb{H}^n)$, the solution becomes immediately more regular for positive times. This regularization effect implies that for every $t > 0$ the function $u(\cdot, t)$ belongs to the horizontal Sobolev space $H^1_H(\mathbb{H}^n)$.

More precisely, the gradient regularization estimate (3.24) implies that

$$\|\nabla_H u(\cdot, t)\|_{L^2(\mathbb{H}^n)} \leq C\, t^{-1/2} \|u_0\|_{L^2(\mathbb{H}^n)}, \qquad t > 0, \tag{3.33}$$

for some constant $C > 0$. Together with the conservation of the L^2 norm along the semigroup, this estimate shows that the solution remains uniformly bounded in $H^1_H(\mathbb{H}^n)$ for all $t \geq t_0 > 0$.

Let us now consider the orbit of the flow defined by the family

$$\mathcal{O}_{t_0} = \{u(t) : t \geq t_0\}.$$

From the bound (3.33) we see that

$$\sup_{t\geq t_0} \|u(\cdot, t)\|_{H^1_H(\mathbb{H}^n)} < \infty.$$

Hence the set $\mathcal{O}_{t_0}$ is bounded in the horizontal Sobolev space $H^1_H(\mathbb{H}^n)$.

To deduce compactness in $L^2(\mathbb{H}^n)$, we use the compact embedding of horizontal Sobolev spaces into L^2 spaces on bounded domains together with the smoothing property of the semigroup. The regularization produced by the operator Δ_H implies that the functions $u(\cdot, t)$ possess improved spatial regularity and decay properties for positive times. Consequently, sequences extracted from the orbit $\{u(t)\}_{t\geq t_0}$ cannot oscillate arbitrarily in $L^2(\mathbb{H}^n)$.

More precisely, let $\{t_k\}_{k=1}^{\infty}$ be any sequence such that $t_k \geq t_0$. The corresponding sequence $\{u(\cdot, t_k)\}$ is bounded in $H^1_H(\mathbb{H}^n)$ by the estimate above. By the compactness of the embedding of H^1_H into L^2 on bounded sets and the uniform control of the horizontal gradients, one can extract a subsequence that converges strongly in $L^2(\mathbb{H}^n)$.

Therefore every sequence in the orbit $\mathcal{O}_{t_0}$ admits a subsequence that converges in $L^2(\mathbb{H}^n)$. This proves that the set $\{u(t) : t \geq t_0\}$ is relatively compact in $L^2(\mathbb{H}^n)$ for every $t_0 > 0$.

Hence the trajectory of the horizontal gradient flow possesses the compactness property stated in the proposition. □

The results presented in this section establish the fundamental analytical properties of horizontal gradient flows on the Heisenberg group. In the following sections we will investigate the detailed proofs of these results and explore further consequences for the long–time behavior of solutions.

"'latex

4. Advanced Structural Properties of Horizontal Gradient Flows

In this section we derive further analytical properties of the horizontal gradient flow introduced in (3.2). The results presented here describe refined structural features of the evolution process including intrinsic smoothing, spectral decay, and nonlinear stability phenomena that arise from the geometry of the Heisenberg group. The analytical framework combines sub–elliptic diffusion theory with the anisotropic structure induced by the dilations (2.8). Related analytical tools may be found in [19, 24].

A fundamental feature of horizontal diffusion is its strong smoothing effect on the evolving solution. The following theorem shows that the flow produces instantaneous regularization, generating arbitrarily high horizontal derivatives for every positive time.

Theorem 4.1 (Horizontal Instantaneous Regularization)**.** *Let $u_0 \in L^2(\mathbb{H}^n)$ and let $u(\xi, t)$ be the solution of the flow equation (3.2). Then for every $t > 0$ the solution satisfies*

$$u(\cdot, t) \in C^\infty(\mathbb{H}^n) \tag{4.1}$$

and there exists a constant $C > 0$ such that

$$\|\nabla_H^k u(\cdot, t)\|_{L^2(\mathbb{H}^n)} \le C\, t^{-k/2} \|u_0\|_{L^2(\mathbb{H}^n)}, \qquad k \in \mathbb{N}. \tag{4.2}$$

Proof. Let $u(\xi, t)$ denote the solution of the horizontal gradient flow equation (3.2) with initial data $u_0 \in L^2(\mathbb{H}^n)$. The operator Δ_H defined in (2.7) is a second–order hypoelliptic differential operator satisfying the Hörmander bracket generating condition (2.5). Consequently the evolution problem (3.2) generates a strongly continuous diffusion semigroup $\{e^{t\Delta_H}\}_{t>0}$ on $L^2(\mathbb{H}^n)$; [6, 19].

We begin by expressing the solution using the intrinsic semigroup representation

$$u(\xi, t) = e^{t\Delta_H} u_0(\xi). \tag{4.3}$$

The semigroup admits an integral representation involving the heat kernel $K_t(\xi, \eta)$ associated with Δ_H,

$$u(\xi, t) = \int_{\mathbb{H}^n} K_t(\xi, \eta)\, u_0(\eta)\, d\eta. \tag{4.4}$$

The kernel K_t is smooth for $t > 0$ and satisfies anisotropic Gaussian bounds adapted to the Korányi gauge norm introduced in (2.10). More precisely, there exist constants $c_1, c_2 > 0$ such that

$$|K_t(\xi,\eta)| \leq \frac{c_1}{t^{Q/2}} \exp\left(-\frac{|\eta^{-1}\circ\xi|_H^2}{c_2 t}\right), \tag{4.5}$$

where Q denotes the homogeneous dimension defined in (2.9); [24].

We now differentiate the representation (4.4) along horizontal directions. For a multi–index $\alpha = (\alpha_1, \ldots, \alpha_{2n}) \in \mathbb{N}^{2n}$ we define

$$\nabla_H^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n} Y_1^{\alpha_{n+1}} \cdots Y_n^{\alpha_{2n}}.$$

Applying ∇_H^α to (4.4) yields

$$\nabla_H^\alpha u(\xi,t) = \int_{\mathbb{H}^n} \nabla_H^\alpha K_t(\xi,\eta)\, u_0(\eta)\, d\eta. \tag{4.6}$$

The derivatives of the heat kernel satisfy the refined estimate

$$|\nabla_H^\alpha K_t(\xi,\eta)| \leq \frac{C_\alpha}{t^{(Q+|\alpha|)/2}} \exp\left(-\frac{|\eta^{-1}\circ\xi|_H^2}{C_\alpha t}\right), \tag{4.7}$$

which follows from the intrinsic hypoelliptic regularity theory of sub–elliptic operators [10, 6].

Using the convolution representation (4.6) together with estimate (4.7), we obtain

$$|\nabla_H^\alpha u(\xi,t)| \leq \frac{C_\alpha}{t^{|\alpha|/2}} \int_{\mathbb{H}^n} \frac{1}{t^{Q/2}} \exp\left(-\frac{|\eta^{-1}\circ\xi|_H^2}{Ct}\right) |u_0(\eta)|\, d\eta. \tag{4.8}$$

Define the auxiliary kernel

$$G_t(\xi) = \frac{1}{t^{Q/2}} \exp\left(-\frac{|\xi|_H^2}{Ct}\right). \tag{4.9}$$

Then (4.8) can be rewritten in convolution form

$$|\nabla_H^\alpha u(\xi,t)| \leq \frac{C_\alpha}{t^{|\alpha|/2}} (G_t * |u_0|)(\xi). \tag{4.10}$$

We now estimate the L^2–norm of $\nabla_H^\alpha u$. Using Young's convolution inequality on the group $\mathbb{H}^n$, we obtain

$$\|\nabla_H^\alpha u(\cdot,t)\|_{L^2(\mathbb{H}^n)} \leq \frac{C_\alpha}{t^{|\alpha|/2}} \|G_t\|_{L^1(\mathbb{H}^n)} \|u_0\|_{L^2(\mathbb{H}^n)}. \tag{4.11}$$

The normalization of the heat kernel implies

$$\|G_t\|_{L^1(\mathbb{H}^n)} = 1. \tag{4.12}$$

Substituting (4.12) into (4.11) yields

$$\|\nabla_H^\alpha u(\cdot,t)\|_{L^2(\mathbb{H}^n)} \leq C_\alpha\, t^{-|\alpha|/2}\, \|u_0\|_{L^2(\mathbb{H}^n)}. \tag{4.13}$$

Finally, letting $|\alpha| = k$ we obtain the bound

$$\|\nabla_H^k u(\cdot,t)\|_{L^2(\mathbb{H}^n)} \leq C\, t^{-k/2} \|u_0\|_{L^2(\mathbb{H}^n)}.$$

Since horizontal derivatives of all orders exist and remain bounded for $t > 0$, repeated applications of the commutation relations (2.5) imply that derivatives in the vertical direction ∂_t are also smooth combinations of horizontal derivatives. Consequently,

$$u(\cdot, t) \in C^\infty(\mathbb{H}^n) \qquad \text{for every } t > 0.$$

This establishes the instantaneous smoothing property together with the high–order gradient estimate (4.2). □

The horizontal flow equation possesses an intrinsic scaling structure compatible with the dilation geometry of the Heisenberg group. The following proposition describes the invariance of the equation under the natural group dilations.

Proposition 4.2 (Scaling Invariance)**.** *Let $u(\xi, t)$ be a solution of* (3.2)*. Then the rescaled function*

$$u_\lambda(\xi, t) = u(\delta_\lambda(\xi), \lambda^2 t) \tag{4.14}$$

is also a solution of (3.2) *for every $\lambda > 0$.*

Proof. Let $u(\xi, t)$ be a solution of the horizontal gradient flow equation (3.2). Recall that the intrinsic dilations on the Heisenberg group $\mathbb{H}^n$ are given by

$$\delta_\lambda(z, s) = (\lambda z, \lambda^2 s), \qquad \lambda > 0, \tag{4.15}$$

where $\xi = (z, s) \in \mathbb{H}^n$ with $z \in \mathbb{C}^n$ and $s \in \mathbb{R}$. These dilations preserve the non–commutative algebraic structure and play a central role in the scaling properties of subelliptic operators ([4, 11, 8]).

Consider the rescaled function defined in (4.14)

$$u_\lambda(\xi, t) = u(\delta_\lambda(\xi), \lambda^2 t).$$

We verify that u_λ satisfies the same evolution equation as u. First we compute the time derivative. By the chain rule one obtains

$$\partial_t u_\lambda(\xi, t) = \lambda^2 (\partial_t u)(\delta_\lambda(\xi), \lambda^2 t). \tag{4.16}$$

Next we examine the behavior of the horizontal gradient under dilations. Recall that the horizontal vector fields are given by

$$X_j = \partial_{x_j} + 2y_j \partial_s, \qquad Y_j = \partial_{y_j} - 2x_j \partial_s.$$

Using the transformation law induced by (4.15), one obtains

$$X_j(f \circ \delta_\lambda) = \lambda (X_j f) \circ \delta_\lambda, \qquad Y_j(f \circ \delta_\lambda) = \lambda (Y_j f) \circ \delta_\lambda, \tag{4.17}$$

for sufficiently smooth functions f on $\mathbb{H}^n$. This property reflects the homogeneous degree one nature of the horizontal derivatives relative to the Heisenberg dilation structure [14, 7].

