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FRACTIONAL BURGERS EQUATIONS WITH MIXED DERIVATIVES

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Abstract. This paper studies a nonlinear fractional Burgers equation involving an Atangana–Baleanu time derivative and a Caputo space derivative. We establish existence and uniqueness results under suitable assumptions and formulate a Laplace variational iteration scheme for constructing approximate solutions. The convergence of the iterative procedure is analyzed under explicit conditions. An illustrative example is presented to examine the influence of the fractional order on the solution behavior and to recover the corresponding classical Burgers model as the fractional order approaches one.

Keywords. fractional Burgers equation, Atangana-Baleanu fractional derivative, Caputo fractional derivative, existence, uniqueness and convergence of solutions, variational iteration method, Laplace transform
2020 Mathematics Subject Classification. 26A33, 35R11, 34A08

1. Introduction

These characteristics have increased interest in recent years in the study of fractional differential equations [10], owing to their ability to model complex phenomena arising in physics, biology, engineering, and finance. Fractional-order formulations permit memory effects and nonlocal behaviour that are often evident in real-world systems, as compared to classical integer-order models.

In this context, the Burgers equation [5, 2, 16] provides a fundamental model for studying nonlinear diffusion and advection processes.

Recent contributions published in the Annals of Mathematics and Computer Science have also addressed fractional differential models and numerical methods. In particular, fractional equations have been investigated from both analytical and computational perspectives, including existence and uniqueness results and numerical approximation techniques [18, 19]. These contributions further motivate the study of fractional Burgers-type models and their numerical treatment.

This paper analyses a generalization of the Burgers equation with mixed fractional derivatives. To be more precise, the time derivative is defined in the sense of

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the Atangana–Baleanu fractional derivative ([9], [8]) with a non-singular Mittag-Leffler kernel so as to have realistic modeling properties, and the spatial derivative is defined in the Caputo fractional derivative sense ([1]), so as to enjoy the classical behavior of initial conditions. The integration provides a more detailed mathematical structure for describing complex dynamics in evolution, which are affected by historical effects and nonlocal interactions.

The primary goal of the work is to prove conclusively the existence, uniqueness, and convergence of solutions related to the suggested nonlinear fractional Burgers equation.

In order to achieve these objectives, we utilize an analytical approach based on the variational iteration method in conjunction with the Laplace transform ([7], [4], [6], [11], [14], [15], [17], [3] and [12]), which is demonstrated to be particularly effective for handling nonlocal fractional equations. The remaining part of this paper is organized as follows.

In the initial section, we recall the fundamental concepts of fractional calculus. In the ensuing section, we introduce the fractional model under consideration and derive the existence and uniqueness results for its solutions. We then develop the variational iteration method combined with the Laplace transform. Finally, an illustrative application is presented to demonstrate the effectiveness of the proposed method and to analyze the influence of the fractional parameters on the solution behaviour.

The studied equation can be written as follows with $\Delta \in [0, \theta]$:

$$ {}^{AB}D^{\alpha}_{\Delta}\varrho(\sigma,\Delta)+\varrho(\sigma,\Delta)\,\varrho_{\sigma}(\sigma,\Delta) = \nu\, {}^{C}D^{\beta}_{\sigma}\varrho(\sigma,\Delta) \quad for \quad 0<\alpha\leq 1 \quad and \quad 1<\beta\leq 2, \tag{1.1} $$

with the initial conditions:

$$ \begin{cases} \varrho(\sigma,0) = h(\sigma), \\ \partial_{\Delta}\varrho(\sigma,0) = p(\sigma). \end{cases} \tag{1.2} $$

Here, $\varrho(\sigma,\Delta)$ denotes the unknown state variable, σ and $\Delta \in [0,\theta]$ represent the spatial and temporal variables, respectively, $\nu > 0$ is the diffusion coefficient, ${}^{AB}D^{\alpha}_{\Delta}$ is the Atangana–Baleanu fractional derivative of order $0 < \alpha \leq 1$ accounting for memory effects, ${}^{C}D^{\beta}_{\sigma}$ is the Caputo fractional derivative of order $1 < \beta \leq 2$ describing nonlocal spatial diffusion, while the nonlinear term $\varrho(\sigma,\Delta)\varrho_{\sigma}(\sigma,\Delta)$ characterizes the convective transport mechanism, and the functions $h(\sigma)$ and $p(\sigma)$ specify the initial state and its initial rate of variation, respectively.

2. Preliminaries

This section presents the fundamental definitions and preliminary results of fractional calculus that will be required in the analysis of the proposed fractional Burgers equation.

