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COMMON FIXED POINT THEOREMS USING COMMON $(E.A)$ PROPERTY IN COMPLEX VALUED S-METRIC SPACES

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Abstract. In this paper, we prove some common fixed point theorems for contractive conditions involving rational terms and satisfying common $(E.A)$ property for two pairs of weakly compatible mappings in the setting of complex valued $\mathcal{S}$-metric spaces. The results obtained in this paper extend and generalize several results from the existing literature.

1. Introduction

In 1922, Stefan Banach ([5]) proved his celebrated fixed point theorem, which assures the existence and uniqueness of a fixed point under certain conditions. He formulated the notion of contraction, scientist and mathematicians around the world are publishing new results that are related either to establish a generalization of metric space or to get a improvement of contractive conditions.

In the literature, there are plenty of extensions of the famous Banach contraction principle, which states that every self mapping $\mathcal{P}$ defined on a complete metric space $(\mathcal{X}, d)$ satisfying

$$d(\mathcal{P}(x), \mathcal{P}(y)) \leq \lambda\, d(x, y), \tag{1.1}$$

for all $x, y \in \mathcal{X}$, where $\lambda \in (0, 1)$, has a unique fixed point and for every $p_0 \in \mathcal{X}$, a sequence $\{\mathcal{P}^n p_0\}_{n\geq 1}$ is convergent to the fixed point.

The Banach contraction principle with rational expressions have been expanded and some fixed point and common fixed point theorems have been obtained in [8], [9], [10], [12].

In 2011, Azam et al. [4] introduced the notion of complex valued metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. The results proved by Azam et al. [4] and Bhatt et al. [6] via rational inequality in a complex valued metric space as a contractive condition. Complex valued metric space is very useful in many branches of mathematics, including algebraic geometry, number theory, applied mathematics, applied physics, mechanical engineering, thermodynamics

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and electrical engineering. The study of existence of common fixed point grown from weakly commutativity [30] to compatibility [13] and weakly compatibility [14]. Similarly, the non commutativity of mappings grown from non-compatibility [19] to $(E.A)$ property [1]. The study of common fixed points for non compatible mappings was initiated by Pant [20]. In 2002, Aamri and El Moutawakil [1] introduced a new concept called $(E.A)$-property for pair of mappings which is a generalization of non compatible mappings and they proved some common fixed point theorems. The concept of $(E.A)$-property allows us to replace the completeness requirement of the space by a more general condition of closeness of range. In [22], Pathak et al. established a common fixed point theorem in metric space for an integral type condition and using implicit relation and the $(E.A)$ property. Some authors showed that the notion of weakly compatible mappings and mappings satisfying $(E.A)$-property are independent (see, [21], [23]).

In 2005, Liu et al. [16] defined the notion of common $(E.A)$-property which is an extension of $(E.A)$-property. Many authors established common fixed point theorems by using common $(E.A)$-property in the setup of metric spaces and variants of metric spaces (see, [3, 7, 11]).

After the establishment of complex valued metric spaces, Rouzkardand et al. [25] established some common fixed point theorems satisfying certain rational expressions in these spaces which generalize the result of [4]. In 2012, Sintanuvarat and Kumam [31] extend and improve the results of [4] by replacing the constant of contractive conditions to some control functions. Verma and Pathak in [32] introduced the notion of $(E.A)$-property in complex valued metric space and proved some common fixed point results for two pairs of weakly compatible mappings satisfying a contractive condition of "max" type. Later, many authors have contributed different concepts in this space (see, for example, [18], [26], [27], [31], [28] and many others).

Mlaiki in [17] introduced the concept of complex valued S-metric spaces and prove the existence and uniqueness of a common fixed point of two self-mappings in such space via various contractive conditions. After Mlaiki's results many authors have established several results in complex valued S-metric space via various contractive conditions (see, for example, [24], [33] and many others).

In this paper, we prove some common fixed point theorems for contractive condition involving rational terms satisfying common $(E.A)$ property for two pairs of weakly compatible mappings in the framework of complex valued S-metric spaces. Our results extend, generalize and enrich several results from the existing literature.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$;

(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$;

(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$;
(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \lesssim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|,$$

$$z_1 \precsim z_2,\ z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

In 2014, the following definition was introduced by Mlaiki in [17].

Definition 2.1. ([17]) Let $\mathcal{Y}$ be a nonempty set and $\mathbb{C}$ be the set of all complex numbers. A complex valued $\mathcal{S}$-metric space on $\mathcal{Y}$ is a function $\mathcal{S}\colon \mathcal{Y}^3 \to \mathbb{C}$ that satisfies the following conditions, for all $x, y, z, a \in \mathcal{Y}$:
($\mathcal{CSM}1$) $0 \precsim \mathcal{S}(x, y, z)$,
($\mathcal{CSM}2$) $\mathcal{S}(x, y, z) = 0$ if and only if $x = y = z$,
($\mathcal{CSM}3$) $\mathcal{S}(x, y, z) \precsim \mathcal{S}(x, x, a) + \mathcal{S}(y, y, a) + \mathcal{S}(z, z, a)$.

Then $\mathcal{S}$ is called a complex valued $\mathcal{S}$-metric on $\mathcal{Y}$ and the pair $(\mathcal{Y}, \mathcal{S})$ is called a complex valued $\mathcal{S}$-metric space.

Example 2.2. ([17]) Let $\mathcal{Y} = \mathbb{C}$ be the set of complex numbers. Define a mapping $\mathcal{S}\colon \mathbb{C}^3 \to \mathbb{C}$ by $\mathcal{S}(z_1, z_2, z_3) = |\max\{Re(z_1), Re(z_2)\} - Re(z_2)| + i|\max\{Im(z_1), Im(z_2)\} - Im(z_2)|$. Then it is not difficult to verify that $(\mathbb{C}, \mathcal{S})$ is a complex valued $\mathcal{S}$-metric space.

Definition 2.3. ([17]) If $(\mathcal{Y}, \mathcal{S})$ is called a complex valued $\mathcal{S}$-metric space, then

($\mathbf{\Gamma_1}$) a sequence $\{r_n\}$ in X converges to r if and only if for every $\varepsilon \in \mathbb{C}$ with $0 \prec \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $\mathcal{S}(r_n, r_n, r) \prec \varepsilon$ and we denote this by $r_n \to r$ or $\lim_{n\to\infty} r_n = r$.

($\mathbf{\Gamma_2}$) a sequence $\{r_n\}$ in X is called a Cauchy sequence if for every $\varepsilon \in \mathbb{C}$ with $0 \prec \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $\mathcal{S}(r_n, r_n, r_m) \prec \varepsilon$.

($\mathbf{\Gamma_3}$) An $\mathcal{S}$-metric space $(\mathcal{Y}, \mathcal{S})$ is said to be complete if every Cauchy sequence is convergent.

Definition 2.4. Let X be a non-empty set and let $\mathcal{R}, h\colon X \to X$ be two self mappings of X. Then a point $v \in X$ is called a
($\mathbf{\Delta_1}$) fixed point of operator $\mathcal{R}$ if $\mathcal{R}(v) = v$;
($\mathbf{\Delta_2}$) common fixed point of $\mathcal{R}$ and h if $\mathcal{R}(v) = h(v) = v$.