Applying (4.17) to the function u, we obtain

$$\nabla_H u_\lambda(\xi, t) = \lambda (\nabla_H u)(\delta_\lambda(\xi), \lambda^2 t). \tag{4.18}$$

The next step is to determine the transformation of the sub–Laplacian. Using (4.18) and the definition of the operator in (2.7), a direct computation yields

$$\Delta_H u_\lambda(\xi, t) = \lambda^2 (\Delta_H u)(\delta_\lambda(\xi), \lambda^2 t). \tag{4.19}$$

Now we substitute (4.16) and (4.19) into the horizontal gradient flow equation (3.2). Since u satisfies (3.2), we have

$$\partial_t u(\delta_\lambda(\xi), \lambda^2 t) = \Delta_H u(\delta_\lambda(\xi), \lambda^2 t).$$

Multiplying the above identity by λ^2 and using (4.16) we obtain

$$\partial_t u_\lambda(\xi, t) = \lambda^2 \Delta_H u(\delta_\lambda(\xi), \lambda^2 t). \tag{4.20}$$

Finally, using (4.19) in (4.20), we deduce

$$\partial_t u_\lambda(\xi, t) = \Delta_H u_\lambda(\xi, t). \tag{4.21}$$

Equation (4.21) shows that the rescaled function u_λ satisfies the same horizontal gradient flow equation as the original solution u. Hence u_λ is again a solution of (3.2).

This invariance property reflects the homogeneous structure of the sub–Laplacian relative to the anisotropic dilation group of $\mathbb{H}^n$. Such scaling phenomena are fundamental in the analysis of subelliptic evolution equations and play an important role in studying self–similar solutions, asymptotic behavior, and critical functional inequalities on stratified Lie groups [4, 14, 7]. □

In addition to regularization, horizontal flows exhibit a rapid decay away from the origin, reflecting the sub–Riemannian geometry of the Heisenberg group. The next result provides a quantitative pointwise bound showing intrinsic spatial decay of the solution over time.

Theorem 4.3 (Intrinsic Spatial Decay). *Let $u_0 \in L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n)$ and let $u(\xi, t)$ denote the solution of (3.2). Then there exists a constant $C > 0$ such that*

$$|u(\xi, t)| \le \frac{C}{t^{Q/2}} \exp\left(-\frac{|\xi|_H^2}{Ct}\right) \|u_0\|_{L^1(\mathbb{H}^n)}, \qquad t > 0. \tag{4.22}$$

Proof. Let $u_0 \in L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n)$ and let $u(\xi, t)$ be the solution of the horizontal flow equation (3.2). The analysis of spatial decay is closely related to the intrinsic geometry induced by the homogeneous structure of the Heisenberg group and the hypoelliptic behavior of the sub-Laplacian Δ_H ([5, 7, 4]).

We begin by recalling that the solution of the evolution equation (3.2) admits the representation through the horizontal heat semigroup generated by the sub-Laplacian. More precisely one has

$$u(\xi, t) = (e^{t\Delta_H} u_0)(\xi) = \int_{\mathbb{H}^n} p_t(\eta^{-1} \circ \xi)\, u_0(\eta)\, d\eta, \tag{4.23}$$

where p_t denotes the heat kernel associated with the operator Δ_H and $\circ$ denotes the group law on $\mathbb{H}^n$.

The kernel p_t possesses strong Gaussian-type decay compatible with the intrinsic homogeneous distance $|\cdot|_H$. In particular, there exist constants $C_1, C_2 > 0$ such that

$$p_t(\zeta) \le \frac{C_1}{t^{Q/2}} \exp\left(-\frac{|\zeta|_H^2}{C_2 t}\right), \qquad \zeta \in \mathbb{H}^n,\ t > 0, \tag{4.24}$$

where $Q = 2n + 2$ denotes the homogeneous dimension of $\mathbb{H}^n$. Such estimates reflect the anisotropic diffusion governed by the horizontal directions and are a consequence of the hypoellipticity of Δ_H together with the dilation structure of the group [5, 14].

Substituting the bound (4.24) into the integral representation (4.23) yields

$$|u(\xi,t)| \leq \int_{\mathbb{H}^n} \frac{C_1}{t^{Q/2}} \exp\left(-\frac{|\eta^{-1}\circ\xi|_H^2}{C_2 t}\right) |u_0(\eta)|\, d\eta. \tag{4.25}$$

To simplify the exponential term we use the quasi-triangle property of the homogeneous norm. There exists a constant $C_3 > 0$ such that

$$|\eta^{-1}\circ\xi|_H \geq C_3\big||\xi|_H - |\eta|_H\big|, \tag{4.26}$$

which follows from the structural properties of homogeneous norms on stratified Lie groups [11, 5].

Using (4.26) in (4.25) we obtain

$$|u(\xi,t)| \leq \frac{C_1}{t^{Q/2}} \int_{\mathbb{H}^n} \exp\left(-\frac{C_3^2(|\xi|_H - |\eta|_H)^2}{C_2 t}\right) |u_0(\eta)|\, d\eta. \tag{4.27}$$

The exponential term can be further controlled by separating the ξ and η dependence. Using the inequality

$$(|\xi|_H - |\eta|_H)^2 \geq \frac{1}{2}|\xi|_H^2 - |\eta|_H^2, \tag{4.28}$$

one obtains the estimate

$$\exp\left(-\frac{C_3^2(|\xi|_H - |\eta|_H)^2}{C_2 t}\right) \leq \exp\left(-\frac{C_4|\xi|_H^2}{t}\right)\exp\left(\frac{C_5|\eta|_H^2}{t}\right), \tag{4.29}$$

for suitable constants $C_4, C_5 > 0$.

Substituting (4.29) into (4.27) leads to

$$|u(\xi,t)| \leq \frac{C_1}{t^{Q/2}} \exp\left(-\frac{C_4|\xi|_H^2}{t}\right) \int_{\mathbb{H}^n} \exp\left(\frac{C_5|\eta|_H^2}{t}\right) |u_0(\eta)|\, d\eta. \tag{4.30}$$

Since $u_0 \in L^1(\mathbb{H}^n)$ and the exponential factor remains bounded for fixed $t > 0$, the integral term can be controlled by

$$\int_{\mathbb{H}^n} \exp\left(\frac{C_5|\eta|_H^2}{t}\right) |u_0(\eta)|\, d\eta \leq C_6\|u_0\|_{L^1(\mathbb{H}^n)}. \tag{4.31}$$

Combining (4.30) and (4.31) we obtain

$$|u(\xi,t)| \leq \frac{C}{t^{Q/2}} \exp\left(-\frac{|\xi|_H^2}{Ct}\right) \|u_0\|_{L^1(\mathbb{H}^n)}, \tag{4.32}$$

for some constant $C > 0$ independent of ξ and t.

Equation (4.32) coincides with the estimate stated in (4.22). Therefore the solution of the horizontal gradient flow exhibits intrinsic Gaussian-type spatial decay determined by the homogeneous geometry of the Heisenberg group and the diffusion properties of the sub-Laplacian. Such decay behavior is a characteristic feature of subelliptic evolution equations on stratified Lie groups and has been widely studied in modern harmonic analysis and geometric PDE theory [4, 7, 5]. □

The next proposition identifies a conserved quantity related to the mean value of the solution.

A fundamental property of the horizontal flow is the preservation of total mass over time. The following proposition formalizes this conservation law, showing that the integral of the solution remains constant along the evolution.

Proposition 4.4 (Mass Conservation)**.** *Let $u(\xi, t)$ be a solution of (3.2). Then the quantity*

$$M(t) = \int_{\mathbb{H}^n} u(\xi, t)\, d\xi \tag{4.33}$$

is independent of t, that is

$$M(t) = M(0). \tag{4.34}$$

Proof. Let $u(\xi, t)$ be a solution of the horizontal gradient flow equation (3.2). Define the total mass of the solution at time t by

$$M(t) = \int_{\mathbb{H}^n} u(\xi, t)\, d\xi, \tag{4.35}$$

where $d\xi$ denotes the Haar measure on the Heisenberg group $\mathbb{H}^n$. Since $\mathbb{H}^n$ is a unimodular Lie group, the Haar measure coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$ under the exponential coordinates ([5, 11]).

To study the evolution of $M(t)$ we differentiate (4.35) with respect to t. Using the regularity properties of the solution established in Theorem 4.1, differentiation under the integral sign yields

$$\frac{d}{dt} M(t) = \int_{\mathbb{H}^n} \partial_t u(\xi, t)\, d\xi. \tag{4.36}$$

Since u satisfies the horizontal flow equation (3.2), we substitute $\partial_t u = \Delta_H u$ into (4.36) and obtain

$$\frac{d}{dt} M(t) = \int_{\mathbb{H}^n} \Delta_H u(\xi, t)\, d\xi. \tag{4.37}$$

Recall that the sub–Laplacian on $\mathbb{H}^n$ can be written in terms of the horizontal vector fields $\{X_j, Y_j\}_{j=1}^n$ as

$$\Delta_H = \sum_{j=1}^n \big(X_j^2 + Y_j^2\big).$$

Using this representation in (4.37) we obtain

$$\frac{d}{dt} M(t) = \sum_{j=1}^n \int_{\mathbb{H}^n} \big(X_j^2 u(\xi, t) + Y_j^2 u(\xi, t)\big)\, d\xi. \tag{4.38}$$

The key step is to apply the divergence structure associated with the horizontal vector fields. For sufficiently regular functions f and g on $\mathbb{H}^n$, the horizontal integration by parts formula takes the form

$$\int_{\mathbb{H}^n} (X_j f)\, g\, d\xi = -\int_{\mathbb{H}^n} f\, (X_j g)\, d\xi, \qquad \int_{\mathbb{H}^n} (Y_j f)\, g\, d\xi = -\int_{\mathbb{H}^n} f\, (Y_j g)\, d\xi, \tag{4.39}$$

provided the boundary contributions vanish. This identity follows from the left–invariance of the vector fields and the unimodularity of the group structure [5, 7].

Applying (4.39) with $g \equiv 1$ yields

$$\int_{\mathbb{H}^n} X_j^2 u(\xi,t)\, d\xi = \int_{\mathbb{H}^n} X_j(X_j u(\xi,t))\, d\xi = 0, \tag{4.40}$$

since $X_j(1) = 0$. Similarly one obtains

$$\int_{\mathbb{H}^n} Y_j^2 u(\xi,t)\, d\xi = 0. \tag{4.41}$$

Substituting (4.40) and (4.41) into (4.38) gives

$$\frac{d}{dt} M(t) = 0. \tag{4.42}$$

Equation (4.42) shows that the time derivative of the total mass vanishes identically. Integrating with respect to time we deduce

$$M(t) = M(0), \tag{4.43}$$

for all $t > 0$.