Definition 2.1. [9]

Let $0 < \alpha < 1$. For $y \in H^2(a,b)$ satisfying the appropriate boundary conditions, the Atangana–Baleanu fractional derivative of order α with a non-singular Mittag–Leffler kernel is defined by

$$ {}^{\mathrm{AB}}_{a}D^{\alpha}_{s}y(s) = \frac{B(\alpha)}{1-\alpha}\int_a^s E_\alpha\left(-\frac{\alpha(s-x)^\alpha}{1-\alpha}\right) y'(x)\,dx, \tag{2.1} $$

where $E_\alpha(\cdot)$ is the one-parameter Mittag-Leffler function, and $B(\alpha)$ is a normalization function such that:

$$ B(0) = B(1) = 1. $$

Proposition 2.2. [9]

(1) ${}^{AB}_{0}D^{\alpha}c = 0$, *for any constant* c.

(2) *The Laplace transform of the AB derivative is given by:*

$$ \mathcal{L}\left\{{}^{AB}_{0}D^{\alpha}y(t)\right\} = \frac{B(\alpha)}{1-\alpha}\frac{s^\alpha Y(s) - s^{\alpha-1}y(0)}{s^\alpha + \frac{\alpha}{1-\alpha}}. $$

Definition 2.3. [9] The fractional integral of order α in the sense of Atangana-Baleanu is defined by:

$$ {}^{\mathrm{AB}}_{a}I^{\alpha}_{s}y(s) = \frac{1-\alpha}{B(\alpha)}y(s) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^s (s-x)^{\alpha-1}y(x)\,dx. \tag{2.2} $$

Proposition 2.4. [9]

(1) ${}^{AB}_{0}I^{\alpha}_{t}\left({}^{AB}_{0}D^{\alpha}_{t}y(t)\right) = y(t) - y(0)$.

(2) ${}^{AB}_{0}I^{\alpha}_{t}c = \frac{c}{B(\alpha)}\left(1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right)$.

(3) ${}^{AB}_{0}I^{\alpha}_{t}t^k = \frac{t^k}{B(\alpha)}\left(1-\alpha+\alpha\frac{\Gamma(k+1)t^\alpha}{\Gamma(\alpha+k+1)}\right)$, *for* $k \in \mathbb{N}$.

Theorem 2.5. (1) *The Atangana–Baleanu fractional derivative is a linear operator.*

(2) *The Atangana–Baleanu fractional derivative is Lipschitz continuous on the space of functions with bounded first derivatives.*

Proof. (1) **Linearity of the Atangana–Baleanu derivative.** Let $\phi, \psi \in C^1([c,d])$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. For all $x \in [0,\theta]$, we have:

$$ \begin{aligned} {}^{\mathrm{AB}}_{c}D^{\alpha}(\lambda_1\phi+\lambda_2\psi)(x) &= \frac{B(\alpha)}{1-\alpha}\int_c^x E_\alpha\Big(-\frac{\alpha(x-t)^\alpha}{1-\alpha}\Big)(\lambda_1\phi+\lambda_2\psi)'(t)\,dt \\ &= \frac{B(\alpha)}{1-\alpha}\int_c^x E_\alpha\Big(-\frac{\alpha(x-t)^\alpha}{1-\alpha}\Big)\big(\lambda_1\phi'(t)+\lambda_2\psi'(t)\big)\,dt \\ &= \lambda_1\,{}^{\mathrm{AB}}_{c}D^{\alpha}\phi(x) + \lambda_2\,{}^{\mathrm{AB}}_{c}D^{\alpha}\psi(x), \end{aligned} $$

which proves the linearity.

(2) **Lipschitz continuity of the Atangana–Baleanu derivative.** Let $\phi, \psi \in C^1([c,d])$ and define $h = \phi - \psi \in C^1([c,d])$. Then, for $x \in [c,d]$:

$$ \begin{aligned} \left|{}^{\mathrm{AB}}_{c}D^{\alpha}h(x)\right| &= \left|\frac{B(\alpha)}{1-\alpha}\int_c^x E_\alpha\Big(-\frac{\alpha(x-t)^\alpha}{1-\alpha}\Big)h'(t)\,dt\right| \\ &\le \frac{B(\alpha)}{1-\alpha}\int_c^x \left|E_\alpha\Big(-\frac{\alpha(x-t)^\alpha}{1-\alpha}\Big)\right|\,|h'(t)|\,dt. \end{aligned} $$

Assume that $\|h'\|_\infty \leq M$ and recall that $|E_\alpha(\cdot)| \leq 1$ for the Mittag?Leffler function with negative argument. Then:

$$\left|{}^{\mathrm{AB}}_{c}D^\alpha h(x)\right| \leq \frac{B(\alpha)M}{1-\alpha}\int_c^x dt = \frac{B(\alpha)M}{1-\alpha}(x-c).$$

Consequently, for all $x \in [c,d]$:

$$\left|{}^{\mathrm{AB}}_{c}D^\alpha \phi(x) - {}^{\mathrm{AB}}_{c}D^\alpha \psi(x)\right| \leq \Theta \, \|\phi' - \psi'\|_\infty,$$

where

$$\Theta = \frac{B(\alpha)(d-c)}{1-\alpha}.$$

Hence, the Atangana–Baleanu derivative is Lipschitz continuous.