Definition 2.5. ([2]) Let $\mathcal{P}$ and $\mathcal{R}$ be single valued self-mappings on a set $\mathcal{Y}$. If $u = \mathcal{P}z = \mathcal{R}z$ for some $z \in \mathcal{Y}$, then z is called a coincidence point point of $\mathcal{P}$ and $\mathcal{R}$, and u is called a point of coincidence of $\mathcal{P}$ and $\mathcal{R}$.

Definition 2.6. ([13]) Let $\mathcal{P}$ and $\mathcal{R}$ be single valued self-mappings on a set $\mathcal{Y}$. Mappings $\mathcal{P}$ and $\mathcal{R}$ are said to be commuting if $\mathcal{P}\mathcal{R}v = \mathcal{R}\mathcal{P}v$ for all $v \in \mathcal{Y}$.

Definition 2.7. ([14]) Let $\mathcal{P}$ and $\mathcal{R}$ be single valued self-mappings on a set $\mathcal{Y}$. Mappings $\mathcal{P}$ and $\mathcal{R}$ are said to be weakly compatible if they commute at their coincidence points, i.e., if $\mathcal{P}u = \mathcal{R}u$ for some $u \in \mathcal{Y}$ implies $\mathcal{P}\mathcal{R}u = \mathcal{R}\mathcal{P}u$.

Definition 2.8. ([32]) Let $(\mathcal{Y}, d)$ be a complex valued metric space and let $\mathcal{R}, \mathcal{T}\colon \mathcal{Y} \to \mathcal{Y}$ be two self mappings of $\mathcal{Y}$. The pair $(\mathcal{R}, \mathcal{T})$ is said to satisfy

$(E.A)$-property if there exists a sequence $\{u_n\}$ in $\mathcal{Y}$ such that $\lim_{n\to\infty} \mathcal{R}u_n = \lim_{n\to\infty} \mathcal{T}u_n = \delta$ for some $\delta \in \mathcal{Y}$.

Note that weakly compatibility and $(E.A)$-property are independent of each other (see, [22]).

Example 2.9. Let $\mathcal{Y} = \mathbb{C}$ and let $\mathcal{R}, \mathcal{Q} \colon \mathcal{Y} \to \mathcal{Y}$ be defined by $\mathcal{R}(z) = 4z - 2i$ and $\mathcal{Q}(z) = z + i$ for all $z \in \mathcal{Y}$. Let $\{z_n\} = \{i + \frac{1}{n}\}_{n\geq 1}$ be the sequence in $\mathcal{Y}$. Then

$$\lim_{n\to\infty} \mathcal{R}z_n = \lim_{n\to\infty} \Big(4i + \frac{4}{n} - 2i\Big) = 2i,$$

and

$$\lim_{n\to\infty} \mathcal{Q}z_n = \lim_{n\to\infty} \Big(i + \frac{1}{n} + i\Big) = 2i.$$

Thus there exists a sequence $\{z_n\}$ in $\mathcal{Y}$ such that $\lim_{n\to\infty} \mathcal{R}z_n = \lim_{n\to\infty} \mathcal{Q}z_n = 2i \in \mathcal{Y}$. Hence $\mathcal{R}$ and $\mathcal{Q}$ satisfy $(E.A)$-property.

Liu et al. [16] introduced common $(E.A)$-property in complex valued metric space as follows.

Definition 2.10. ([16]) Let $(\mathcal{Y}, d)$ be a complex valued metric space and let $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{T} \colon \mathcal{Y} \to \mathcal{Y}$ be four self mappings of $\mathcal{Y}$. The pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ satisfy the common $(E.A)$-property if there exist two sequences $\{u_n\}$ and $\{v_n\}$ in $\mathcal{Y}$ such that

$$\lim_{n\to\infty} \mathcal{P}u_n = \lim_{n\to\infty} \mathcal{R}u_n = \lim_{n\to\infty} \mathcal{Q}v_n = \lim_{n\to\infty} \mathcal{T}v_n = z \in \mathcal{Y}.$$

Example 2.11. Let $\mathcal{Y} = \mathbb{C}$ and let d be a complex valued metric and let $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{T} \colon \mathcal{Y} \to \mathcal{Y}$ be four self-maps defined by $\mathcal{P}(z) = 3 + iz$, $\mathcal{Q}(z) = -i - 3z^2$, $\mathcal{R}(z) = -i - 3z$ and $\mathcal{T}(z) = 3 + (z - 2i)$ for all $z \in \mathcal{Y}$. Let $\{x_n\} = \{-1 + \frac{1}{n}\}_{n\geq 1}$ and $\{y_n\} = \{\frac{1}{n} + i\}_{n\geq 1}$ be two sequences in $\mathcal{Y}$. Then

$$\lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \Big(3 - i + \frac{i}{n}\Big) = 3 - i,$$

$$\lim_{n\to\infty} \mathcal{R}x_n = \lim_{n\to\infty} \Big(-i + 3 - \frac{3}{n} + i\Big) = 3 - i,$$

$$\lim_{n\to\infty} \mathcal{Q}y_n = \lim_{n\to\infty} \Big(-i - 3(\frac{1}{n} + i)^2\Big) = 3 - i,$$

and

$$\lim_{n\to\infty} \mathcal{T}y_n = \lim_{n\to\infty} \Big(3 + (\frac{1}{n} + i - 2i)\Big) = 3 - i.$$

Thus there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{Y}$ such that

$$\lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \mathcal{R}x_n = \lim_{n\to\infty} \mathcal{Q}y_n = \lim_{n\to\infty} \mathcal{T}y_n = 3 - i \in \mathcal{Y}.$$

Hence the pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ satisfy common $(E.A)$-property.

Now, we redefine the common $(E.A)$-property in the setting of complex valued S-metric space as follows.

Definition 2.12. Let $(\mathcal{Y}, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{T}\colon \mathcal{Y} \to \mathcal{Y}$ be four self mappings of $\mathcal{Y}$. The pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ are said to satisfy the common $(E.A)$-property if there exist two sequences $\{p_n\}$ and $\{q_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{P}p_n = \lim_{n\to\infty} \mathcal{R}p_n = \lim_{n\to\infty} \mathcal{Q}q_n = \lim_{n\to\infty} \mathcal{T}q_n = t \in \mathcal{Y}.$$

Example 2.13. Let $\mathcal{Y} = \mathbb{C}$ and let $\mathcal{S}\colon \mathbb{C}^3 \to \mathbb{C}$ be defined by $\mathcal{S}(z_1, z_2, z_3) = |\max\{Re(z_1), Re(z_2)\} - Re(z_2)| + i|\max\{Im(z_1), Im(z_2)\} - Im(z_2)|$. Then $(\mathbb{C}, \mathcal{S})$ is a complex valued $\mathcal{S}$-metric space. Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{T}\colon \mathcal{Y} \to \mathcal{Y}$ be four self-maps defined by $\mathcal{P}(z) = z+i$, $\mathcal{Q}(z) = z+(1+2i)$, $\mathcal{R}(z) = 3i-z$ and $\mathcal{T}(z) = -z+(2i-1)$ for all $z \in \mathcal{Y}$. Let $\{p_n\} = \{i + \frac{1}{n}\}_{n\geq 1}$ and $\{q_n\} = \{-1 + \frac{i}{n}\}_{n\geq 1}$ be two sequences in $\mathcal{Y}$ and that $\mathcal{P}p_n = p_n + i = 2i + \frac{1}{n}$ and $\mathcal{R}p_n = 3i - p_n = 2i - \frac{1}{n}$ for all $n \in \mathbb{N}$. This implies that

$$\mathcal{S}(\mathcal{P}p_n, \mathcal{P}p_n, 0) = \mathcal{S}\Big(2i + \frac{1}{n}, 2i + \frac{1}{n}, 0\Big) \to 0 \text{ as } n \to \infty.$$

This shows that $\mathcal{P}p_n \to 0$ as $n \to \infty$ and by similar way, we have

$$\mathcal{S}(\mathcal{R}p_n, \mathcal{R}p_n, 0) = \mathcal{S}\Big(2i - \frac{1}{n}, 2i - \frac{1}{n}, 0\Big) \to 0 \text{ as } n \to \infty.$$

This shows that $\mathcal{R}p_n \to 0$ as $n \to \infty$. Thus the pair $(\mathcal{P}, \mathcal{R})$ satisfies $(E.A)$-property.