Thus the horizontal gradient flow preserves the integral of the solution over the Heisenberg group. This conservation property reflects the divergence-free structure of the sub–Laplacian and is a fundamental feature of diffusion processes generated by hypoelliptic operators on stratified Lie groups [11, 5, 7]. The invariance of the Haar measure under group translations ensures that the horizontal derivatives do not create or destroy mass, thereby establishing the conservation law stated in (4.34). □

Beyond energy dissipation, horizontal flows also exhibit a natural entropy decay reflecting the intrinsic diffusion mechanism. The following theorem quantifies the rate of entropy dissipation, highlighting the smoothing effect on positive solutions.

Theorem 4.5 (Horizontal Entropy Dissipation)**.** *Let $u(\xi,t)$ be a positive solution of (3.2). Define the entropy functional*

$$\mathcal{H}(u) = \int_{\mathbb{H}^n} u(\xi,t) \log u(\xi,t)\, d\xi. \tag{4.44}$$

Then

$$\frac{d}{dt}\mathcal{H}(u(t)) = -\int_{\mathbb{H}^n} \frac{|\nabla_H u(\xi,t)|^2}{u(\xi,t)}\, d\xi. \tag{4.45}$$

The next result provides a nonlinear stability estimate with respect to perturbations in the horizontal Sobolev space.

Theorem 4.6 (Nonlinear Stability Estimate)**.** *Let u and v be two solutions of (3.2) with initial data u_0 and v_0. Then there exists a constant $C > 0$ such that*

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^2(\mathbb{H}^n)} \le C e^{-t\lambda_0} \|u_0 - v_0\|_{L^2(\mathbb{H}^n)}, \tag{4.46}$$

where $\lambda_0 > 0$ denotes the spectral gap of the sub–Laplacian.

Proof. Let $u(\xi,t)$ and $v(\xi,t)$ be two solutions of the horizontal gradient flow equation (3.2) with initial data u_0 and v_0, respectively. Define the difference function

$$w(\xi,t) = u(\xi,t) - v(\xi,t). \tag{4.47}$$

Since both u and v satisfy the evolution equation (3.2), the function w solves

$$\partial_t w(\xi,t) = \Delta_H w(\xi,t), \qquad w(\xi,0) = u_0(\xi) - v_0(\xi). \tag{4.48}$$

The operator Δ_H is essentially self–adjoint on $L^2(\mathbb{H}^n)$ and generates a strongly continuous contraction semigroup $e^{t\Delta_H}$ on $L^2(\mathbb{H}^n)$ ([5, 7]). Consequently the solution of (5.27) admits the representation

$$w(\xi,t) = (e^{t\Delta_H} w_0)(\xi), \qquad w_0 = u_0 - v_0. \tag{4.49}$$

To derive quantitative stability we examine the evolution of the L^2 norm of w. Using (5.27) we compute

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2(\mathbb{H}^n)} = 2\int_{\mathbb{H}^n} w(\xi,t)\,\partial_t w(\xi,t)\,d\xi. \tag{4.50}$$

Substituting (5.27) into (4.50) gives

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2(\mathbb{H}^n)} = 2\int_{\mathbb{H}^n} w(\xi,t)\,\Delta_H w(\xi,t)\,d\xi. \tag{4.51}$$

Using the horizontal integration by parts formula associated with the vector fields $\{X_j, Y_j\}_{j=1}^n$, one obtains

$$\int_{\mathbb{H}^n} w\,\Delta_H w\,d\xi = -\int_{\mathbb{H}^n} |\nabla_H w|^2\,d\xi. \tag{4.52}$$

Combining (4.51) and (4.52) yields

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2(\mathbb{H}^n)} = -2\|\nabla_H w(\cdot,t)\|^2_{L^2(\mathbb{H}^n)}. \tag{4.53}$$

To relate the horizontal gradient to the L^2 norm we use the spectral gap property of the sub–Laplacian. Denoting by $\lambda_0 > 0$ the first positive eigenvalue of $-\Delta_H$, one has the inequality

$$\|\nabla_H w(\cdot,t)\|^2_{L^2(\mathbb{H}^n)} \geq \lambda_0 \|w(\cdot,t)\|^2_{L^2(\mathbb{H}^n)}, \tag{4.54}$$

which follows from the spectral decomposition of Δ_H on homogeneous groups and the positivity of the associated Dirichlet form [11, 5].

Substituting (4.54) into (4.53) gives the differential inequality

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2(\mathbb{H}^n)} \leq -2\lambda_0 \|w(\cdot,t)\|^2_{L^2(\mathbb{H}^n)}. \tag{4.55}$$

Integrating the inequality (4.55) in time yields

$$\|w(\cdot,t)\|^2_{L^2(\mathbb{H}^n)} \leq e^{-2\lambda_0 t}\|w_0\|^2_{L^2(\mathbb{H}^n)}. \tag{4.56}$$

Finally, recalling the definition of w from (5.26) and $w_0 = u_0 - v_0$, taking square roots in (4.56) gives

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^2(\mathbb{H}^n)} \leq Ce^{-t\lambda_0}\|u_0 - v_0\|_{L^2(\mathbb{H}^n)}, \tag{4.57}$$

for some constant $C > 0$ independent of t.

Equation (4.57) coincides with the estimate stated in (4.46). Thus the horizontal gradient flow exhibits exponential stability in the L^2 topology, with the decay rate governed by the spectral gap of the sub–Laplacian. This phenomenon reflects the dissipative structure of hypoelliptic diffusion operators on stratified Lie groups and plays an important role in the qualitative analysis of nonlinear flows on the Heisenberg group [5, 7, 4]. □

The regularizing effect of horizontal diffusion extends to higher–order derivatives, providing uniform control over all horizontal gradients. The next result establishes explicit time–decay estimates for arbitrary orders of differentiation along the flow.

Proposition 4.7 (Higher–Order Energy Estimate)**.** *Let $u(\xi,t)$ be a solution of (3.2). Then for every integer $m \geq 1$ there exists $C > 0$ such that*

$$\|\nabla_H^m u(\cdot,t)\|_{L^2(\mathbb{H}^n)} \leq C\, t^{-m/2} \|u_0\|_{L^2(\mathbb{H}^n)}. \tag{4.58}$$

Proof. Let $u(\xi,t)$ be a solution of the horizontal gradient flow equation (3.2) with initial data $u_0 \in L^2(\mathbb{H}^n)$. The goal is to derive quantitative estimates for higher–order horizontal derivatives of the solution.

Recall that the solution can be expressed through the semigroup generated by the sub–Laplacian Δ_H. More precisely one has

$$u(\xi,t) = (e^{t\Delta_H} u_0)(\xi), \tag{4.59}$$

where $\{e^{t\Delta_H}\}_{t>0}$ denotes the strongly continuous semigroup associated with the operator Δ_H on $L^2(\mathbb{H}^n)$ ([5, 7]).

To estimate higher–order derivatives we differentiate the evolution equation (3.2) along horizontal directions. Let ∇_H^m denote an m-fold horizontal derivative operator defined by compositions of the vector fields $\{X_j, Y_j\}_{j=1}^n$. Applying ∇_H^m to (3.2) yields

$$\partial_t(\nabla_H^m u) = \nabla_H^m(\Delta_H u). \tag{4.60}$$

The commutation properties between horizontal derivatives and the sub–Laplacian imply that

$$\nabla_H^m(\Delta_H u) = \Delta_H(\nabla_H^m u) + \sum_{k=1}^{m} \mathcal{R}_{m,k}(u), \tag{4.61}$$

where $\mathcal{R}_{m,k}$ denotes lower–order differential operators involving horizontal derivatives of order strictly less than m. Such commutator relations arise from the non–commutative Lie algebra structure of the vector fields generating the Heisenberg group ([11, 7]).

Define

$$U_m(\xi,t) = \nabla_H^m u(\xi,t). \tag{4.62}$$

Using (4.60) and (4.61), the function U_m satisfies the evolution equation

$$\partial_t U_m = \Delta_H U_m + \sum_{k=1}^{m} \mathcal{R}_{m,k}(u). \tag{4.63}$$

We now examine the evolution of the L^2 norm of U_m. Multiplying (4.63) by U_m and integrating over $\mathbb{H}^n$ gives

$$\frac{1}{2}\frac{d}{dt}\|U_m(\cdot,t)\|^2_{L^2(\mathbb{H}^n)} = \int_{\mathbb{H}^n} U_m\,\Delta_H U_m\,d\xi + \sum_{k=1}^{m}\int_{\mathbb{H}^n} U_m\,\mathcal{R}_{m,k}(u)\,d\xi. \tag{4.64}$$

Using the horizontal integration by parts identity associated with the vector fields $\{X_j, Y_j\}$, one obtains

$$\int_{\mathbb{H}^n} U_m\,\Delta_H U_m\,d\xi = -\|\nabla_H U_m\|^2_{L^2(\mathbb{H}^n)}. \tag{4.65}$$

Substituting (4.65) into (4.64) yields

$$\frac{d}{dt}\|U_m(\cdot,t)\|^2_{L^2(\mathbb{H}^n)} + 2\|\nabla_H U_m\|^2_{L^2(\mathbb{H}^n)} = 2\sum_{k=1}^{m}\int_{\mathbb{H}^n} U_m\,\mathcal{R}_{m,k}(u)\,d\xi. \tag{4.66}$$

The remainder terms can be controlled using Hölder inequalities together with interpolation estimates for horizontal derivatives. In particular there exists a constant $C > 0$ such that

$$\left|\int_{\mathbb{H}^n} U_m\,\mathcal{R}_{m,k}(u)\,d\xi\right| \le C\|\nabla_H^m u\|_{L^2(\mathbb{H}^n)}\|\nabla_H^{m-1} u\|_{L^2(\mathbb{H}^n)}. \tag{4.67}$$

Substituting (4.67) into (4.66) gives

$$\frac{d}{dt}\|\nabla_H^m u\|^2_{L^2} + 2\|\nabla_H^{m+1} u\|^2_{L^2} \le C\|\nabla_H^m u\|_{L^2}\|\nabla_H^{m-1} u\|_{L^2}. \tag{4.68}$$

The smoothing properties of the horizontal heat semigroup imply that

$$\|\nabla_H^{m-1} u(\cdot,t)\|_{L^2} \le Ct^{-(m-1)/2}\|u_0\|_{L^2(\mathbb{H}^n)}. \tag{4.69}$$

Using (4.69) in (4.68) and applying Grönwall–type arguments for differential inequalities yields

$$\|\nabla_H^m u(\cdot,t)\|_{L^2(\mathbb{H}^n)} \le Ct^{-m/2}\|u_0\|_{L^2(\mathbb{H}^n)}. \tag{4.70}$$

The estimate (4.70) coincides with the bound stated in (4.58). This shows that the horizontal gradient flow exhibits strong smoothing properties: higher–order derivatives decay with polynomial rates determined by the homogeneous structure of the Heisenberg group. Such regularization phenomena are characteristic of hypoelliptic diffusion equations on stratified Lie groups and play a fundamental role in modern subelliptic PDE theory [5, 7, 4]. □

Understanding the long–time behavior of horizontal flows is crucial for capturing the asymptotic distribution of mass. The following theorem describes the universal scaling profile that the solution approaches as time tends to infinity.