□

3. Fractional model

By applying the Atangana–Baleanu integral operator to equation 1.1, we first specify the functional framework used below. Let

$$X = L^2(a,b)$$

be endowed with the norm

$$\|u\|_X = \left(\int_a^b |u(\sigma)|^2 \, d\sigma\right)^{1/2}.$$

For the spatial fractional derivative, we additionally assume that the functions under consideration belong to a fractional Sobolev space $H^\beta(a,b)$, with $1 < \beta \leq 2$, so that the Caputo derivative ${}^{C}D^\beta_\sigma u$ is well defined in $L^2(a,b)$.

Throughout this section, we assume that the admissible functions satisfy boundary conditions compatible with the definition of the spatial Caputo operator. Moreover, all constants appearing below are assumed to be finite and independent of the functions under consideration.

$$\varrho(\sigma,\Delta) - \varrho(\sigma,0) = \frac{1-\alpha}{B(\alpha)}\left(\nu \, {}^{C}D^\beta_\sigma \varrho - \varrho \, \varrho_\sigma\right) + \frac{\alpha}{B(\alpha)\,\Gamma(\alpha)}\int_0^\Delta (\Delta-\xi)^{\alpha-1}\left(\nu \, {}^{C}D^\beta_\sigma \varrho - \varrho \, \varrho_\sigma\right) d\xi. \tag{3.1}$$

Let us set,

$$\chi(\sigma,\Delta,\varrho) = \nu \, {}^{C}D^\beta_\sigma \varrho(\sigma,\Delta) - \varrho(\sigma,\Delta) \, \varrho_\sigma(\sigma,\Delta).$$

Then, equation 3.1 becomes

$$\varrho(\sigma,\Delta) - \varrho(\sigma,0) = \frac{1-\alpha}{B(\alpha)}\chi(\sigma,\Delta,\varrho) + \frac{\alpha}{B(\alpha)\,\Gamma(\alpha)}\int_0^\Delta (\Delta-\xi)^{\alpha-1}\chi(\sigma,\xi,\varrho)\, d\xi. \tag{3.2}$$

We assume that the nonlinear operator

$$\chi(\sigma,\Delta,\varrho) = \nu \, {}^{C}D^\beta_\sigma \varrho(\sigma,\Delta) - \varrho(\sigma,\Delta)\varrho_\sigma(\sigma,\Delta)$$

satisfies a Lipschitz condition on the chosen admissible set. More precisely, there exists a constant $\kappa > 0$ such that

$$\|\chi(\cdot,\Delta,\varrho_1) - \chi(\cdot,\Delta,\varrho_2)\|_{L^2(a,b)} \leq \kappa \|\varrho_1 - \varrho_2\|_{\mathcal{X}},$$

for all $\varrho_1, \varrho_2 \in \mathcal{X}$, where $\mathcal{X}$ denotes the functional space of admissible solutions equipped with a norm sufficiently strong to control both the Caputo fractional derivative and the nonlinear convective term. We impose the following Lipschitz assumption.

Assumption 3.1. There exists a constant $\kappa > 0$ such that, for every $\varrho_1, \varrho_2 \in \mathcal{X}$ and every $\Delta \in [0, \theta]$,

$$\|\chi(\cdot, \Delta, \varrho_1) - \chi(\cdot, \Delta, \varrho_2)\|_{L^2(a,b)} \leq \kappa \|\varrho_1 - \varrho_2\|_{\mathcal{X}}.$$

Here, $\mathcal{X}$ is chosen so that ${}^{C}D^{\beta}_{\sigma}\varrho$ and $\varrho\varrho_\sigma$ belong to $L^2(a, b)$.

This assumption is sufficient for the fixed-point argument developed below.

Theorem 3.1. *Let $\mathcal{X}$ be the Banach space of admissible functions associated with the problem 1.1, and assume that Assumption 3.1 holds. Define the operator $\tau : \mathcal{X} \to \mathcal{X}$ by*

$$\begin{aligned} \tau[\varrho](\sigma, \Delta) = \varrho(\sigma, 0) &+ \frac{1-\alpha}{B(\alpha)} \chi(\sigma, \Delta, \varrho) \\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^\Delta (\Delta - \xi)^{\alpha-1} \chi(\sigma, \xi, \varrho)\, d\xi. \end{aligned} \tag{3.3}$$

Then, for $\varrho_1, \varrho_2 \in \mathcal{X}$ satisfying the same initial conditions,

$$\|\tau\varrho_1 - \tau\varrho_2\|_{\mathcal{X}} \leq L_\tau \|\varrho_1 - \varrho_2\|_{\mathcal{X}},$$

where

$$L_\tau = \left[\frac{1-\alpha}{B(\alpha)} + \frac{\alpha\theta^\alpha}{B(\alpha)\Gamma(\alpha+1)} \right] \kappa.$$

Consequently, τ is Lipschitz continuous on $\mathcal{X}$.