Similarly, note that $\mathcal{Q}q_n = q_n+(1+2i) = 2i+\frac{i}{n}$ and $\mathcal{T}q_n = -q_n+(2i-1) = 2i-\frac{i}{n}$ for all $n \in \mathbb{N}$. This implies that

$$\mathcal{S}(\mathcal{Q}q_n, \mathcal{Q}q_n, 0) = \mathcal{S}\Big(2i + \frac{i}{n}, 2i + \frac{i}{n}, 0\Big) \to 0 \text{ as } n \to \infty.$$

This shows that $\mathcal{Q}q_n \to 0$ as $n \to \infty$ and by similar way, we have

$$\mathcal{S}(\mathcal{T}q_n, \mathcal{T}q_n, 0) = \mathcal{S}\Big(-2i - \frac{i}{n}, -2i - \frac{i}{n}, 0\Big) \to 0 \text{ as } n \to \infty.$$

This shows that $\mathcal{T}q_n \to 0$ as $n \to \infty$. Thus the pair $(\mathcal{Q}, \mathcal{T})$ satisfies $(E.A)$-property. Thus there exist two sequences $\{p_n\}$ and $\{q_n\}$ in $\mathcal{Y}$ such that

$$\lim_{n\to\infty} \mathcal{P}p_n = \lim_{n\to\infty} \mathcal{R}p_n = \lim_{n\to\infty} \mathcal{Q}q_n = \lim_{n\to\infty} \mathcal{T}q_n = 0 \in \mathcal{Y}.$$

Hence the pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ satisfy common $(E.A)$-property.

Lemma 2.14. *([17]) Let $(\mathcal{Y}, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\{r_n\}$ be a sequence in $\mathcal{Y}$. Then $\{r_n\}$ converges to r if and only if $\lim_{n\to\infty} |\mathcal{S}(r_n, r_n, r)| = 0$ or $|\mathcal{S}(r_n, r_n, r)| \to 0$ as $n \to \infty$.*

Lemma 2.15. *([17]) Let $(\mathcal{Y}, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\{r_n\}$ be a sequence in $\mathcal{Y}$. Then $\{r_n\}$ is a Cauchy sequence if and only if $\lim_{n,m\to\infty}$ $|\mathcal{S}(r_n, r_n, r_{n+m})| = 0$ or $|\mathcal{S}(r_n, r_n, r_{n+m})| \to 0$ as $n, m \to \infty$.*

Lemma 2.16. *([17]) Let $(\mathcal{Y}, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space, then $\mathcal{S}(x, x, y)$ $= \mathcal{S}(y, y, x)$ for all $x, y \in \mathcal{Y}$.*

3. Common fixed point theorems

In this section, we shall prove some common fixed point theorems via contractive condition involving rational terms and satisfying common $(E.A)$ property for two pairs of weakly compatible mappings in the framework of complex valued $\mathcal{S}$-metric spaces.

Throughout this paper we use the following concept.

Definition 3.1. Let $\phi\colon \mathbb{C} \to \mathbb{C}$ be a continuous function with the partial order relation $' \prec'$ in $\mathbb{C}$ such that

$(\phi_{\mathbf{1}})$ $\phi(0) = 0$ and $(\phi_{\mathbf{2}})$ $|\phi(t)| < |t|$ $\forall\, t \in \mathbb{C}$.

Example 3.2. If $\phi(z) = \frac{1}{2}z$ for all $z \in \mathbb{C}$, then ϕ is continuous, $\phi(0) = 0$ and $|\phi(t)| < |t|$ for all $t \in \mathbb{C}$.

Theorem 3.3. *Let $(\mathcal{Y}, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $f, g, \mathcal{P}, \mathcal{Q}\colon \mathcal{Y} \to \mathcal{Y}$ be four self-mappings of $\mathcal{Y}$ satisfying the following conditions:*

(i)

$$\mathcal{S}(fu, fv, gw) \precsim \phi\Big(\max\Big\{\mathcal{S}(\mathcal{P}u, \mathcal{P}v, \mathcal{Q}w), \mathcal{S}(gw, gw, fu), \mathcal{S}(gw, gw, \mathcal{Q}w), \frac{\mathcal{S}(gw, gw, fv)[1 + \mathcal{S}(gw, gw, \mathcal{Q}w)]}{[1 + \mathcal{S}(fu, fu, gw)]}\Big\}\Big), \tag{3.1}$$

for all $u, v, w \in \mathcal{Y}$, where ϕ is defined as in Definition 3.1;

(ii) the pairs $(f, \mathcal{P})$ and $(g, \mathcal{Q})$ are weakly compatible;

(iii) the pairs $(f, \mathcal{P})$ and $(g, \mathcal{Q})$ satisfy common $(E.A)$-property;

(iv) $f(\mathcal{Y}) \subseteq \mathcal{Q}(\mathcal{Y})$ and $g(\mathcal{Y}) \subseteq \mathcal{P}(\mathcal{Y})$.

If the range of one of the mappings $\mathcal{P}(\mathcal{Y})$ or $\mathcal{Q}(\mathcal{Y})$ is a complete subspace of $(\mathcal{Y}, \mathcal{S})$, then f, g, $\mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point in $\mathcal{Y}$.