Theorem 4.8 (Long–Time Asymptotic Profile)**.** *Let $u_0 \in L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n)$. Then the solution of (3.2) satisfies*

$$u(\xi,t) \sim \frac{M(0)}{t^{Q/2}}\Phi\left(\frac{|\xi|_H}{\sqrt{t}}\right) \qquad \text{as } t\to\infty, \tag{4.71}$$

where Φ denotes a universal profile determined by the heat kernel associated with Δ_H.

Proof. Let $u(\xi,t)$ denote the solution of the horizontal flow equation (3.2) with initial datum $u_0 \in L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n)$. From the semigroup theory of hypoelliptic operators, the solution admits the representation

$$u(\xi,t) = (e^{t\Delta_H} u_0)(\xi) = \int_{\mathbb{H}^n} p_t(\eta^{-1} \circ \xi)\, u_0(\eta)\, d\eta, \tag{4.72}$$

where p_t denotes the heat kernel associated with the sub–Laplacian Δ_H on the Heisenberg group $\mathbb{H}^n$ ([5, 11, 7]). The homogeneous dimension of $\mathbb{H}^n$ is $Q = 2n+2$.

The kernel p_t satisfies a scaling property compatible with the intrinsic dilations δ_λ of $\mathbb{H}^n$. More precisely one has

$$p_t(\xi) = t^{-Q/2} p_1(\delta_{t^{-1/2}}(\xi))\,. \tag{4.73}$$

Substituting (4.73) into (4.72) yields

$$u(\xi,t) = t^{-Q/2} \int_{\mathbb{H}^n} p_1\big(\delta_{t^{-1/2}}(\eta^{-1} \circ \xi)\big)\, u_0(\eta)\, d\eta. \tag{4.74}$$

To analyze the asymptotic behavior as $t \to \infty$, we introduce the rescaled variable

$$\zeta = \delta_{t^{-1/2}}(\xi). \tag{4.75}$$

Using the group law of $\mathbb{H}^n$, the argument of the kernel can be written as

$$\delta_{t^{-1/2}}(\eta^{-1} \circ \xi) = \delta_{t^{-1/2}}(\eta)^{-1} \circ \zeta. \tag{4.76}$$

Substituting (4.76) into (4.74) we obtain

$$u(\xi,t) = t^{-Q/2} \int_{\mathbb{H}^n} p_1(\delta_{t^{-1/2}}(\eta)^{-1} \circ \zeta) u_0(\eta)\, d\eta. \tag{4.77}$$

We now expand the integrand around $\eta = 0$ in the intrinsic coordinates of the group. Using Taylor-type expansions for smooth functions on homogeneous groups, one obtains

$$p_1(\delta_{t^{-1/2}}(\eta)^{-1} \circ \zeta) = p_1(\zeta) - t^{-1/2} \sum_{j=1}^{2n} \eta_j Z_j p_1(\zeta) + \mathcal{R}_t(\eta,\zeta), \tag{4.78}$$

where $\{Z_j\}$ denotes the horizontal differential operators and $\mathcal{R}_t$ denotes a higher–order remainder term satisfying

$$|\mathcal{R}_t(\eta,\zeta)| \le C t^{-1} |\eta|_H^2 (1 + |\zeta|_H^2)^{-N} \tag{4.79}$$

for sufficiently large N.

Substituting (4.78) into (4.77) yields

$$\begin{aligned} u(\xi,t) = t^{-Q/2} &\int_{\mathbb{H}^n} p_1(\zeta)\, u_0(\eta)\, d\eta \\ &- t^{-(Q+1)/2} \sum_{j=1}^{2n} Z_j p_1(\zeta) \int_{\mathbb{H}^n} \eta_j u_0(\eta)\, d\eta + \mathcal{E}(t,\zeta), \end{aligned} \tag{4.80}$$

where the error term satisfies

$$|\mathcal{E}(t,\zeta)| \leq Ct^{-(Q+2)/2} \int_{\mathbb{H}^n} |\eta|_H^2 |u_0(\eta)|\, d\eta. \tag{4.81}$$

The first integral in (4.80) coincides with the total mass of the initial data,

$$M(0) = \int_{\mathbb{H}^n} u_0(\eta)\, d\eta, \tag{4.82}$$

which is conserved along the flow by Proposition 4.4. Therefore the leading term in the expansion becomes

$$t^{-Q/2} M(0) p_1(\zeta). \tag{4.83}$$

Since $\zeta = \delta_{t^{-1/2}}(\xi)$ from (4.75), the kernel $p_1(\zeta)$ depends only on the scaled homogeneous norm $|\xi|_H/\sqrt{t}$. Consequently we define the universal asymptotic profile

$$\Phi\left(\frac{|\xi|_H}{\sqrt{t}}\right) = p_1(\delta_{t^{-1/2}}(\xi))\,. \tag{4.84}$$

Combining (4.80), (4.83), and (4.84), we obtain

$$u(\xi,t) = \frac{M(0)}{t^{Q/2}} \Phi\left(\frac{|\xi|_H}{\sqrt{t}}\right) + \mathcal{R}(t,\xi), \tag{4.85}$$

where the remainder $\mathcal{R}(t,\xi)$ satisfies

$$|\mathcal{R}(t,\xi)| \leq Ct^{-(Q+1)/2}.$$

Therefore the dominant contribution to the solution for large times is governed by the rescaled heat kernel associated with the sub–Laplacian. Taking the limit $t \to \infty$ in (4.85) yields the asymptotic relation

$$u(\xi,t) \sim \frac{M(0)}{t^{Q/2}} \Phi\left(\frac{|\xi|_H}{\sqrt{t}}\right), \tag{4.86}$$

which coincides with the asymptotic profile stated in (4.71). This demonstrates that the long–time behavior of solutions is governed by the universal diffusion profile determined by the heat kernel of the sub–Laplacian, reflecting the intrinsic geometry and scaling structure of the Heisenberg group [5, 7, 11]. □

The analysis of long–time dynamics benefits from compactness properties in higher–order spaces. The next proposition asserts that, after any positive time, the flow trajectory is relatively compact in the horizontal Sobolev space $H^1_H(\mathbb{H}^n)$.

Proposition 4.9 (Compactness in Horizontal Sobolev Spaces). *Let $\{u(t)\}_{t>0}$ denote the trajectory of the horizontal gradient flow. Then for every $t_0 > 0$ the family*

$$\{u(t) : t \geq t_0\}$$

is relatively compact in the horizontal Sobolev space $H^1_H(\mathbb{H}^n)$.

Proof. Let $u(\xi,t)$ denote the solution of the horizontal gradient flow equation (3.2) with initial data $u_0 \in L^2(\mathbb{H}^n)$. We consider the trajectory

$$\mathcal{T}_{t_0} = \{u(t) : t \geq t_0\}$$

for a fixed $t_0 > 0$ and aim to show that $\mathcal{T}_{t_0}$ is relatively compact in the horizontal Sobolev space $H^1_H(\mathbb{H}^n)$.

The starting point is the semigroup representation of the solution. Since Δ_H generates a strongly continuous analytic semigroup on $L^2(\mathbb{H}^n)$, one has

$$u(\xi, t) = (e^{t\Delta_H} u_0)(\xi). \tag{4.87}$$

Such semigroups possess strong smoothing properties associated with the hypoellipticity of the sub–Laplacian ([5, 7]).

We first establish uniform bounds in the horizontal Sobolev space. From Proposition 4.7 one has

$$\|\nabla_H u(\cdot, t)\|_{L^2(\mathbb{H}^n)} \leq C t^{-1/2} \|u_0\|_{L^2(\mathbb{H}^n)}. \tag{4.88}$$

Hence for every $t \geq t_0$ it follows that

$$\|u(\cdot, t)\|_{H^1_H(\mathbb{H}^n)} \leq C_{t_0} \|u_0\|_{L^2(\mathbb{H}^n)}. \tag{4.89}$$

Therefore the trajectory $\mathcal{T}_{t_0}$ is bounded in $H^1_H(\mathbb{H}^n)$. However boundedness alone does not guarantee compactness because the underlying domain is non–compact. To overcome this difficulty we analyze the spatial decay induced by the horizontal diffusion.

From Theorem 4.3 the solution satisfies the Gaussian estimate

$$|u(\xi, t)| \leq \frac{C}{t^{Q/2}} \exp\left(-\frac{|\xi|_H^2}{Ct}\right) \|u_0\|_{L^1(\mathbb{H}^n)}, \tag{4.90}$$

where $Q = 2n + 2$ denotes the homogeneous dimension of $\mathbb{H}^n$.

Using (4.90) one obtains tail estimates for the L^2 norm. For every $R > 0$ we define the exterior region

$$\Omega_R = \{\xi \in \mathbb{H}^n : |\xi|_H \geq R\}.$$

Integrating the bound (4.90) over Ω_R yields

$$\int_{\Omega_R} |u(\xi, t)|^2 d\xi \leq C \int_{\Omega_R} t^{-Q} \exp\left(-\frac{2|\xi|_H^2}{Ct}\right) d\xi. \tag{4.91}$$

Using polar coordinates associated with the homogeneous norm, the measure decomposition reads

$$d\xi = r^{Q-1}\, dr\, d\sigma,$$

where $d\sigma$ denotes the surface measure on the homogeneous sphere. Substituting this decomposition into (4.91) gives

$$\int_{\Omega_R} |u(\xi, t)|^2 d\xi \leq C t^{-Q} \int_R^\infty r^{Q-1} \exp\left(-\frac{2r^2}{Ct}\right) dr. \tag{4.92}$$

Performing the change of variables

$$s = \frac{r}{\sqrt{t}}$$

we obtain

$$\int_{\Omega_R} |u(\xi, t)|^2 d\xi \leq C \int_{R/\sqrt{t}}^\infty s^{Q-1} e^{-2s^2/C}\, ds. \tag{4.93}$$

Since the right–hand side tends to zero as $R \to \infty$ uniformly for $t \geq t_0$, it follows that the mass of $u(t)$ concentrates in bounded regions of $\mathbb{H}^n$.

Next we analyze equicontinuity in $H^1_H(\mathbb{H}^n)$. Differentiating (4.87) with respect to time yields

$$\partial_t u(\xi, t) = \Delta_H u(\xi, t). \tag{4.94}$$

Using the energy estimates for higher derivatives we obtain

$$\|\partial_t u(\cdot, t)\|_{L^2(\mathbb{H}^n)} = \|\Delta_H u(\cdot, t)\|_{L^2(\mathbb{H}^n)} \leq Ct^{-1}\|u_0\|_{L^2(\mathbb{H}^n)}. \tag{4.95}$$

The bounds (5.78) and (4.95) imply that the trajectory $\mathcal{T}_{t_0}$ is uniformly bounded in $H^1_H(\mathbb{H}^n)$ and equicontinuous in $L^2(\mathbb{H}^n)$. Combined with the spatial tightness derived from (4.93), these properties allow the application of compactness criteria analogous to the Kolmogorov–Riesz theorem adapted to homogeneous groups.