Proof. Let $\varrho_1, \varrho_2 \in \mathcal{X}$ satisfy the same initial conditions. Using 3.3 and Assumption 3.1, we obtain

$$\begin{aligned} \|\tau\varrho_1 - \tau\varrho_2\|_{\mathcal{X}} \leq\ & \frac{1-\alpha}{B(\alpha)} \|\chi(\cdot, \Delta, \varrho_1) - \chi(\cdot, \Delta, \varrho_2)\|_{L^2} \\ & + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^\Delta (\Delta - \xi)^{\alpha-1} \|\chi(\cdot, \xi, \varrho_1) - \chi(\cdot, \xi, \varrho_2)\|_{L^2}\, d\xi \\ \leq\ & \frac{1-\alpha}{B(\alpha)} \kappa \|\varrho_1 - \varrho_2\|_{\mathcal{X}} \\ & + \frac{\alpha\kappa}{B(\alpha)\Gamma(\alpha)} \|\varrho_1 - \varrho_2\|_{\mathcal{X}} \int_0^\Delta (\Delta - \xi)^{\alpha-1}\, d\xi. \end{aligned}$$

Since

$$\int_0^\Delta (\Delta - \xi)^{\alpha-1}\, d\xi = \frac{\Delta^\alpha}{\alpha},$$

we obtain

$$\|\tau\varrho_1 - \tau\varrho_2\|_{\mathcal{X}} \leq \left[\frac{1-\alpha}{B(\alpha)} + \frac{\kappa\theta^\alpha}{B(\alpha)\Gamma(\alpha+1)} \right] \|\varrho_1 - \varrho_2\|_{\mathcal{X}}.$$

Thus τ is Lipschitz continuous.

∎

4. Study of the Existence and Uniqueness of the Solution

To study the existence and uniqueness of the solution of equation 1.1, we can write:

$$\begin{cases} \varrho_0(\sigma,\Delta) = h(\sigma), \\ \varrho_{n+1}(\sigma,\Delta) = h(\sigma) + \dfrac{1-\alpha}{B(\alpha)}\chi(\sigma,\Delta,\varrho_n) \\ \qquad\qquad + \dfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\displaystyle\int_0^{\Delta}(\Delta-\xi)^{\alpha-1}\chi(\sigma,\xi,\varrho_n)\,d\xi, \end{cases} \tag{4.1}$$

Set

$$e_n = \varrho_n - \varrho_{n-1}, \qquad n \geq 1.$$

Using Assumption 3.1, we obtain

$$\|e_{n+1}\|_{\mathcal{X}} \leq \left[\frac{1-\alpha}{B(\alpha)} + \frac{\alpha\theta^{\alpha}}{B(\alpha)\Gamma(\alpha+1)}\right]\kappa\|e_n\|_{\mathcal{X}}.$$

Define

$$q = \kappa\left[\frac{1-\alpha}{B(\alpha)} + \frac{\theta^{\alpha}}{B(\alpha)\Gamma(\alpha+1)}\right]. \tag{4.2}$$

If

$$q < 1, \tag{4.3}$$

then

$$\|e_{n+1}\|_{\mathcal{X}} \leq q\|e_n\|_{\mathcal{X}},$$

and therefore

$$\|e_n\|_{\mathcal{X}} \leq q^{n-1}\|e_1\|_{\mathcal{X}}. \tag{4.4}$$

Hence the sequence $\{\varrho_n\}$ is Cauchy in $\mathcal{X}$.

Let us write the difference between successive terms as follows:

$$\begin{aligned} \varrho_n(\sigma,\Delta) &= \varrho(\sigma,0) + \frac{1-\alpha}{B(\alpha)}\chi(\sigma,\Delta,\varrho_{n-1}) \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{\Delta}(\Delta-\xi)^{\alpha-1}\chi(\sigma,\xi,\varrho_{n-1})\,d\xi. \end{aligned} \tag{4.5}$$

Consequently,

$$\varrho_n = \sum_{i=1}^{n}\theta_i. \tag{4.6}$$

Thus,

$$\begin{aligned} \|\Theta_{n+1}(\sigma,\Delta)\| &= \|\varrho_{n+1}(\sigma,\Delta) - \varrho_n(\sigma,\Delta)\| \\ &= \|\frac{1-\alpha}{B(\alpha)}\{\chi(\sigma,\Delta,\varrho_n) - \chi(\sigma,\Delta,\varrho_{n-1})\} \\ &\quad + \frac{\alpha}{B(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^{\Delta}(\Delta-\xi)^{\alpha-1}\{\chi(\sigma,\xi,\varrho_n) - \chi(\sigma,\xi,\varrho_{n-1})\}\,d\xi\|. \end{aligned}$$