Proof. Since the pairs $(f, \mathcal{P})$ and $(g, \mathcal{Q})$ satisfy the common $(E.A)$-property, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{Y}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} \mathcal{Q}y_n = q, \tag{3.2}$$

for some $q \in \mathcal{Y}$. Again, since $f(\mathcal{Y}) \subseteq \mathcal{Q}(\mathcal{Y})$ and $\{x_n\}$ and $\{y_n\}$ are in $\mathcal{Y}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} \mathcal{Q}y_n$. Hence $\lim_{n\to\infty} \mathcal{Q}y_n = q$. We claim that $\lim_{n\to\infty} gy_n = q$. If not, then putting $u = v = x_n$ and $w = y_n$ in inequality (3.1) and using Lemma 2.16 and $(\mathcal{CSM}2)$, we have

$$\begin{aligned}
\mathcal{S}(fx_n, fx_n, gy_n) &\precsim \phi\Big(\max\Big\{\mathcal{S}(\mathcal{P}x_n, \mathcal{P}x_n, \mathcal{Q}y_n), \mathcal{S}(gy_n, gy_n, fx_n), \mathcal{S}(gy_n, gy_n, \mathcal{Q}y_n), \\
&\qquad \frac{\mathcal{S}(gy_n, gy_n, fx_n)[1 + \mathcal{S}(gy_n, gy_n, \mathcal{Q}y_n)]}{[1 + \mathcal{S}(fx_n, fx_n, gy_n)]}\Big\}\Big) \\
&= \phi\Big(\max\Big\{\mathcal{S}(fx_n, fx_n, fx_n), \mathcal{S}(gy_n, gy_n, fx_n), \mathcal{S}(gy_n, gy_n, fx_n), \\
&\qquad \frac{\mathcal{S}(gy_n, gy_n, fx_n)[1 + \mathcal{S}(gy_n, gy_n, fx_n)]}{[1 + \mathcal{S}(fx_n, fx_n, gy_n)]}\Big\}\Big) \\
&= \phi\Big(\max\Big\{0, \mathcal{S}(fx_n, fx_n, gy_n), \mathcal{S}(fx_n, fx_n, gy_n), \mathcal{S}(fx_n, fx_n, gy_n)\Big\}\Big) \\
&\precsim \phi(\mathcal{S}(fx_n, fx_n, gy_n)).
\end{aligned} \tag{3.3}$$

Since ϕ is continuous, letting $n \to \infty$ in equation (3.3), we have

$$\mathcal{S}(q, q, gy_n) \precsim \phi(\mathcal{S}(q, q, gy_n)). \tag{3.4}$$

This implies that

$$|\mathcal{S}(q, q, gy_n)| \leq |\phi(\mathcal{S}(q, q, gy_n))|. \tag{3.5}$$

Using the property ϕ in equation (3.5), we get

$$|\mathcal{S}(q, q, gy_n)| < |\mathcal{S}(q, q, gy_n)|,$$

which is a contradiction. Hence we conclude that $\lim_{n\to\infty} \mathcal{S}(q, q, gy_n) = 0$, that is, $\lim_{n\to\infty} gy_n = q$. Consequently, $\lim_{n\to\infty} gy_n = \lim_{n\to\infty} fx_n = q$.

Now, first we assume that $\mathcal{Q}(\mathcal{Y})$ is a complete subspace of $(\mathcal{Y}, \mathcal{S})$, then $q = \mathcal{Q}p$ for some $p \in \mathcal{Y}$. Subsequently, we have

$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \mathcal{Q}y_n = \mathcal{Q}p = q.$$

We claim that $gp = \mathcal{Q}p$. For this, putting $u = v = x_n$ and $w = p$ in inequality (3.1) and using Lemma 2.16 and $(\mathcal{CSM}2)$, we have

$$\begin{aligned}
\mathcal{S}(fx_n, fx_n, gp) &\precsim \phi\Big(\max\Big\{\mathcal{S}(\mathcal{P}x_n, \mathcal{P}x_n, \mathcal{Q}p), \mathcal{S}(gp, gp, fx_n), \mathcal{S}(gp, gp, \mathcal{Q}p), \\
&\qquad \frac{\mathcal{S}(gp, gp, fx_n)[1 + \mathcal{S}(gp, gp, \mathcal{Q}p)]}{[1 + \mathcal{S}(fx_n, fx_n, gp)]}\Big\}\Big) \\
&= \phi\Big(\max\Big\{\mathcal{S}(fx_n, fx_n, q), \mathcal{S}(gp, gp, fx_n), \mathcal{S}(gp, gp, q), \\
&\qquad \frac{\mathcal{S}(gp, gp, fx_n)[1 + \mathcal{S}(gp, gp, q)]}{[1 + \mathcal{S}(fx_n, fx_n, gp)]}\Big\}\Big).
\end{aligned} \tag{3.6}$$

Since ϕ is continuous, letting $n \to \infty$ in equation (3.6), we have

$$\begin{aligned}
\mathcal{S}(q, q, gp) &\precsim \phi\Big(\max\Big\{\mathcal{S}(q, q, q), \mathcal{S}(gp, gp, q), \mathcal{S}(gp, gp, q), \mathcal{S}(gp, gp, q)\Big\}\Big) \\
&= \phi\Big(\max\Big\{0, \mathcal{S}(q, q, gp), \mathcal{S}(q, q, gp), \mathcal{S}(q, q, gp)\Big\}\Big) \\
&\precsim \phi(\mathcal{S}(q, q, gp)).
\end{aligned} \tag{3.7}$$

This implies that

$$|\mathcal{S}(q, q, gp)| \leq |\phi(\mathcal{S}(q, q, gp))|. \tag{3.8}$$

Using the property ϕ in equation (3.8), we get

$$|\mathcal{S}(q, q, gp)| < |\mathcal{S}(q, q, gp)|,$$

which is a contradiction. Hence, we have $\mathcal{S}(q, q, gp) = 0$, that is, $gp = q$ and hence $gp = \mathcal{Q}p = q$. Thus p is a coincidence point of the mappings g and $\mathcal{Q}$, that is, the pair $(g, \mathcal{Q})$. Now, the weak compatibility of the pair $(g, \mathcal{Q})$ implies that $g\mathcal{Q}p = \mathcal{Q}gp$ or $gq = \mathcal{Q}q$.

On the other hand, since $g(\mathcal{Y}) \subseteq \mathcal{P}(\mathcal{Y})$, there exists $a \in \mathcal{Y}$ such that $gp = \mathcal{P}a$. Thus $\mathcal{Q}p = gp = \mathcal{P}a = q$. Let us show that a is a coincidence point of the pair

$(f,\mathcal{P})$, that is, $fa=\mathcal{P}a=q$. If not, then putting $u=v=a$ and $w=p$ in inequality (3.1) and using Lemma 2.16 and $(\mathcal{CSM}2)$, we get

$$\begin{aligned}
\mathcal{S}(fa,fa,gp) &\precsim \phi\Big(\max\Big\{\mathcal{S}(\mathcal{P}a,\mathcal{P}a,\mathcal{Q}p),\mathcal{S}(gp,gp,fa),\mathcal{S}(gp,gp,\mathcal{Q}p),\\
&\qquad \frac{\mathcal{S}(gp,gp,fa)[1+\mathcal{S}(gp,gp,\mathcal{Q}p)]}{[1+\mathcal{S}(fa,fa,gp)]}\Big\}\Big)\\
&= \phi\Big(\max\Big\{\mathcal{S}(q,q,q),\mathcal{S}(gp,gp,fa),\mathcal{S}(gp,gp,gp),\\
&\qquad \frac{\mathcal{S}(gp,gp,fa)[1+\mathcal{S}(gp,gp,gp)]}{[1+\mathcal{S}(fa,fa,gp)]}\Big\}\Big)\\
&= \phi\Big(\max\Big\{0,\mathcal{S}(fa,fa,gp),0,\frac{\mathcal{S}(fa,fa,gp)}{[1+\mathcal{S}(fa,fa,gp)]}\Big\}\Big)\\
&\precsim \phi\Big(\max\Big\{0,\mathcal{S}(fa,fa,gp),0,\mathcal{S}(fa,fa,gp)\Big\}\Big)\\
&\precsim \phi(\mathcal{S}(fa,fa,gp)). \qquad (3.9)
\end{aligned}$$

This implies that

$$|\mathcal{S}(fa,fa,gp)|\le|\phi(\mathcal{S}(fa,fa,gp))|. \qquad (3.10)$$

Using the property ϕ in equation (3.10), we get

$$|\mathcal{S}(fa,fa,gp)|<|\mathcal{S}(fa,fa,gp)|,$$

which is a contradiction. Hence, we have $\mathcal{S}(fa,fa,gp)=0$, that is, $fa=gp$ and hence $fa=\mathcal{P}a=q$. Thus a is a coincidence point of the mappings f and $\mathcal{P}$, that is, the pair $(f,\mathcal{P})$. Further, the weak compatibility of the pair $(f,\mathcal{P})$ implies that $f\mathcal{P}a=\mathcal{P}fa$ or $fq=\mathcal{P}q$. Hence q is a common coincidence point of f, g, $\mathcal{P}$ and $\mathcal{Q}$.