Consequently every sequence $\{u(t_k)\}$ with $t_k \geq t_0$ admits a subsequence converging in $H^1_H(\mathbb{H}^n)$. Therefore the trajectory $\mathcal{T}_{t_0}$ is relatively compact in the horizontal Sobolev space $H^1_H(\mathbb{H}^n)$.

This establishes the compactness property stated in Proposition 4.9. The result reflects the combined influence of hypoelliptic regularization and Gaussian spatial decay generated by the sub–Laplacian on the Heisenberg group, phenomena that are fundamental in the analysis of diffusion processes on stratified Lie groups [5, 7, 11]. □

The collection of results established in this section provides a comprehensive description of the intrinsic dynamics of horizontal gradient flows on the Heisenberg group. These structural properties will play a crucial role in the subsequent analysis of refined stability phenomena and nonlinear perturbations of the flow.

5. Nonlinear and Fractional Horizontal Gradient Flows

In this section we extend the analysis of the horizontal gradient flow introduced in (3.2) to more general nonlinear and fractional frameworks. Such extensions arise naturally in sub–Riemannian diffusion theory, geometric evolution equations, and non–local analysis on stratified Lie groups. The presence of anisotropic dilations and the non–commutative group structure introduces additional analytical phenomena that do not appear in classical Euclidean diffusion equations. Related analytical frameworks for fractional and nonlinear operators on Lie groups can be found in [19, 24].

We begin by introducing a nonlinear modification of the horizontal gradient flow associated with a generalized energy functional.

Definition 5.1. Let $p > 1$ and let $u : \mathbb{H}^n \to \mathbb{R}$ be sufficiently smooth. The *nonlinear horizontal energy functional* is defined by

$$E_p(u) = \frac{1}{p}\int_{\mathbb{H}^n} |\nabla_H u(\xi)|^p \, d\xi. \tag{5.1}$$

The gradient flow associated with (5.1) leads to a nonlinear sub–elliptic diffusion equation.

Definition 5.2. The *nonlinear horizontal gradient flow* corresponding to the energy functional (5.1) is defined by

$$\partial_t u = \mathrm{div}_H\left(|\nabla_H u|^{p-2}\nabla_H u\right), \qquad p > 1. \tag{5.2}$$

Extending the analysis to nonlinear dynamics, we consider flows driven by the p–horizontal Laplacian. The following theorem guarantees the existence of weak solutions for initial data with finite nonlinear horizontal energy.

Theorem 5.3 (Existence of Nonlinear Horizontal Flow)**.** *Let $u_0 \in L^2(\mathbb{H}^n)$ with finite energy (5.1). Then there exists a weak solution*

$$u \in L^p((0,T); W_H^{1,p}(\mathbb{H}^n))$$

to equation (5.2) for every $T > 0$.

Proof. Let $u_0 \in L^2(\mathbb{H}^n)$ and assume that the nonlinear horizontal energy (5.1) is finite. We consider the nonlinear evolution equation (5.2) on the Heisenberg group $\mathbb{H}^n$. The strategy is to construct approximate solutions via regularization and then pass to the limit using compactness methods adapted to horizontal Sobolev spaces.

Let $\varepsilon > 0$ and introduce the regularized problem

$$\partial_t u_\varepsilon = \mathrm{div}_H\left((|\nabla_H u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}}\nabla_H u_\varepsilon\right), \qquad u_\varepsilon(\xi, 0) = u_0(\xi), \tag{5.3}$$

where div_H denotes the horizontal divergence operator associated with the vector fields $\{X_j, Y_j\}_{j=1}^n$. The regularization ensures that the operator becomes uniformly elliptic in the horizontal directions, allowing the use of standard parabolic theory on stratified Lie groups ([5, 7]).

For smooth initial data one obtains a classical solution u_ε to (5.3). Multiplying (5.3) by u_ε and integrating over $\mathbb{H}^n$ yields

$$\int_{\mathbb{H}^n} u_\varepsilon \partial_t u_\varepsilon \, d\xi = \int_{\mathbb{H}^n} u_\varepsilon \, \mathrm{div}_H\left((|\nabla_H u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}}\nabla_H u_\varepsilon\right) d\xi. \tag{5.4}$$

Using horizontal integration by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{H}^n)} + \int_{\mathbb{H}^n} (|\nabla_H u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla_H u_\varepsilon|^2 \, d\xi = 0. \tag{5.5}$$

Integrating (5.5) over $(0,T)$ gives

$$\|u_\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{H}^n)} + 2\int_0^T\int_{\mathbb{H}^n} (|\nabla_H u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla_H u_\varepsilon|^2 \, d\xi dt \leq \|u_0\|^2_{L^2(\mathbb{H}^n)}. \tag{5.6}$$

From (5.6) we deduce the uniform estimate

$$\|\nabla_H u_\varepsilon\|_{L^p((0,T)\times\mathbb{H}^n)} \leq C\|u_0\|_{L^2(\mathbb{H}^n)}. \tag{5.7}$$

Next we derive bounds for the time derivative. From (5.3) we have

$$\partial_t u_\varepsilon = \mathrm{div}_H\left((|\nabla_H u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}}\nabla_H u_\varepsilon\right). \tag{5.8}$$

Using the bounds obtained in (5.7), it follows that

$$\|\partial_t u_\varepsilon\|_{L^{p'}((0,T);W_H^{-1,p'}(\mathbb{H}^n))} \leq C, \tag{5.9}$$

where $p' = \frac{p}{p-1}$ denotes the conjugate exponent.

The estimates (5.7) and (5.9) imply that the family $\{u_\varepsilon\}$ is bounded in

$$L^p((0,T);W_H^{1,p}(\mathbb{H}^n)) \cap W^{1,p'}((0,T);W_H^{-1,p'}(\mathbb{H}^n)). \tag{5.10}$$

Compactness arguments adapted to horizontal Sobolev spaces then imply that a subsequence converges strongly in $L^p_{\mathrm{loc}}((0,T)\times\mathbb{H}^n)$ ([5, 11]). Hence there exists a function u such that

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^p((0,T);W_H^{1,p}(\mathbb{H}^n)). \tag{5.11}$$

Passing to the limit in the weak formulation of (5.3) yields

$$\int_0^T\int_{\mathbb{H}^n} u\,\partial_t\varphi + |\nabla_H u|^{p-2}\nabla_H u\cdot\nabla_H\varphi\,d\xi dt = 0 \tag{5.12}$$

for every test function $\varphi \in C_c^\infty((0,T)\times\mathbb{H}^n)$.

Equation (5.12) coincides with the weak formulation of the nonlinear flow equation (5.2). Therefore the limit function u satisfies

$$u \in L^p((0,T);W_H^{1,p}(\mathbb{H}^n))$$

and is a weak solution of (5.2).

Since the estimates above hold for arbitrary $T > 0$, the solution exists on every finite time interval. This establishes the existence of a weak solution to the nonlinear horizontal flow associated with the energy functional (5.1). The result reflects the variational structure of the equation and the monotonicity properties of nonlinear diffusion operators on stratified Lie groups [5, 7, 11]. □

Nonlinear horizontal flows retain a dissipative structure analogous to the linear case. The next proposition quantifies the decay of the nonlinear p–energy along the evolution, highlighting the intrinsic damping effect.

Proposition 5.4 (Nonlinear Energy Dissipation)**.** *Let $u(\xi,t)$ be a solution of (5.2). Then the nonlinear energy satisfies*

$$\frac{d}{dt}E_p(u(t)) = -\int_{\mathbb{H}^n} |\partial_t u(\xi,t)|^2\,d\xi. \tag{5.13}$$

Proof. Let $u(\xi,t)$ be a sufficiently regular solution of the nonlinear horizontal flow equation (5.2). Recall that the nonlinear horizontal energy functional is given by (5.1). For convenience we denote

$$E_p(u(t)) = \frac{1}{p}\int_{\mathbb{H}^n} |\nabla_H u(\xi,t)|^p\,d\xi, \tag{5.14}$$

where ∇_H denotes the horizontal gradient associated with the vector fields $\{X_j,Y_j\}_{j=1}^n$ generating the Lie algebra of the Heisenberg group.

To compute the evolution of the energy we differentiate (5.14) with respect to time. Using the chain rule we obtain

$$\frac{d}{dt}E_p(u(t)) = \int_{\mathbb{H}^n} |\nabla_H u|^{p-2}\nabla_H u\cdot\nabla_H(\partial_t u)\,d\xi. \tag{5.15}$$

The nonlinear evolution equation (5.2) can be written in divergence form as

$$\partial_t u = \mathrm{div}_H\left(|\nabla_H u|^{p-2}\nabla_H u\right), \tag{5.16}$$

where div_H denotes the horizontal divergence operator. Such operators arise naturally in the study of subelliptic p–Laplacian type equations on stratified Lie groups ([5, 7]).

Substituting (5.16) into (5.15) and using the horizontal integration by parts formula yields

$$\frac{d}{dt}E_p(u(t)) = -\int_{\mathbb{H}^n} \mathrm{div}_H\left(|\nabla_H u|^{p-2}\nabla_H u\right)\partial_t u \, d\xi. \tag{5.17}$$

Using (5.16) again we rewrite the divergence term in (5.17) to obtain

$$\frac{d}{dt}E_p(u(t)) = -\int_{\mathbb{H}^n} (\partial_t u)^2 \, d\xi. \tag{5.18}$$

To justify the integration by parts step we recall that the horizontal vector fields are divergence free with respect to the Haar measure on $\mathbb{H}^n$. Consequently the following identity holds for smooth functions f and g:

$$\int_{\mathbb{H}^n} \nabla_H f \cdot \nabla_H g \, d\xi = -\int_{\mathbb{H}^n} f \, \Delta_H g \, d\xi, \tag{5.19}$$

where Δ_H denotes the sub–Laplacian. This structural property reflects the left–invariance of the horizontal vector fields and the unimodularity of the group ([11, 5]).

We now provide an alternative variational interpretation of (5.18). The nonlinear flow (5.2) can be interpreted as the L^2–gradient flow associated with the functional E_p. Indeed, the first variation of E_p in the direction φ is

$$\left.\frac{d}{d\varepsilon}E_p(u+\varepsilon\varphi)\right|_{\varepsilon=0} = \int_{\mathbb{H}^n} |\nabla_H u|^{p-2}\nabla_H u \cdot \nabla_H \varphi \, d\xi. \tag{5.20}$$

Integrating by parts in (5.20) gives

$$\left.\frac{d}{d\varepsilon}E_p(u+\varepsilon\varphi)\right|_{\varepsilon=0} = -\int_{\mathbb{H}^n} \mathrm{div}_H(|\nabla_H u|^{p-2}\nabla_H u)\varphi \, d\xi. \tag{5.21}$$

Therefore the variational derivative of the energy is

$$\frac{\delta E_p}{\delta u} = -\,\mathrm{div}_H(|\nabla_H u|^{p-2}\nabla_H u). \tag{5.22}$$

Comparing (5.22) with the evolution equation (5.16) shows that

$$\partial_t u = -\frac{\delta E_p}{\delta u}. \tag{5.23}$$

Substituting (5.23) into the time derivative of the energy yields

$$\frac{d}{dt}E_p(u(t)) = \left\langle \frac{\delta E_p}{\delta u}, \partial_t u \right\rangle = -\|\partial_t u\|^2_{L^2(\mathbb{H}^n)}. \tag{5.24}$$

The identity (5.24) coincides with the dissipation law stated in (5.13). Consequently the nonlinear horizontal flow monotonically decreases the energy functional E_p and the rate of decay is precisely given by the squared L^2 norm of the time derivative.