Using the triangle inequality,

$$
\begin{aligned}
\|\Theta_n(\sigma,\Delta)\| &\leq \frac{1-\alpha}{B(\alpha)}\|\chi(\sigma,\Delta,\varrho_{n-1})-\chi(\sigma,\Delta,\varrho_{n-2})\| \\
&\quad + \frac{\alpha}{B(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^{\Delta}(\Delta-\xi)^{\alpha-1}\|\chi(\sigma,\xi,\varrho_{n-1})-\chi(\sigma,\xi,\varrho_{n-2})\|\,d\xi \\
&\leq \frac{1-\alpha}{B(\alpha)}\kappa\|\varrho_{n-1}-\varrho_{n-2}\| + \frac{\alpha}{B(\alpha)}\frac{1}{\Gamma(\alpha)}\kappa\int_0^{\Delta}(\Delta-\xi)^{\alpha-1}\|\varrho_{n-1}-\varrho_{n-2}\|\,d\xi.
\end{aligned}
$$

Theorem 4.1. *Assume that Assumption 3.1 holds and that*

$$
q=\kappa\left[\frac{1-\alpha}{B(\alpha)}+\frac{\theta^{\alpha}}{B(\alpha)\Gamma(\alpha+1)}\right]<1. \tag{4.5}
$$

Then the iterative sequence defined by 4.1 converges in $\mathcal{X}$ to a solution ϱ of equation 1.1.

Proof. From (4.4), we have

$$
\|\varrho_{n+1}-\varrho_n\|_{\mathcal{X}} \leq q^n\|\varrho_1-\varrho_0\|_{\mathcal{X}}.
$$

For $m>n$,

$$
\begin{aligned}
\|\varrho_m-\varrho_n\|_{\mathcal{X}} &\leq \sum_{j=n}^{m-1}\|\varrho_{j+1}-\varrho_j\|_{\mathcal{X}} \\
&\leq \|\varrho_1-\varrho_0\|_{\mathcal{X}}\sum_{j=n}^{m-1}q^j \\
&\leq \frac{q^n}{1-q}\|\varrho_1-\varrho_0\|_{\mathcal{X}}.
\end{aligned}
$$

Since $0<q<1$, the right-hand side tends to zero as $n\to\infty$. Consequently, $\{\varrho_n\}$ is a Cauchy sequence in the Banach space $\mathcal{X}$. Hence, there exists $\varrho\in\mathcal{X}$ such that

$$
\varrho_n \longrightarrow \varrho \quad \text{in } \mathcal{X}.
$$

Passing to the limit in the integral equation 4.1, using the Lipschitz continuity of χ, shows that ϱ satisfies 1.1. Therefore, ϱ is a solution of the fractional Burgers problem. □

Theorem 4.2. *Under the assumptions of Theorem 4.1, the solution of problem 1.1 is unique.*

Proof. Let $\varrho_1,\varrho_2\in\mathcal{X}$ be two solutions of 1.1. Since they satisfy the same initial conditions, we obtain

$$
\begin{aligned}
\|\varrho_1-\varrho_2\|_{\mathcal{X}} &\leq \left[\frac{1-\alpha}{B(\alpha)}+\frac{\theta^{\alpha}}{B(\alpha)\Gamma(\alpha+1)}\right]\kappa\|\varrho_1-\varrho_2\|_{\mathcal{X}} \\
&= q\|\varrho_1-\varrho_2\|_{\mathcal{X}}.
\end{aligned}
$$

Since $q<1$, it follows that

$$
(1-q)\|\varrho_1-\varrho_2\|_{\mathcal{X}} \leq 0.
$$

Therefore,

$$\|\varrho_1 - \varrho_2\|_{\mathcal{X}} = 0,$$

and hence

$$\varrho_1 = \varrho_2.$$

Thus, the solution is unique. $\square$

5. Laplace Variational Iteration Method

We consider the following equation:

$$^{AB}D^{\alpha}_{\Delta}\varrho(\sigma,\Delta)+\varrho(\sigma,\Delta)\,\varrho_{\sigma}(\sigma,\Delta) = \nu\, ^{C}D^{\beta}_{\sigma}\varrho(\sigma,\Delta) \quad for \quad 0<\alpha\leq 1 \quad and \quad 1<\beta\leq 2, \tag{5.1}$$

with the initial conditions:

$$\begin{cases} \varrho(\sigma,0) = h(\sigma), \\ \partial_{\Delta}\varrho(\sigma,0) = p(\sigma). \end{cases} \tag{5.2}$$

Applying the Laplace transform, we obtain:

$$L\left\{^{AB}D^{\alpha}_{\Delta}\varrho(\sigma,\Delta)+\varrho(\sigma,\Delta)\,\varrho_{\sigma}(\sigma,\Delta)\right\} = L\{\nu\, ^{C}D^{\beta}_{\sigma}\varrho(\sigma,\Delta)\},$$

that is,

$$\frac{B(\alpha)s^{\alpha}}{s^{\alpha}(1-\alpha)+\alpha}L\{\varrho(\sigma,\Delta)\} - \frac{s^{\alpha-1}}{s^{\alpha}(1-\alpha)+\alpha}L\{\varrho(\sigma,0)\} - L\{\nu\, ^{C}D^{\beta}_{\sigma}\varrho(\sigma,\Delta) - \varrho(\sigma,\Delta)\,\varrho_{\sigma}(\sigma,\Delta)\} = 0.$$