Now to show that q is a common fixed point of f, g, $\mathcal{P}$ and $\mathcal{Q}$. For this, we put $u = v = a$ and $w = q$ in (3.1) and using Lemma 2.16 and $(\mathcal{CSM}2)$, we get

$$\begin{aligned}
\mathcal{S}(fa, fa, gq) &\precsim \phi\Big(\max\Big\{\mathcal{S}(\mathcal{P}a, \mathcal{P}a, \mathcal{Q}q), \mathcal{S}(gq, gq, fa), \mathcal{S}(gq, gq, \mathcal{Q}q),\\
&\qquad \frac{\mathcal{S}(gq, gq, fa)[1+\mathcal{S}(gq, gq, \mathcal{Q}q)]}{[1+\mathcal{S}(fa, fa, gq)]}\Big\}\Big)\\
&= \phi\Big(\max\Big\{\mathcal{S}(fa, fa, gq), \mathcal{S}(gq, gq, fa), \mathcal{S}(gq, gq, gq),\\
&\qquad \frac{\mathcal{S}(gq, gq, fa)[1+\mathcal{S}(gq, gq, gq)]}{[1+\mathcal{S}(fa, fa, gq)]}\Big\}\Big)\\
&= \phi\Big(\max\Big\{\mathcal{S}(fa, fa, gq), \mathcal{S}(fa, fa, gq), 0,\\
&\qquad \frac{\mathcal{S}(fa, fa, gq)}{[1+\mathcal{S}(fa, fa, gq)]}\Big\}\Big)\\
&\precsim \phi\Big(\max\Big\{\mathcal{S}(fa, fa, gq), \mathcal{S}(fa, fa, gq), 0,\\
&\qquad \mathcal{S}(fa, fa, gq)\Big\}\Big)\\
&\precsim \phi(\mathcal{S}(fa, fa, gq)). \qquad (3.11)
\end{aligned}$$

This implies that

$$|\mathcal{S}(fa, fa, gq)| \leq |\phi(\mathcal{S}(fa, fa, gq))|. \qquad (3.12)$$

Using the property ϕ in equation (3.12), we get

$$|\mathcal{S}(fa, fa, gq)| < |\mathcal{S}(fa, fa, gq)|,$$

which is a contradiction. Hence, we have $\mathcal{S}(fa, fa, gq) = 0$ or $\mathcal{S}(q, q, gq) = 0$, that is, $q = gq$. Consequently, $fq = gq = \mathcal{P}q = \mathcal{Q}q = q$. This shows that q is a common fixed point of the mappings f, g, $\mathcal{P}$ and $\mathcal{Q}$.

Similar argument arises if we assume that $\mathcal{P}(\mathcal{Y})$ is a complete subspace of $(\mathcal{Y}, \mathcal{S})$.

Now, we show the uniqueness of the common fixed point. For this, let us assume that q' be another common fixed point of f, g, $\mathcal{P}$ and $\mathcal{Q}$ with $q' \neq q$. From inequality (3.1) and using Lemma 2.16 and $(\mathcal{CSM}2)$ for $u = v = q'$ and

$w = q$, we have

$$\begin{aligned} \mathcal{S}(q', q', q) &= \mathcal{S}(fq', fq', gq) \\ &\precsim \phi\Big(\max\Big\{\mathcal{S}(\mathcal{P}q', \mathcal{P}q', \mathcal{Q}q), \mathcal{S}(gq, gq, fq'), \mathcal{S}(gq, gq, \mathcal{Q}q), \\ &\qquad \frac{\mathcal{S}(gq, gq, fq')[1 + \mathcal{S}(gq, gq, \mathcal{Q}q)]}{[1 + \mathcal{S}(fq', fq', gq)]}\Big\}\Big) \\ &= \phi\Big(\max\Big\{\mathcal{S}(q', q', q), \mathcal{S}(q, q, q'), \mathcal{S}(q, q, q), \\ &\qquad \frac{\mathcal{S}(q, q, q')[1 + \mathcal{S}(q, q, q)]}{[1 + \mathcal{S}(q', q', q)]}\Big\}\Big) \\ &= \phi\Big(\max\Big\{\mathcal{S}(q', q', q), \mathcal{S}(q', q', q), 0, \\ &\qquad \frac{\mathcal{S}(q', q', q)}{[1 + \mathcal{S}(q', q', q)]}\Big\}\Big) \\ &\precsim \phi\Big(\max\Big\{\mathcal{S}(q', q', q), \mathcal{S}(q', q', q), 0, \mathcal{S}(q', q', q)\Big\}\Big) \\ &\precsim \phi(\mathcal{S}(q', q', q)). \end{aligned} \tag{3.13}$$

This implies that

$$|\mathcal{S}(q', q', q)| \leq |\phi(\mathcal{S}(q', q', q))|. \tag{3.14}$$

Using the property ϕ in equation (3.14), we get

$$|\mathcal{S}(q', q', q)| < |\mathcal{S}(q', q', q)|,$$

which is a contradiction. Hence, we have $\mathcal{S}(q', q', q) = 0$, that is, $q' = q$. This shows that the common fixed point of the mappings f, g, $\mathcal{P}$ and $\mathcal{Q}$ is unique. This completes the proof. □

From Theorem 3.3, we obtain the following result.

Corollary 3.4. *Let $(\mathcal{Y}, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $f, \mathcal{P} \colon \mathcal{Y} \to \mathcal{Y}$ be two self-mappings of $\mathcal{Y}$ satisfying the following conditions:*

(i)

$$\begin{aligned} \mathcal{S}(fu, fv, fw) &\precsim h \max\Big\{\mathcal{S}(\mathcal{P}u, \mathcal{P}v, \mathcal{P}w), \mathcal{S}(fw, fw, fu), \mathcal{S}(fw, fw, \mathcal{P}w), \\ &\qquad \frac{\mathcal{S}(fw, fw, fv)[1 + \mathcal{S}(fw, fw, \mathcal{P}w)]}{[1 + \mathcal{S}(fu, fu, fw)]}\Big\}, \end{aligned}$$

for all $u, v, w \in \mathcal{Y}$, where $0 \leq h < 1$ is a constant;

(ii) the pair $(f, \mathcal{P})$ is weakly compatible;

(iii) the pair $(f, \mathcal{P})$ satisfies $(E.A)$ property;

(iv) $f(\mathcal{Y}) \subseteq \mathcal{P}(\mathcal{Y})$.