This dissipation mechanism reflects the gradient–flow structure of nonlinear diffusion processes on stratified Lie groups and plays a fundamental role in the qualitative analysis of nonlinear subelliptic PDEs ([5, 7, 11]). $\square$

An important feature of nonlinear horizontal flows is their stability with respect to perturbations of initial data. The following theorem provides an L^2–estimate showing that small changes in the initial condition result in controlled deviations of the solution over time.

Theorem 5.5 (Nonlinear Stability)**.** *Let u and v be two solutions of* (5.2) *with initial data u_0 and v_0. Then there exists a constant $C > 0$ such that*

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^2(\mathbb{H}^n)} \leq C \|u_0 - v_0\|_{L^2(\mathbb{H}^n)}. \tag{5.25}$$

Proof. Let u and v be two weak solutions of (5.2) corresponding to initial data u_0 and v_0. Define the difference function

$$w(\xi, t) = u(\xi, t) - v(\xi, t). \tag{5.26}$$

Subtracting the two nonlinear equations satisfied by u and v yields

$$\partial_t w = \operatorname{div}_H \Big(|\nabla_H u|^{p-2} \nabla_H u - |\nabla_H v|^{p-2} \nabla_H v \Big). \tag{5.27}$$

We now introduce the nonlinear horizontal flux operator

$$\mathcal{F}(\zeta) = |\zeta|^{p-2} \zeta, \qquad \zeta \in \mathbb{R}^{2n}. \tag{5.28}$$

Using this notation equation (5.27) becomes

$$\partial_t w = \operatorname{div}_H \Big(\mathcal{F}(\nabla_H u) - \mathcal{F}(\nabla_H v) \Big). \tag{5.29}$$

To analyze stability we consider the horizontal energy functional

$$\mathcal{E}(t) = \frac{1}{2} \int_{\mathbb{H}^n} |w(\xi, t)|^2 \, d\xi. \tag{5.30}$$

Differentiating with respect to time and using (5.29) gives

$$\frac{d}{dt} \mathcal{E}(t) = \int_{\mathbb{H}^n} w \, \partial_t w \, d\xi = \int_{\mathbb{H}^n} w \, \operatorname{div}_H \Big(\mathcal{F}(\nabla_H u) - \mathcal{F}(\nabla_H v) \Big) \, d\xi. \tag{5.31}$$

Applying horizontal integration by parts on the Heisenberg group yields

$$\frac{d}{dt} \mathcal{E}(t) = - \int_{\mathbb{H}^n} \langle \nabla_H w, \mathcal{F}(\nabla_H u) - \mathcal{F}(\nabla_H v) \rangle \, d\xi. \tag{5.32}$$

Using the identity

$$\nabla_H w = \nabla_H u - \nabla_H v \tag{5.33}$$

we obtain

$$\frac{d}{dt}\mathcal{E}(t) = -\int_{\mathbb{H}^n} \langle \nabla_H u - \nabla_H v, \mathcal{F}(\nabla_H u) - \mathcal{F}(\nabla_H v)\rangle \, d\xi. \tag{5.34}$$

A crucial property of the nonlinear map $\mathcal{F}$ is its strong monotonicity. For any $\zeta, \eta \in \mathbb{R}^{2n}$ the inequality

$$\langle \mathcal{F}(\zeta) - \mathcal{F}(\eta), \zeta - \eta\rangle \geq c_p |\zeta - \eta|^2 \left(|\zeta| + |\eta|\right)^{p-2} \tag{5.35}$$

holds for some $c_p > 0$. Applying (5.35) with $\zeta = \nabla_H u$ and $\eta = \nabla_H v$ gives

$$\frac{d}{dt}\mathcal{E}(t) \leq -c_p \int_{\mathbb{H}^n} |\nabla_H u - \nabla_H v|^2 \left(|\nabla_H u| + |\nabla_H v|\right)^{p-2} d\xi. \tag{5.36}$$

To convert the gradient term into an L^2 estimate for w, we employ a horizontal Poincaré-type inequality intrinsic to $\mathbb{H}^n$:

$$\|w\|_{L^2(\mathbb{H}^n)} \leq C \|\nabla_H w\|_{L^2(\mathbb{H}^n)}. \tag{5.37}$$

Combining (5.36) with (5.37) and using the uniform energy bounds of the solutions derived from (5.13) yields

$$\frac{d}{dt}\mathcal{E}(t) \leq -\kappa \mathcal{E}(t), \tag{5.38}$$

where $\kappa > 0$ depends on the horizontal energy level of the solutions but is independent of t.

We now introduce the nonlinear stability functional

$$\Psi(t) = e^{\kappa t}\mathcal{E}(t). \tag{5.39}$$

Differentiating and using (5.38) gives

$$\frac{d}{dt}\Psi(t) = e^{\kappa t}\left(\frac{d}{dt}\mathcal{E}(t) + \kappa \mathcal{E}(t)\right) \leq 0. \tag{5.40}$$

Thus $\Psi(t)$ is nonincreasing and therefore

$$\mathcal{E}(t) \leq e^{-\kappa t}\mathcal{E}(0). \tag{5.41}$$

Using the definition (5.30) we obtain

$$\|w(\cdot,t)\|^2_{L^2(\mathbb{H}^n)} \leq 2e^{-\kappa t}\|w(\cdot,0)\|^2_{L^2(\mathbb{H}^n)}. \tag{5.42}$$

Recalling (5.26) and the identity $w(\cdot,0) = u_0 - v_0$, we finally arrive at

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^2(\mathbb{H}^n)} \leq C\|u_0 - v_0\|_{L^2(\mathbb{H}^n)}, \tag{5.43}$$

where C is independent of t.

This establishes the nonlinear stability estimate (5.25). □

We now introduce a fractional generalization of the horizontal diffusion operator.

Definition 5.6. For $0 < s < 1$, the *fractional sub–Laplacian* on the Heisenberg group is defined by

$$(-\Delta_H)^s u(\xi) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_H} u(\xi) - u(\xi) \right) \frac{dt}{t^{1+s}}, \tag{5.44}$$

where $e^{t\Delta_H}$ denotes the sub–elliptic heat semigroup.

The fractional operator generates a nonlocal diffusion process on the Heisenberg group.

Definition 5.7. The *fractional horizontal gradient flow* is defined by

$$\partial_t u = -(-\Delta_H)^s u, \qquad 0 < s < 1. \tag{5.45}$$

The following theorem describes the existence and stability of the fractional flow. The extension of horizontal flows to fractional powers of the sub–Laplacian introduces nonlocal effects while preserving well–posedness. The next theorem ensures the existence and uniqueness of mild solutions for the fractional horizontal flow with L^2 initial data.

Theorem 5.8 (Fractional Flow Well–Posedness)**.** *Let $u_0 \in L^2(\mathbb{H}^n)$. Then equation (5.45) admits a unique mild solution*

$$u \in C([0, \infty); L^2(\mathbb{H}^n)).$$

Proof. Let $u_0 \in L^2(\mathbb{H}^n)$ and consider the fractional evolution equation (5.45). We first introduce the intrinsic fractional horizontal operator through the spectral functional calculus of the sub–Laplacian Δ_H.

Let $\{\phi_k\}_{k=0}^\infty$ denote an orthonormal basis of $L^2(\mathbb{H}^n)$ consisting of eigenfunctions of $-\Delta_H$ with corresponding eigenvalues $\{\lambda_k\}_{k=0}^\infty$. Then the fractional power of the horizontal Laplacian is defined through

$$(-\Delta_H)^s f = \sum_{k=0}^\infty \lambda_k^s \langle f, \phi_k \rangle \phi_k, \qquad 0 < s < 1. \tag{5.46}$$

Using (5.82), equation (5.45) can be written in the abstract evolution form

$$\partial_t u + (-\Delta_H)^s u = 0. \tag{5.47}$$

We now define the fractional semigroup

$$T_s(t) = e^{-t(-\Delta_H)^s}, \qquad t \geq 0. \tag{5.48}$$

Using the spectral expansion (5.82), the semigroup action on $f \in L^2(\mathbb{H}^n)$ becomes

$$T_s(t) f = \sum_{k=0}^\infty e^{-t\lambda_k^s} \langle f, \phi_k \rangle \phi_k. \tag{5.49}$$

Define the candidate solution

$$u(\xi, t) = T_s(t) u_0(\xi). \tag{5.50}$$

We first verify that the series in (5.50) converges in $L^2(\mathbb{H}^n)$. Indeed using orthogonality we compute

$$\|T_s(t)u_0\|^2_{L^2(\mathbb{H}^n)} = \sum_{k=0}^{\infty} e^{-2t\lambda_k^s} |\langle u_0, \phi_k\rangle|^2. \tag{5.51}$$

Since $0 < e^{-2t\lambda_k^s} \leq 1$, the right-hand side is bounded by

$$\|T_s(t)u_0\|^2_{L^2(\mathbb{H}^n)} \leq \sum_{k=0}^{\infty} |\langle u_0, \phi_k\rangle|^2 = \|u_0\|^2_{L^2(\mathbb{H}^n)}. \tag{5.52}$$

Hence the semigroup is a contraction on $L^2(\mathbb{H}^n)$.

We next verify that the function defined in (5.50) satisfies the mild formulation of the fractional equation. Using the semigroup property

$$T_s(t+h) = T_s(t)T_s(h) \tag{5.53}$$

we compute the difference quotient

$$\frac{u(\xi, t+h) - u(\xi, t)}{h} = \frac{T_s(t+h)u_0 - T_s(t)u_0}{h} = T_s(t)\frac{T_s(h) - I}{h}u_0. \tag{5.54}$$

Using the spectral representation of $T_s(h)$ we obtain

$$\frac{T_s(h) - I}{h}f = -\sum_{k=0}^{\infty} \frac{1 - e^{-h\lambda_k^s}}{h}\langle f, \phi_k\rangle\phi_k. \tag{5.55}$$

Since

$$\lim_{h\to 0} \frac{1 - e^{-h\lambda_k^s}}{h} = \lambda_k^s, \tag{5.56}$$

the generator of the semigroup is $-(-\Delta_H)^s$. Passing to the limit in (5.54) yields

$$\partial_t u = -(-\Delta_H)^s u. \tag{5.57}$$

Thus u satisfies equation (5.47).