The iteration formula is given by:

$$\begin{aligned} \varrho_{n+1}(s) = \varrho_n(s) + \lambda(s)[&\frac{B(\alpha)s^{\alpha}L\{\varrho_n(\sigma,\Delta)\}}{s^{\alpha}(1-\alpha)+\alpha} - \frac{s^{\alpha-1}L\{\varrho(\sigma,0)\}}{s^{\alpha}(1-\alpha)+\alpha} \\ &- L\{\nu\, ^{C}D^{\beta}_{\sigma}\varrho(\sigma,\Delta) - \varrho(\sigma,\Delta)\,\varrho_{\sigma}(\sigma,\Delta)\}], \end{aligned}$$

where $\lambda(s)$ is the Lagrange multiplier.

We impose,

$$\frac{\delta\varrho_{n+1}(s)}{\delta\varrho_n(s)} = 0,$$

which yields,

$$1 + \frac{B(\alpha)s^{\alpha}\lambda(s)}{s^{\alpha}(1-\alpha)+\alpha} = 0.$$

Hence,

$$\lambda(s) = -\frac{s^{\alpha}(1-\alpha)+\alpha}{B(\alpha)s^{\alpha}}.$$

Using the inverse Laplace transform, we obtain:

$$\varrho_{n+1}(\sigma,\Delta) = \varrho(\sigma,0) - L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\left[\nu\, ^{C}D^{\beta}_{\sigma}\varrho_n - \varrho_n(\varrho_n)_{\sigma}\right]\right\}. \tag{5.3}$$

Finally, the solution of equation 1.1 is given by:

$$\varrho = \lim_{n\to\infty}\varrho_n(\sigma,\Delta).$$

6. Application

We consider the following equation:

$$ {}^{AB}D^{\alpha}_{\Delta}\varrho(\sigma,\Delta)+\varrho(\sigma,\Delta)\,\varrho_{\sigma}(\sigma,\Delta)=\nu\, {}^{C}D^{\beta}_{\sigma}\varrho(\sigma,\Delta)\quad for\quad 0<\alpha\leq 1\quad and\quad 1<\beta\leq 2, \tag{6.1}$$

With,

$$\varrho(\sigma,0)=h(\sigma)\qquad and\qquad \varrho_{\Delta}(\sigma,0)=p(\sigma). \tag{6.2}$$

From the equations, we obtain:

$$\varrho_{n+1}(\sigma,\Delta)=\varrho(\sigma,0)+L^{-1}\left(\left[1-\alpha+\frac{\alpha}{s^{\alpha}}\right]L\{\nu\, {}^{C}D^{\beta}_{\sigma}(\varrho_n)-(\varrho_n)(\varrho_n)_{\sigma}\}\right).$$

Consequently, let us consider the initial function

$$h(\sigma)=\sigma(1-\sigma),$$

which is compatible with the homogeneous boundary conditions

$$h(0)=h(1)=0.$$

We set

$$\varrho_0(\sigma,\Delta)=h(\sigma)=\sigma(1-\sigma)$$

and $\beta=2$.

Then,

$$\begin{aligned}\varrho_1(\sigma,\Delta)&=\sigma+L^{-1}\left(\left[1-\alpha+\frac{\alpha}{s^{\alpha}}\right]L\{\nu\, {}^{C}D^{\beta}_{\sigma}(\sigma)-\sigma\}\right)\\&=\sigma+L^{-1}\left(\left[1-\alpha+\frac{\alpha}{s^{\alpha}}\right]L\{0-\sigma\}\right)\\&=\sigma-\sigma\left(1-\alpha+\frac{\alpha\Delta^{\alpha}}{\Gamma(\alpha+1)}\right),\end{aligned}$$

and,

$$\begin{aligned}\varrho_2(\sigma,\Delta)&=\sigma-L^{-1}\left(\left[1-\alpha+\frac{\alpha}{s^{\alpha}}\right]L\{\nu\, {}^{C}D^{\beta}_{\sigma}(\varrho_1)-(\varrho_1)(\varrho_1)_{\sigma}\}\right)\\&=\sigma-\sigma L^{-1}\left[\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\left(\left(\alpha-\frac{\alpha\Delta^{\alpha}}{\Gamma(\alpha+1)}\right)^{2}\right)\right]\\&=\sigma-\sigma[(1-\alpha)\alpha^2-2\alpha^2(1-\alpha)\frac{\Delta^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2(1-\alpha)\Delta^{2\alpha}}{\Gamma^2(\alpha+1)}+\alpha^3\frac{\Delta^{\alpha}}{\Gamma(\alpha+1)}\\&-\frac{2\alpha^3\Delta^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3\Delta^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)}].\end{aligned}$$