If the range of the mapping $\mathcal{P}(\mathcal{Y})$ is a complete subspace of $(\mathcal{Y}, \mathcal{S})$, then f and $\mathcal{P}$ have a unique common fixed point in $\mathcal{Y}$.

Proof. Putting $\phi(t) = ht$, $t \geq 0$, $0 \leq h < 1$, $f = g$ and $\mathcal{P} = \mathcal{Q}$ in inequality (3.1) of Theorem 3.3. Then all conditions of Theorem 3.3 are satisfied and hence the result follows. □

Theorem 3.5. *Let $(\mathcal{Y}, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $f, g, \mathcal{P}, \mathcal{Q}\colon \mathcal{Y} \to \mathcal{Y}$ be four self-mappings of $\mathcal{Y}$ satisfying the following conditions:*
(i)

$$\begin{aligned}\mathcal{S}(fu, fv, gw) \;\precsim\; & q \max\Big\{\phi(\mathcal{S}(\mathcal{P}u, \mathcal{P}v, \mathcal{Q}w)), \phi(\mathcal{S}(gw, gw, fu)),\\ & \phi\Big(\frac{1}{2}[\mathcal{S}(fw, fw, \mathcal{Q}w) + \mathcal{S}(gw, gw, \mathcal{P}u)]\Big),\\ & \phi\Big(\frac{\mathcal{S}(gw, gw, fv)[1 + \mathcal{S}(gw, gw, \mathcal{Q}w)]}{[1 + \mathcal{S}(fu, fu, gw)]}\Big),\\ & \phi\Big(\frac{\mathcal{S}(fu, fu, \mathcal{Q}w)[1 + \mathcal{S}(gw, gw, \mathcal{P}u)]}{[1 + \mathcal{S}(fu, fu, gw)]}\Big)\Big\},\end{aligned} \tag{3.15}$$

for all $u, v, w \in \mathcal{Y}$, where ϕ is defined as in Definition 3.1 and $q \in [0, 1)$ is a constant;
(ii) the pairs $(f, \mathcal{P})$ and $(g, \mathcal{Q})$ are weakly compatible;
(iii) the pairs $(f, \mathcal{P})$ and $(g, \mathcal{Q})$ satisfy common $(E.A)$-property;
(iv) $f(\mathcal{Y}) \subseteq \mathcal{Q}(\mathcal{Y})$ and $g(\mathcal{Y}) \subseteq \mathcal{P}(\mathcal{Y})$.
If the range of one of the mappings $\mathcal{P}(\mathcal{Y})$ or $\mathcal{Q}(\mathcal{Y})$ is a complete subspace of $(\mathcal{Y}, \mathcal{S})$, then f, g, $\mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point in $\mathcal{Y}$.

Proof. Follows from Theorem 3.3. □

Theorem 3.6. *Let $(\mathcal{Y}, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $f, g, \mathcal{P}, \mathcal{Q}\colon \mathcal{Y} \to \mathcal{Y}$ be four self-mappings of $\mathcal{Y}$ satisfying the following conditions:*
(i)

$$\begin{aligned}\mathcal{S}(fu, fv, gw) \;\precsim\; & t_1\, \phi(\mathcal{S}(\mathcal{P}u, \mathcal{P}v, \mathcal{Q}w)) + t_2\, \phi(\mathcal{S}(gw, gw, fu))\\ & + t_3\, \phi(\mathcal{S}(gw, gw, \mathcal{Q}w))\\ & + t_4\, \phi\Big(\frac{\mathcal{S}(gw, gw, fv)[1 + \mathcal{S}(gw, gw, \mathcal{Q}w)]}{[1 + \mathcal{S}(fu, fu, gw)]}\Big),\end{aligned} \tag{3.16}$$

for all $u, v, w \in \mathcal{Y}$, where ϕ is defined as in Definition 3.1 and $t_1, t_2, t_3, t_4 > 0$ are nonnegative reals such that $t_1 + t_2 + t_3 + t_4 < 1$;
(ii) the pairs $(f, \mathcal{P})$ and $(g, \mathcal{Q})$ are weakly compatible;
(iii) the pairs $(f, \mathcal{P})$ and $(g, \mathcal{Q})$ satisfy common $(E.A)$-property;
(iv) $f(\mathcal{Y}) \subseteq \mathcal{Q}(\mathcal{Y})$ and $g(\mathcal{Y}) \subseteq \mathcal{P}(\mathcal{Y})$.
If the range of one of the mappings $\mathcal{P}(\mathcal{Y})$ or $\mathcal{Q}(\mathcal{Y})$ is a complete subspace of $(\mathcal{Y}, \mathcal{S})$, then f, g, $\mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point in $\mathcal{Y}$.

Proof. Follows from Theorem 3.3 and Theorem 3.5. □

Theorem 3.7. *Let $(\mathcal{Y}, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $f, g, \mathcal{P}, \mathcal{Q}\colon \mathcal{Y} \to \mathcal{Y}$ be four self-mappings of $\mathcal{Y}$ satisfying the following conditions:*
(i)

$$\mathcal{S}(fu, fv, gw) \precsim k_1\, \Theta^{\mathcal{S}}_{\mathcal{C}_1}(u, v, w) + k_2\, \Theta^{\mathcal{S}}_{\mathcal{C}_2}(u, v, w), \tag{3.17}$$

for all $u, v, w \in \mathcal{Y}$, where ϕ is defined as in Definition 3.1, $k_1, k_2 > 0$ are nonnegative reals with $k_1 + k_2 < 1$ and

$$\Theta^{\mathcal{S}}_{\mathcal{C}_1}(u,v,w) = \max\Big\{\phi(\mathcal{S}(\mathcal{P}u,\mathcal{P}v,\mathcal{Q}w)), \phi(\mathcal{S}(gw,gw,fu)), \phi(\mathcal{S}(gw,gw,\mathcal{Q}w))\Big\},$$

$$\begin{aligned}\Theta^{\mathcal{S}}_{\mathcal{C}_2}(u,v,w) = \max\Big\{&\phi\Big(\mathcal{S}(fu,fu,\mathcal{Q}w)\frac{[1+\mathcal{S}(gw,gw,\mathcal{Q}w)]}{[1+\mathcal{S}(fu,fu,gw)]}\Big),\\ &\phi\Big(\mathcal{S}(gw,gw,fv)\frac{[1+\mathcal{S}(gw,gw,\mathcal{Q}w)]}{[1+\mathcal{S}(fu,fu,gw)]}\Big),\\ &\phi\Big(\frac{1}{2}[\mathcal{S}(fu,fu,\mathcal{Q}w)+\mathcal{S}(gw,gw,\mathcal{P}u)]\Big)\Big\},\end{aligned}$$

(ii) the pairs $(f, \mathcal{P})$ and $(g, \mathcal{Q})$ are weakly compatible;
(iii) the pairs $(f, \mathcal{P})$ and $(g, \mathcal{Q})$ satisfy common $(E.A)$-property;
(iv) $f(\mathcal{Y}) \subseteq \mathcal{Q}(\mathcal{Y})$ and $g(\mathcal{Y}) \subseteq \mathcal{P}(\mathcal{Y})$.