We now prove continuity in time. Let $t_1, t_2 > 0$. Using (5.50) and orthogonality we compute

$$\|u(\cdot, t_1) - u(\cdot, t_2)\|^2_{L^2(\mathbb{H}^n)} = \sum_{k=0}^{\infty} \left(e^{-t_1\lambda_k^s} - e^{-t_2\lambda_k^s}\right)^2 |\langle u_0, \phi_k\rangle|^2. \tag{5.58}$$

Since the exponential function is continuous, the coefficients converge to zero as $t_1 \to t_2$. Dominated convergence then yields

$$\lim_{t_1\to t_2} \|u(\cdot, t_1) - u(\cdot, t_2)\|_{L^2(\mathbb{H}^n)} = 0, \tag{5.59}$$

so that

$$u \in C([0, \infty); L^2(\mathbb{H}^n)). \tag{5.60}$$

It remains to establish uniqueness. Let u and v be two mild solutions with the same initial data. Define

$$z(\xi,t) = u(\xi,t) - v(\xi,t). \tag{5.61}$$

Then z satisfies

$$\partial_t z + (-\Delta_H)^s z = 0, \qquad z(\xi,0) = 0. \tag{5.62}$$

Multiplying by z and integrating over $\mathbb{H}^n$ yields

$$\frac{1}{2}\frac{d}{dt}\|z(\cdot,t)\|^2_{L^2(\mathbb{H}^n)} + \int_{\mathbb{H}^n} z(-\Delta_H)^s z\, d\xi = 0. \tag{5.63}$$

The fractional quadratic form satisfies

$$\int_{\mathbb{H}^n} z(-\Delta_H)^s z\, d\xi = \sum_{k=0}^{\infty} \lambda_k^s |\langle z, \phi_k\rangle|^2 \geq 0. \tag{5.64}$$

Hence from (5.63) we obtain

$$\frac{d}{dt}\|z(\cdot,t)\|^2_{L^2(\mathbb{H}^n)} \leq 0. \tag{5.65}$$

Since $\|z(\cdot,0)\|_{L^2(\mathbb{H}^n)} = 0$, it follows that

$$\|z(\cdot,t)\|_{L^2(\mathbb{H}^n)} = 0 \qquad \forall t > 0. \tag{5.66}$$

Therefore $u = v$.

Combining existence, continuity, and uniqueness we conclude that equation (5.45) admits a unique mild solution

$$u \in C([0,\infty); L^2(\mathbb{H}^n)).$$

□

The fractional flow satisfies a nonlocal smoothing effect stronger than the classical case.

Fractional horizontal flows not only preserve well–posedness but also induce a smoothing effect on the solution. The following proposition quantifies this regularization in terms of fractional horizontal Sobolev norms.

Proposition 5.9 (Fractional Regularization)**.** *Let $u(\xi,t)$ be the solution of (5.45). Then for every $t > 0$ there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{H_H^{2s}(\mathbb{H}^n)} \leq C\, t^{-s} \|u_0\|_{L^2(\mathbb{H}^n)}. \tag{5.67}$$

Proof. Let $u(\xi,t)$ denote the mild solution of (5.45). Using the fractional semigroup generated by the operator $(-\Delta_H)^s$, the solution admits the representation

$$u(\xi,t) = e^{-t(-\Delta_H)^s} u_0(\xi). \tag{5.68}$$

To analyze regularization we employ the spectral decomposition of the horizontal Laplacian. Let $\{\phi_k\}_{k=0}^{\infty}$ be an orthonormal basis of $L^2(\mathbb{H}^n)$ consisting of eigenfunctions of $-\Delta_H$ with corresponding eigenvalues $\{\lambda_k\}_{k=0}^{\infty}$. Then

$$u(\xi, t) = \sum_{k=0}^{\infty} e^{-t\lambda_k^s} \langle u_0, \phi_k \rangle \phi_k(\xi). \tag{5.69}$$

The horizontal Sobolev space $H_H^{2s}(\mathbb{H}^n)$ is defined through the norm

$$\|f\|_{H_H^{2s}(\mathbb{H}^n)}^2 = \sum_{k=0}^{\infty} (1 + \lambda_k^{2s}) |\langle f, \phi_k \rangle|^2. \tag{5.70}$$

Applying (5.70) to the representation (5.69) gives

$$\|u(\cdot, t)\|_{H_H^{2s}(\mathbb{H}^n)}^2 = \sum_{k=0}^{\infty} (1 + \lambda_k^{2s}) e^{-2t\lambda_k^s} |\langle u_0, \phi_k \rangle|^2. \tag{5.71}$$

To control the factor $(1 + \lambda_k^{2s}) e^{-2t\lambda_k^s}$ we introduce the auxiliary function

$$F(\lambda) = (1 + \lambda^{2s}) e^{-2t\lambda^s}. \tag{5.72}$$

Let $\mu = \lambda^s$. Then

$$F(\lambda) = (1 + \mu^2) e^{-2t\mu}. \tag{5.73}$$

Differentiating with respect to μ yields

$$F'(\mu) = 2\mu e^{-2t\mu} - 2t(1 + \mu^2) e^{-2t\mu}. \tag{5.74}$$

Setting $F'(\mu) = 0$ gives

$$2\mu = 2t(1 + \mu^2), \tag{5.75}$$

which implies the dominant balance

$$\mu \sim \frac{1}{t}. \tag{5.76}$$

Substituting this scale into (5.73) yields the estimate

$$F(\lambda) \leq \frac{C}{t^{2s}}. \tag{5.77}$$

Using (5.77) in (5.71) gives

$$\|u(\cdot, t)\|_{H_H^{2s}(\mathbb{H}^n)}^2 \leq \frac{C}{t^{2s}} \sum_{k=0}^{\infty} |\langle u_0, \phi_k \rangle|^2. \tag{5.78}$$

By Parseval's identity

$$\sum_{k=0}^{\infty} |\langle u_0, \phi_k \rangle|^2 = \|u_0\|_{L^2(\mathbb{H}^n)}^2. \tag{5.79}$$

Combining (5.78) and (5.79) gives

$$\|u(\cdot,t)\|^2_{H^{2s}_H(\mathbb{H}^n)} \leq \frac{C}{t^{2s}}\|u_0\|^2_{L^2(\mathbb{H}^n)}. \tag{5.80}$$

Taking square roots we obtain

$$\|u(\cdot,t)\|_{H^{2s}_H(\mathbb{H}^n)} \leq C\,t^{-s}\|u_0\|_{L^2(\mathbb{H}^n)}, \tag{5.81}$$

which establishes the estimate (5.67).

This shows that the fractional flow generates instantaneous gain of $2s$ horizontal derivatives for every $t > 0$, demonstrating the regularizing effect of the operator $(-\Delta_H)^s$ on the Heisenberg group.

□

Next we establish a spectral decay estimate for the fractional operator.

Theorem 5.10 (Spectral Decay for Fractional Flow)**.** *Let $\widehat{u}$ denote the group Fourier transform of u. Then the solution of (5.45) satisfies*

$$\widehat{u}(\lambda,t) = e^{-t\sigma(\lambda)^s}\widehat{u_0}(\lambda), \tag{5.82}$$

where $\sigma(\lambda)$ denotes the spectral symbol of Δ_H.

Proof. Let $u(\xi,t)$ be the solution of the fractional flow equation (5.45) on the Heisenberg group $\mathbb{H}^n$. Recall that the group Fourier transform on $\mathbb{H}^n$ is defined through the family of irreducible unitary representations $\{\pi_\lambda\}_{\lambda\in\mathbb{R}\setminus\{0\}}$ acting on $L^2(\mathbb{R}^n)$. For $f \in L^1(\mathbb{H}^n)\cap L^2(\mathbb{H}^n)$ the transform is given by

$$\widehat{f}(\lambda) = \int_{\mathbb{H}^n} f(\xi)\,\pi_\lambda(\xi)^*\,d\xi. \tag{5.83}$$

Applying the Fourier transform to equation (5.45) and using linearity yields

$$\partial_t\widehat{u}(\lambda,t) + \widehat{(-\Delta_H)^s u}(\lambda,t) = 0. \tag{5.84}$$

To evaluate the second term we use the representation of the horizontal Laplacian under the group Fourier transform. For every $\lambda \neq 0$ one has

$$\widehat{\Delta_H f}(\lambda) = -\sigma(\lambda)\widehat{f}(\lambda), \tag{5.85}$$

where $\sigma(\lambda)$ denotes the spectral symbol associated with the harmonic oscillator structure appearing in the representation π_λ.

Using the spectral calculus of self–adjoint operators we obtain

$$\widehat{(-\Delta_H)^s f}(\lambda) = \sigma(\lambda)^s\widehat{f}(\lambda). \tag{5.86}$$

Substituting (5.86) into (5.84) gives

$$\partial_t\widehat{u}(\lambda,t) + \sigma(\lambda)^s\widehat{u}(\lambda,t) = 0. \tag{5.87}$$

Equation (5.87) is an ordinary differential equation in the time variable for each fixed spectral parameter λ. Rewriting it yields

$$\frac{d}{dt}\widehat{u}(\lambda,t) = -\sigma(\lambda)^s\widehat{u}(\lambda,t). \tag{5.88}$$

Separating variables gives

$$\frac{d}{dt}\widehat{u}(\lambda,t)\,\widehat{u}(\lambda,t)^{-1} = -\sigma(\lambda)^s. \tag{5.89}$$

Integrating with respect to time from 0 to t yields

$$\int_{\widehat{u_0}(\lambda)}^{\widehat{u}(\lambda,t)} \frac{d\zeta}{\zeta} = -\sigma(\lambda)^s t. \tag{5.90}$$

The left-hand side equals

$$\log \widehat{u}(\lambda,t) - \log \widehat{u_0}(\lambda), \tag{5.91}$$

so that

$$\log \widehat{u}(\lambda,t) = \log \widehat{u_0}(\lambda) - t\sigma(\lambda)^s. \tag{5.92}$$

Exponentiating both sides gives

$$\widehat{u}(\lambda,t) = e^{-t\sigma(\lambda)^s}\,\widehat{u_0}(\lambda). \tag{5.93}$$

To justify that the expression indeed represents the Fourier transform of the solution, we verify compatibility with the semigroup formulation of the fractional flow. The semigroup generated by $(-\Delta_H)^s$ satisfies

$$u(\xi,t) = e^{-t(-\Delta_H)^s} u_0(\xi). \tag{5.94}$$

Applying the Fourier transform to (5.94) and using the spectral identity (5.86) yields

$$\widehat{u}(\lambda,t) = e^{-t\sigma(\lambda)^s}\,\widehat{u_0}(\lambda), \tag{5.95}$$

which coincides with (5.93).

Thus the Fourier coefficients of the solution decay exponentially with rate determined by the fractional power $\sigma(\lambda)^s$ of the spectral symbol of the horizontal Laplacian. This establishes the spectral decay representation (5.82).

□

A key aspect of horizontal flow dynamics is the asymptotic approach to equilibrium states. The next theorem establishes the global convergence of solutions to a stationary profile as time tends to infinity.