From the above, we conclude that the approximate solution is:

$$\begin{aligned}\varrho(\sigma,\Delta)&=\sigma-\sigma[(1-\alpha)\alpha^2-2\alpha^2(1-\alpha)\frac{\Delta^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2(1-\alpha)\Delta^{2\alpha}}{\Gamma^2(\alpha+1)}+\alpha^3\frac{\Delta^{\alpha}}{\Gamma(\alpha+1)}\\&-\frac{2\alpha^3\Delta^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3\Delta^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)}]+\ldots\end{aligned}$$

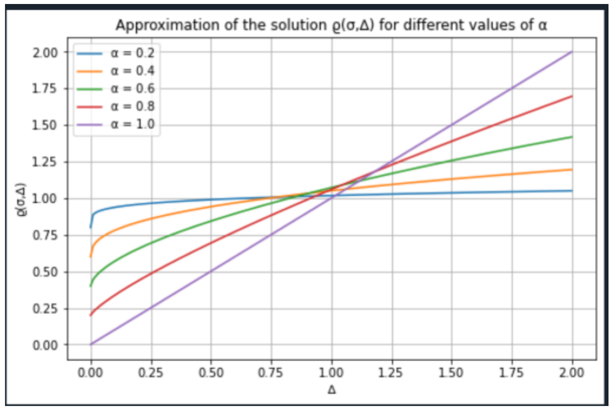

FIGURE 1. Comparaison of the solutions for different values of α.

Setting $\alpha = 1$, we obtain:

$$\begin{aligned}\varrho(\sigma,\Delta) &= \sigma(1-\Delta+\Delta^2+...)\\ &= \frac{\sigma}{1+\Delta}.\end{aligned}$$

6.1. **Numerical error and convergence analysis.** To assess the accuracy of the truncated variational iteration approximation, let

$$\varrho^{(N)}(\sigma,\Delta) = \sum_{n=0}^{N} \varrho_n(\sigma,\Delta)$$

denote the N-term approximation. The corresponding residual is defined by

$$R_N(\sigma,\Delta) = {}^{AB}D^{\alpha}_{\Delta}\varrho^{(N)}(\sigma,\Delta) + \varrho^{(N)}(\sigma,\Delta)\frac{\partial \varrho^{(N)}}{\partial \sigma}(\sigma,\Delta) - \nu\, {}^{C}D^{\beta}_{\sigma}\varrho^{(N)}(\sigma,\Delta).$$

We measure the approximation error using the norms

$$E_{L^\infty} = \max_{(\sigma,\Delta)\in\Omega} |R_N(\sigma,\Delta)|$$

and

$$E_{L^2} = \left(\int_{\Omega} |R_N(\sigma,\Delta)|^2 \, d\sigma \, d\Delta\right)^{1/2}.$$

According to the convergence estimate established in Section 4, increasing N reduces the truncation error geometrically whenever $q < 1$.

The qualitative behavior of the approximate solution $\rho(\sigma,\Delta)$ is examined across various fractional orders α to evaluate the model's sensitivity to memory effects. Figure 1 illustrates a comparative analysis of the solution profiles, highlighting the significant influence of the fractional parameter on the underlying dynamics. To provide a more comprehensive view of the spatio-temporal evolution of the system, several 3D surfaces are presented. Specifically, Figure 2 depicts the solution surface for the classical case where $\alpha = 1$, while the impacts of non-local interactions and historical memory are visualized through the surfaces in Figure 3 ($\alpha = 0.8$) , Figure 4 ($\alpha = 0.5$) , and Figure 5 ($\alpha = 0.2$). These numerical

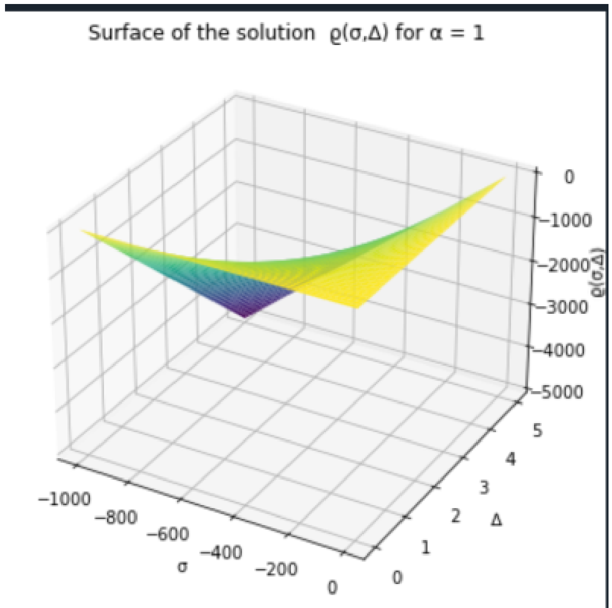

Figure 2. Three-dimensional surface of the approximate solution $\varrho(\sigma, \Delta)$ for $\alpha = 1$.