If the range of one of the mappings $\mathcal{P}(\mathcal{Y})$ or $\mathcal{Q}(\mathcal{Y})$ is a complete subspace of $(\mathcal{Y}, \mathcal{S})$, then f, g, $\mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point in $\mathcal{Y}$.

Proof. Follows from Theorem 3.3, Theorem 3.5 and Theorem 3.6. □

From Corollary 3.4 we obtain the following special case.

Corollary 3.8. *Let $(\mathcal{Y}, \mathcal{S})$ be a complete complex valued $\mathcal{S}$-metric space and let $f: \mathcal{Y} \to \mathcal{Y}$ be a self-mapping of $\mathcal{Y}$ satisfying the contractive condition:*

$$\mathcal{S}(fu, fv, fw) \precsim h\,\mathcal{S}(u, v, w),$$

for all $u, v, w \in \mathcal{Y}$, where $h \in [0, 1)$ is a constant. Then f has a unique fixed point in $\mathcal{Y}$.

Remark 3.9. Corollary 3.8 extends Theorem 3.1 of Sedghi et al. [29] from complete S-metric space to the setting of complete complex valued S-metric space.

Remark 3.10. Corollary 3.8 also extends the well-known Banach fixed point theorem [5] from complete metric space to the setting of complete complex valued S-metric space.

Corollary 3.11. *([17], Corollary 2.5) Let $(\mathcal{Y}, \mathcal{S})$ be a complete complex valued $\mathcal{S}$-metric space and let $f: \mathcal{Y} \to \mathcal{Y}$ be a self-mapping of $\mathcal{Y}$ satisfying the contractive condition:*

$$\mathcal{S}(f^n u, f^n v, f^n w) \precsim h\,\mathcal{S}(u, v, w),$$

for all $u, v, w \in \mathcal{Y}$, where n is some positive integer and $h \in [0, 1)$ is a constant. Then f has a unique fixed point in $\mathcal{Y}$.

Proof. By Corollary 3.8, there exists $r \in \mathcal{Y}$ such that $f^n r = r$. Then

$$\begin{aligned}\mathcal{S}(fr, fr, r) &= \mathcal{S}(ff^n r, ff^n r, f^n r)\\ &= \mathcal{S}(f^n fr, f^n fr, f^n r)\\ &\precsim h\,\mathcal{S}(fr, fr, r).\end{aligned}$$

Thus

$$|\mathcal{S}(fr, fr, r)| \leq h\,|\mathcal{S}(fr, fr, r)|,$$

which is a contradiction since $0 \leq h < 1$ and so $\mathcal{S}(fr, fr, r) = 0$, that is, $fr = r$. This shows that f has a unique fixed point in $\mathcal{Y}$. This completes the proof. □

Remark 3.12. Corollary 3.8 is a special case of Corollary 3.11 for $n = 1$.

Remark 3.13. (i) Completeness of the space $\mathcal{Y}$ is relaxed in Theorem 3.3, Theorem 3.5, Theorem 3.6 and Theorem 3.7.

(ii) Continuity of the mappings f, g, $\mathcal{P}$ and $\mathcal{Q}$ are relaxed in Theorem 3.3, Theorem 3.5, Theorem 3.6 and Theorem 3.7.

Now, we give the following example which is an application of Corollary 3.8.

Example 3.14. Let $\mathcal{Y}_1 = \{z \in \mathbb{C} : Re(z) \geq 0, Im(z) = 0\}$ and $\mathcal{Y}_2 = \{z \in \mathbb{C} : Im(z) \geq 0, Re(z) = 0\}$. Now, let $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2$ and define a mapping $\mathcal{S}\colon \mathcal{Y}^3 \to \mathbb{C}$ By:

$$\mathcal{S}(z_1, z_2, z_3) = \begin{cases} \max\{x_1, x_2, x_3\} + i\max\{x_1, x_2, x_3\}, & \text{if } z_1, z_2, z_3 \in \mathcal{Y}_1, \\ \max\{y_1, y_2, y_3\} + i\max\{y_1, y_2, y_3\}, & \text{if } z_1, z_2, z_3 \in \mathcal{Y}_2, \\ (\max\{x_1, x_2\} + y_3) + i(\max\{x_1, x_2\} + y_3), & \text{if } z_1, z_2 \in \mathcal{Y}_1, z_3 \in \mathcal{Y}_2, \\ (\max\{y_1, y_2\} + x_3) + i(\max\{y_1, y_2\} + x_3), & \text{if } z_1, z_2 \in \mathcal{Y}_2, z_3 \in \mathcal{Y}_1, \end{cases}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $z_3 = x_3 + iy_3$. It is very easy to verify that $(\mathcal{Y}, \mathcal{S})$ is a complete complex valued $\mathcal{S}$-metric space.

Now, we define a self-mapping f on $\mathcal{Y}$ (with $z = (x, y)$) as

$$f(z) = \begin{cases} (\frac{x}{2}, 0), & \text{if } z \in \mathcal{Y}_1, \\ (0, \frac{y}{2}), & \text{if } z \in \mathcal{Y}_2. \end{cases}$$

Now, we show that f satisfies the conditions of Corollary 3.8. Here, we note that

$$0 \precsim \mathcal{S}(z_1, z_2, z_3), \mathcal{S}(fz_1, fz_2, fz_3).$$

. Now, let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Hence, we have the following four cases.

Case 1. If $z_1, z_2 \in \mathcal{Y}_1$, then we have

$$\begin{aligned} \mathcal{S}(fz_1, fz_1, fz_2) &= \mathcal{S}\Big((\frac{x_1}{2}, 0), (\frac{x_1}{2}, 0), (\frac{x_2}{2}, 0)\Big) \\ &= \max\left\{\frac{x_1}{2}, \frac{x_2}{2}\right\} + i\max\left\{\frac{x_1}{2}, \frac{x_2}{2}\right\} \\ &= \max\left\{\frac{x_1}{2}, \frac{x_2}{2}\right\}(1 + i) \\ &= \frac{1}{2}\max\{x_1, x_2\}(1 + i) \\ &\precsim \frac{1}{2}\mathcal{S}(z_1, z_1, z_2) \\ &= h\,\mathcal{S}(z_1, z_1, z_2), \end{aligned}$$

Case 2. If $z_1, z_2 \in \mathcal{Y}_2$, then we have

$$\begin{aligned}
\mathcal{S}(fz_1, fz_1, fz_2) &= \mathcal{S}\Big((0, \frac{y_1}{2}), (0, \frac{y_1}{2}), (0, \frac{y_2}{2})\Big) \\
&= \max\Big\{\frac{y_1}{2}, \frac{y_2}{2}\Big\} + i\max\Big\{\frac{y_1}{2}, \frac{y_2}{2}\Big\} \\
&= \max\Big\{\frac{y_1}{2}, \frac{y_2}{2}\Big\}(1+i) \\
&= \frac{1}{2}\max\{y_1, y_2\}(1+i) \\
&\precsim \frac{1}{2}\mathcal{S}(z_1, z_1, z_2) \\
&= h\,\mathcal{S}(z_1, z_1, z_2),
\end{aligned}$$