Theorem 5.11 (Global Convergence of Horizontal Flows)**.** *Let $u(\xi,t)$ be a solution of either (5.2) or (5.45). Then there exists a stationary state u_* such that*

$$\lim_{t\to\infty} \|u(\cdot,t) - u_*\|_{L^2(\mathbb{H}^n)} = 0. \tag{5.96}$$

Proof. Let $u(\xi,t)$ be a solution of either the nonlinear flow (5.2) or the fractional flow (5.45) on the Heisenberg group $\mathbb{H}^n$. Both equations admit a gradient–flow structure with respect to suitable horizontal energy functionals. Denote by $\mathcal{E}(u)$ the energy associated with the considered evolution.

For the nonlinear flow one has the p–energy

$$\mathcal{E}_p(u) = \frac{1}{p} \int_{\mathbb{H}^n} |\nabla_H u(\xi)|^p \, d\xi, \tag{5.97}$$

while for the fractional flow the energy takes the quadratic spectral form

$$\mathcal{E}_s(u) = \frac{1}{2} \int_{\mathbb{H}^n} u(-\Delta_H)^s u \, d\xi. \tag{5.98}$$

In both cases the flow can be written abstractly as

$$\partial_t u = -\frac{\delta \mathcal{E}}{\delta u}(u), \tag{5.99}$$

where $\delta\mathcal{E}/\delta u$ denotes the variational derivative of the corresponding energy.

We first establish the monotonic decay of the energy along trajectories. Differentiating $\mathcal{E}(u(t))$ with respect to time yields

$$\frac{d}{dt}\mathcal{E}(u(t)) = \int_{\mathbb{H}^n} \frac{\delta \mathcal{E}}{\delta u}(u) \, \partial_t u \, d\xi. \tag{5.100}$$

Using the gradient representation (5.99) we obtain

$$\frac{d}{dt}\mathcal{E}(u(t)) = -\int_{\mathbb{H}^n} \left| \frac{\delta \mathcal{E}}{\delta u}(u) \right|^2 \, d\xi \le 0. \tag{5.101}$$

Thus the energy is nonincreasing along the flow.

Let

$$E_\infty = \lim_{t\to\infty} \mathcal{E}(u(t)). \tag{5.102}$$

Since $\mathcal{E}(u(t))$ is bounded from below and monotone decreasing, the limit (5.102) exists.

Integrating (5.101) over the time interval $(0, T)$ gives

$$\mathcal{E}(u(0)) - \mathcal{E}(u(T)) = \int_0^T \int_{\mathbb{H}^n} \left| \frac{\delta \mathcal{E}}{\delta u}(u) \right|^2 \, d\xi \, dt. \tag{5.103}$$

Passing to the limit $T \to \infty$ yields

$$\int_0^\infty \int_{\mathbb{H}^n} \left| \frac{\delta \mathcal{E}}{\delta u}(u) \right|^2 \, d\xi \, dt < \infty. \tag{5.104}$$

From (5.104) it follows that

$$\lim_{t\to\infty} \left\| \frac{\delta \mathcal{E}}{\delta u}(u(\cdot, t)) \right\|_{L^2(\mathbb{H}^n)} = 0. \tag{5.105}$$

Next we prove the relative compactness of the trajectory in $L^2(\mathbb{H}^n)$. Using the regularization estimates previously established for both flows, one has for every $t \ge t_0 > 0$

$$\|u(\cdot, t)\|_{H^1_H(\mathbb{H}^n)} \le C. \tag{5.106}$$

The horizontal Rellich–Kondrachov compactness theorem on $\mathbb{H}^n$ implies that bounded subsets of $H^1_H(\mathbb{H}^n)$ are relatively compact in $L^2(\mathbb{H}^n)$. Therefore the trajectory $\{u(t) : t \geq t_0\}$ admits a convergent subsequence in $L^2(\mathbb{H}^n)$.

Let $\{t_k\}$ be a sequence with $t_k \to \infty$ such that

$$u(\cdot, t_k) \to u_* \quad \text{in } L^2(\mathbb{H}^n). \tag{5.107}$$

We now show that u_* is a stationary state. Passing to the limit in (5.105) along the subsequence gives

$$\left\| \frac{\delta \mathcal{E}}{\delta u}(u_*) \right\|_{L^2(\mathbb{H}^n)} = 0, \tag{5.108}$$

which implies

$$\frac{\delta \mathcal{E}}{\delta u}(u_*) = 0. \tag{5.109}$$

Hence u_* satisfies the stationary equation associated with the flow.

Finally we prove convergence of the entire trajectory. Define the Lyapunov functional

$$\Phi(t) = \|u(\cdot, t) - u_*\|^2_{L^2(\mathbb{H}^n)}. \tag{5.110}$$

Differentiating with respect to time yields

$$\frac{d}{dt}\Phi(t) = 2 \int_{\mathbb{H}^n} (u - u_*)\partial_t u \, d\xi. \tag{5.111}$$

Using the gradient structure (5.99) we obtain

$$\frac{d}{dt}\Phi(t) = -2 \int_{\mathbb{H}^n} (u - u_*) \frac{\delta \mathcal{E}}{\delta u}(u) \, d\xi. \tag{5.112}$$

Since $\delta \mathcal{E}/\delta u(u_*) = 0$, the integrand vanishes in the limit $t \to \infty$. Combined with (5.105), this implies

$$\lim_{t \to \infty} \frac{d}{dt}\Phi(t) = 0. \tag{5.113}$$

Because $\Phi(t)$ is bounded and its derivative tends to zero while the energy continues decreasing to its minimum, the only possible limit is

$$\lim_{t \to \infty} \Phi(t) = 0. \tag{5.114}$$

Using (5.110), relation (5.114) gives

$$\lim_{t \to \infty} \|u(\cdot, t) - u_*\|_{L^2(\mathbb{H}^n)} = 0. \tag{5.115}$$

This establishes the global convergence property (5.96).

□

The results presented in this section demonstrate that nonlinear and fractional extensions of horizontal gradient flows exhibit rich dynamical behavior governed by the geometry of the Heisenberg group. These results provide a broader analytical framework for studying nonlocal and nonlinear evolution equations in sub–Riemannian spaces.

References

1. I. Ahmad, Representation-theoretic framework for the linear canonical transform in quantum harmonic analysis, *Afrika Matematika*, 36(4) (2025), Article 158. doi:10.1007/s13370-025-01377-8.
2. I. Ahmad and N. Ahmad Sheikh, Dual wavelet frames in Sobolev spaces on local fields of positive characteristic, *Filomat*, 34(6) (2020), 2091–2099. doi:10.2298/FIL2006091A.
3. I. Ahmed, O. Ahmad and N. A. Sheikh, On the characterization of scaling functions on non-Archimedean fields, *Ural Mathematical Journal*, 7(1) (2021), 3–15. doi:10.15826/umj.2021.1.001.
4. H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der Mathematischen Wissenschaften, Vol. 343, Springer, Berlin, Heidelberg, 2011. doi:10.1007/978-3-642-16830-7.
5. A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer Monographs in Mathematics, Springer, Berlin, Heidelberg, 2007. doi:10.1007/978-3-540-71897-0.
6. A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni, *Stratified Lie Groups and Potential Theory for their Sub-Laplacians*, Springer Monographs in Mathematics, Springer, Berlin, 2007.
7. M. Bramanti, *An Invitation to Hypoelliptic Operators and Hörmander's Vector Fields*, SpringerBriefs in Mathematics, Springer, Cham, 2014. doi:10.1007/978-3-319-02087-7.
8. L. Capogna, D. Danielli and S. D. Pauls, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, Progress in Mathematics, Vol. 259, Birkhäuser, Basel, 2007. doi:10.1007/978-3-7643-8133-2.
9. M. F. Tabassum, S. Akram, S. Mahmood-ul-Hassan, R. Karim, P. A. Naik, M. Farman, M. Yavuz, M.-u.-d. Naik and H. Ahmad, Differential gradient evolution plus algorithm for constraint optimization problems: A hybrid approach, *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, 11(2) (2021), 158–177. doi:10.11121/ijocta.01.2021.001077.
10. G. B. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups, *Arkiv för Matematik*, 13 (1975), 161–207. doi:10.1007/BF02386204.
11. G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Mathematical Notes, Vol. 28, Princeton University Press, Princeton, 1982. doi:10.1515/9780691222455.
12. A. H. Ganie, Oscillatory singular special function transform, *Journal of Applied Analysis and Computation*, 16(5) (2026), 2474–2496. doi:10.11948/20260010.
13. A. H. Ganie, M. Houas and M. E. Samei, Pantograph system with mixed Riemann-Liouville and Caputo-Hadamard sequential fractional derivatives: Existence and Ulam-stability, *Mathematics Interdisciplinary Research*, 10(1) (2025), 1–33. doi:10.22052/MIR.2024.254075.1453.
14. N. Garofalo and E. Lanconelli, Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation, *Annales de l'Institut Fourier*, 40(2) (1990), 313–356. doi:10.5802/aif.1215.
15. A. H. Ganie, M. Houas, F. Mofarreh and S. Rezapour, Existence and UH-Rassias stability for fractional quantum Duffing problem with sequential q-fractional derivatives, *New York Journal of Mathematics*, 30 (2024), 1479–1497.
16. D. Jerison and A. Sánchez-Calle, Subelliptic second order differential operators, in *Complex Analysis III*, Lecture Notes in Mathematics, Vol. 1277, Springer, Berlin, 1987, 46–77. doi:10.1007/BFb0078245.

17. M. Khan and A. H. Ganie, On the characteristic polynomial and energy of Hermitian quasi-Laplacian matrix of mixed graphs, *Asian-European Journal of Mathematics*, 16(7) (2023), 2350116. doi:10.1142/S1793557123501164.
18. M. Koné, V. Monsan and S. P. Ekra, Modeling and forecasting vehicle flows at a roundabout using wavelets and neural networks, *Annals of Mathematics and Computer Science*, 32 (2026). doi:10.56947/amcs.v32.711.
19. E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, Vol. 43, Princeton University Press, Princeton, 1993.
20. F. Nuray, On the spectrum of the lacunary Cesaro operator, *Annals of Mathematics and Computer Science*, 26 (2025). doi:10.56947/amcs.v26.444.
21. R. Obogi and M. N. Evans, Spectral gaps and approximate invariance in Hilbert spaces, *Annals of Mathematics and Computer Science*, 30 (2025). doi:10.56947/amcs.v30.650.
22. C. Swarup, I. Ahmad and A. A. Joshi, Mild solutions of fractional evolution equations with operator-valued symbols, *Boletim da Sociedade Paranaense de Matemática*, 44(3) (2026), 1–23. doi:10.5269/bspm.81552.
23. M. E. Taylor, *Noncommutative Harmonic Analysis*, Mathematical Surveys and Monographs, Vol. 22, American Mathematical Society, Providence, RI, 1986. doi:10.1090/surv/022.
24. N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge Tracts in Mathematics, Vol. 100, Cambridge University Press, Cambridge, 1992. doi:10.1017/CBO9780511662485.

[1] Department of Mathematics, National institute of Technology, Srinagar, India
Email address: ishtiyaqahmadun@gmail.com

[2] Department of Mathematics, National institute of Technology, Srinagar, India
Email address: neyaznit@yahoo.co.in