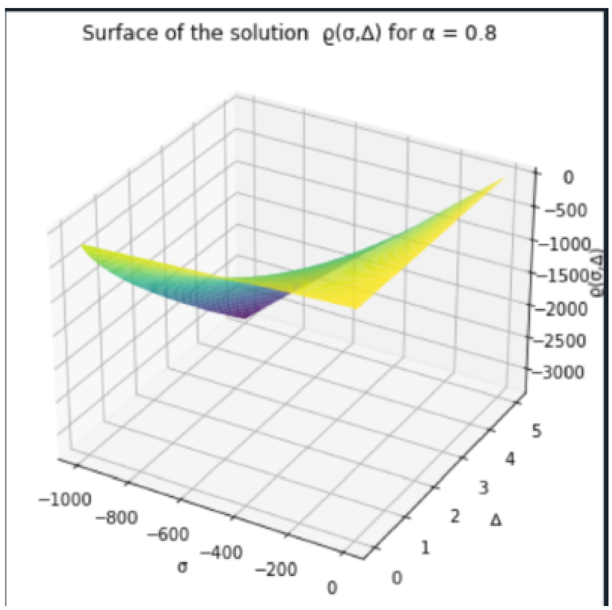

Figure 3. Three-dimensional surface of the approximate solution $\varrho(\sigma, \Delta)$ for $\alpha = 0.8$.

illustrations confirm that the proposed method effectively captures the complex dynamical behavior inherent in fractional-order Burgers equations.

7. Conclusion

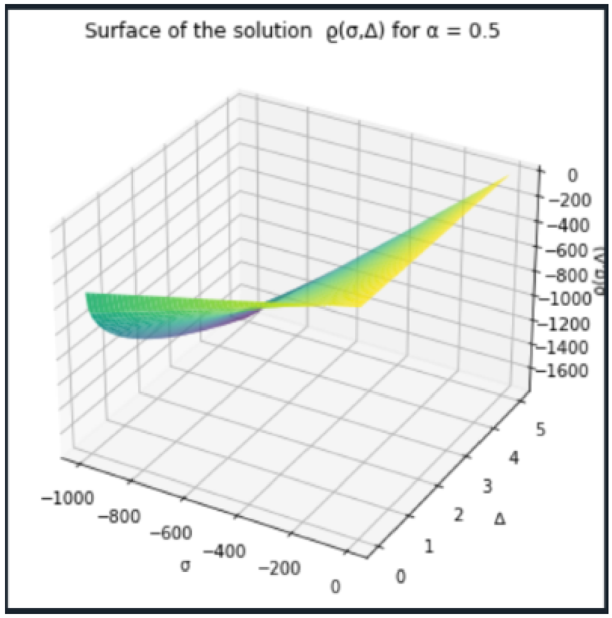

FIGURE 4. Three-dimensional surface of the approximate solution $\varrho(\sigma, \Delta)$ for $\alpha = 0.5$.

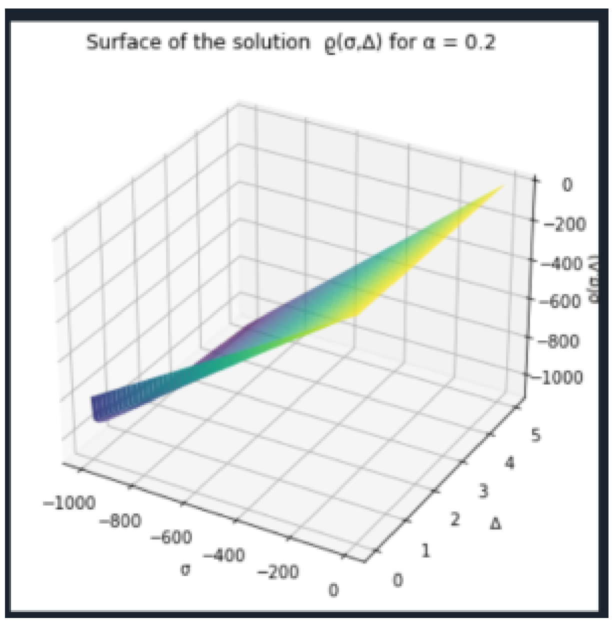

FIGURE 5. Three-dimensional surface of the approximate solution $\varrho(\sigma, \Delta)$ for $\alpha = 0.2$.

8. Conclusion

In this work, we studied a nonlinear Burgers equation involving an Atangana–Baleanu fractional derivative in time and a Caputo fractional derivative in space. A functional framework was specified for the analysis, and existence and uniqueness of the solution were established under an explicit Lipschitz assumption on the nonlinear operator. A Laplace variational iteration scheme was then formulated, and a geometric convergence estimate together with an explicit truncation-error bound was obtained.

An illustrative example was considered to investigate the influence of the fractional order on the behavior of the solution. The numerical results show the effect of the fractional parameter on the dynamics and complement the theoretical convergence analysis. In the limiting case $\alpha \to 1$, the fractional time derivative approaches the corresponding classical description, providing consistency with the classical Burgers model.

Future work will focus on developing fully discretized finite-difference and spectral schemes, together with rigorous stability, consistency, and convergence analyses, in order to provide a more comprehensive computational study of mixed fractional Burgers equations.

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