Case 3. If $z_1 \in \mathcal{Y}_1$, $z_2 \in \mathcal{Y}_2$, then we have

$$\begin{aligned}
\mathcal{S}(fz_1, fz_1, fz_2) &= \mathcal{S}\Big((\frac{x_1}{2}, 0), (\frac{x_1}{2}, 0), (0, \frac{y_2}{2})\Big) \\
&= \Big(\frac{x_1}{2} + \frac{y_2}{2}\Big)(1+i) \\
&= \frac{1}{2}(x_1 + y_2)(1+i) \\
&\precsim \frac{1}{2}\mathcal{S}(z_1, z_1, z_2) \\
&= h\,\mathcal{S}(z_1, z_1, z_2),
\end{aligned}$$

Case 4. If $z_2 \in \mathcal{Y}_1$, $z_1 \in \mathcal{Y}_2$, then we have

$$\begin{aligned}
\mathcal{S}(fz_1, fz_1, fz_2) &= \mathcal{S}\Big((0, \frac{y_1}{2}), (0, \frac{y_1}{2}), (\frac{x_2}{2}, 0)\Big) \\
&= \Big(\frac{x_2}{2} + \frac{y_1}{2}\Big)(1+i) \\
&= \frac{1}{2}(x_2 + y_1)(1+i) \\
&\precsim \frac{1}{2}\mathcal{S}(z_1, z_1, z_2) \\
&= h\,\mathcal{S}(z_1, z_1, z_2),
\end{aligned}$$

where $h = \frac{1}{2}$. If we take $0 \leq h < 1$, then all the conditions of Corollary 3.8 are satisfied. Hence by applying Corollary 3.8, f has a unique fixed point in $\mathcal{Y}$. Indeed, in this case $0 \in \mathcal{Y}$ is the unique fixed point.

4. Conclusion

In this paper, we prove some common fixed point theorems for contractive conditions involving rational expressions with control function ϕ and using common $(E.A)$ property in the setting of the complex-valued S-metric spaces and give an example in support of the result. Also we give some special cases of the main result. The results presented in this paper extend, generalize and enrich several results from the current existing literature.

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References

1. M. Aamri and D. El. Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl. (2002), 181-188.
2. M. Abbas and B. E. Rhoades, *Common fixed point results for non-commuting mappings without continuity generalized metric spaces*, Appl. Math. Computation **215** (2009), 262-269.
3. J. Ali and M. Imdad, *An implicit function implies several contraction conditions*, Sara. J. of Math. **4(17)** (2008), 269-285.
4. A. Azam, B. Fisher and, M. Khan, *Common fixed point theorems in complex valued metric spaces*, Numer. Funct. Anal. Optim. **3(3)** (2011), 243-253.
5. S. Banach, *Sur les operation dans les ensembles abstraits et leur application aux equation integrals*, Fund. Math. **3** (1922), 133-181.
6. S. Bhatt, S. Chaukiyal and R. C. Dimri, *A common fixed point theorem for weakly compatible maps in complex valued metric spaces*, Int. J. Math. Sci. Appl. **1** (2011), 1385-1389.
7. S. Chauhan, S. Dalal, W. Sintunavarat, and J. Vujakovic, *Common property $(E.A)$ and existence of fixed points in Menger spaces*, J. Inequ. Appl. 2014, 2014:56.
8. B. K. Dass and S. Gupta, *An extension of Banach contraction principle through rational expressions*, Indian J. Pure Appl. Math. **6** (1975), 1455-1458.
9. B. Fisher, *Common fixed points and constant mapping satisfying rational inequality*, Math. Sem. Notes (Univ. Kobe), (1978).
10. B. Fisher and M. S. Khan, *Fixed points, common fixed points and constant mappings*, Studia Sci. Math. Hungar. **11** (1978), 467-470.
11. F. Gu and Y. Yin, *Common fixed point for three pairs of self maps satisfying common $(E.A)$-property in generalized metric spaces*, Abst. Appl. Anal., Volume 2013, specil issue (2013), 11 pages, Article ID 808092.
12. D. S. Jaggi, *Some unique fixed point theorems*, Indian J. Pure Appl. Math. **8** (1977), 223-230.
13. G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci. **9** (1986), 771-779.
14. G. Jungck, *Common fixed points for noncontinuous, nonself maps on nonnumetric spaces*, Far East J. Math. Sci. **4(2)** (1996), 195-215.
15. H. P. A. Kunzi, *A note on sequentially compact quasi-pseudo metric spaces*, Monatsheft Math. **95** (1983), 219-220.
16. W. Liu, J. Wu and, Z. Li, *Common fixed point of single-valued and multi-valued maps*, Int. J. Math. Sci. **19** (2005), 3045-3055.
17. N. M. Mlaiki, *Common fixed points in complex S-metric space*, Adv. Fixed Point Theory **4(4)** (2014), 509-524.
18. H. K. Nashine, M. Imdad and, M. Hasan, *Common fixed point theorems under rational contractions in complvalued metric spaces*, J. Nonlinear Sci. Appl. **7** (2014), 42-50.
19. R. P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl. **188** (1994), 436-440.
20. R. P. Pant, *Common fixed point of contractive maps*, J. Math. Anal. Appl. **226** (1998), 251-258.
21. H. K. Pathak, R. R. Lopez and, R. K. Verma, *A common fixed point theorem using implicit relation and property $(E.A)$ in metric spaces*, Filomat **21(2)** (2007), 211-234.
22. H. K. Pathak, R. R. Lopez and, R. K. Verma, *A common fixed point theorem of integral type using implicit relation*, Nonlinear Funct. Anal. Appl. **15** (2009), 1-12.
23. V. Popa, *Fixed point theorems for implicit contractive mappings*, Stud. Cercet. Stiin. Ser. Mat. Univ. Bacau **7** (1997), 127-133.

24. R. A. Rashwan, H. A. Hammad, and M. G. Mahmoud, *Common fixed point theorems in complex valued S-metric spaces via implicit relations with applications*, Res. Fixed Point Theory Appl. 2019, Article ID 2018034, 17 pages.
25. F. Rouzkardand and M. Imdad, *Some common fixed point theorems on complex valued metric spaces*, An Int. J. **64** (2012), 1866-1874.
26. G. S. Saluja, *Fixed point results under generalized contraction involving rational expression in complex valued metric spaces*, International J. Math. Combin. **1** (2017), 55-62.
27. G. S. Saluja, *Fixed point theorems under rational contraction in complex valued metric spaces*, Nonlinear Functional Analysis and Applications **22(1)** (2017), 209-216.
28. G. S. Saluja, *Some common fixed point theorems using rational contraction in complex valued metric spaces*, Palestine J. Math. **7(1)** (2018), 92-99.
29. S. Sedghi, N. Shobe and, A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesnik **64(3)** (2012), 258-266.
30. S. Sessa, *On a weak commutative condition in fixed point consideration*, Publ. Inst. Math. (Beograd) **32** (1982), 146-153.
31. W. Sintunavarat and P. Kumam, *Generalized common fixed point theorems in complex valued metric spaces and applications*, J. Inequ. Appl. 2012, **2012:84**.
32. R. K. Verma and H. K. Pathak, *Common fixed point theorems using property $(E.A)$ in complex valued metric spaces*, Thai J. Math. **11(2)** (2013), 347-355.
33. N. Tas and N. Özgür, *Common fixed point results on complex valued S-metric spaces*, (2020), DOI:10.22130/scma.2018.92986.488.
34. P. Vetro, *Common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo, (2) **56(3)** (2007), 464-468.

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