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DEVANEY CHAOS OF A PROPORTIONAL CAPUTO DERIVATIVE

EL-MAHDI NAFIA*, HASNAA ALATOUNE, AND M'HAMED EL OMARI

Abstract. In this article, we analyze the dynamical behaviour of a proportional Caputo-type complex fractional derivative. We prove, by using the Bayart–Grivaux eigenvector-field theorem, that this operator is Devaney chaotic in a suitable weighted Mittag–Leffler–Caputo space.

Keywords. Caputo fractional derivative; proportional derivative; Devaney chaos; Mittag–Leffler function; Köthe sequence space and eigenvector field.
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1. Introduction

Fractional calculus has become a central tool in the mathematical analysis of systems whose evolution is influenced by memory effects. In contrast with local integer-order differentiation, fractional derivatives involve the past history of the state variable through singular integral kernels. This non-local structure makes them particularly useful in the modelling of anomalous diffusion, viscoelasticity, hereditary processes, control systems and several classes of evolution equations; see, among others, [14, 16, 15, 8, 4]. Within this broad family of operators, the Caputo derivative occupies a special position because it vanishes on constants and is compatible with classical initial conditions formulated in terms of integer-order derivatives [15, 16, 6].

Parallel to these developments, the theory of linear dynamics has shown that many natural operators acting on infinite-dimensional spaces display highly unstable behaviour. Notions such as hypercyclicity, topological transitivity and Devaney chaos have become fundamental tools for studying this instability; see the monographs [3, 7]. Several recent works have also emphasized the usefulness of spectral and operator-theoretic methods in the study of chaotic dynamics for evolution equations. Nafia and Elomari investigated chaotic dynamics for an initial and boundary value problem associated with the viscous van Wijngaarden–Eringen model, which describes acoustic wave propagation in bubbly liquids [11]. In another direction, Nafia and Elomari developed spectral conditions for chaos in

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*Corresponding author: El-mahdi Nafia.
El-mahdi Nafia and M'hamed El Omari contributed equally to this work.

strongly continuous cosine operator families, thereby extending chaos criteria to abstract second-order evolution equations governed by cosine functions [12]. More recently, Nafia and Elomari studied chaotic dynamics of hyperbolic bioheat models in a Banach space, showing that eigenvector-field methods can also be applied to hyperbolic thermal models [13]. These contributions illustrate the growing role of spectral methods, semigroup and cosine-family techniques, and eigenvector-field criteria in the modern analysis of chaotic linear dynamics. In particular, the usual differentiation operator is a classical source of chaotic dynamics on spaces of holomorphic functions. It is therefore natural to ask whether fractional derivatives, which are non-local analogues of differentiation, can also generate chaotic linear dynamics on suitably chosen spaces.

This question has recently been addressed for the complex Caputo derivative in [10]. There, the authors constructed a Mittag-Leffler-Caputo type space and proved chaos by combining three ingredients: the explicit eigenvectors of the Caputo derivative, a Köthe sequence space adapted to the induced weighted backward shift, and an eigenvector field criterion from linear dynamics [2, 1, 5]. This result provides a starting point for the study of chaotic phenomena generated by fractional operators. However, it leaves open the question of whether more general fractional derivatives, especially those involving proportional or exponentially weighted effects, can be treated within a comparable dynamical framework.

Motivated by this observation, the natural question considered in the present paper is the following: does the chaotic behaviour known for the complex Caputo derivative persist when the Caputo operator is replaced by a proportional Caputo-type fractional derivative ? In other words, can the proportional Caputo-type derivative generate Devaney chaos on a suitable space of holomorphic functions, in the same spirit as the classical Caputo derivative ? This question is not merely a formal extension, because the proportional term modifies both the fractional memory operator and the functional structure on which the dynamics has to be studied.

The aim of the present paper is to give an affirmative answer to this problem for a proportional Caputo-type fractional derivative. More precisely, for $0 < \alpha < 1$ and $\rho > 0$, we study the operator

$$(\mathfrak{D}^{\alpha,\rho} f)(t) = \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_0^t \mathrm{e}^{\frac{\rho-1}{\rho}(t-\tau)}(t-\tau)^{-\alpha}(\mathfrak{D}_\rho f)(\tau)\, d\tau,$$

where

$$(\mathfrak{D}_\rho f)(t) = \rho f'(t) + (1-\rho) f(t).$$

Previous studies on generalized proportional fractional dynamical systems have shown that the proportional parameter may have a significant influence on the qualitative behaviour of the corresponding solutions. In generalized proportional Caputo models, variations of this parameter have been associated with changes in stability properties, asymptotic behaviour, equilibrium dynamics, and numerical solution profiles. For instance, generalized proportional Caputo models of gene regulatory networks exhibit parameter-dependent equilibrium stability and generalized exponential stability [17]. Practical and exponential practical stability have also been investigated for delayed proportional Caputo systems by means of Lyapunov–Razumikhin techniques [18]. In a different setting, generalized proportional fractional logistic equations display changes in the evolution of the solution profiles and provide numerical evidence of the influence of the proportional structure on the underlying dynamics [19]. Related numerical investigations for impulsive pantograph problems further illustrate the effect of proportional fractional parameters on solution trajectories [20]. These studies indicate that the proportional parameter is not merely an auxiliary quantity in fractional models, but can affect the concrete dynamical realization of the system. This motivates our investigation of its influence on the dynamics generated by the proportional Caputo operator.

The paper is organized as follows. Section 2 recalls the background on complex fractional calculus, Köthe spaces and linear dynamics. Section 3 introduces the proportional Caputo-type operator and proves the conjugacy formula. Section 4 constructs the weighted Mittag-Leffler-Caputo space associated with the proportional structure. In Section 5, we prove that the operator is well defined and continuous on this space. Section 6 constructs the eigenvector field, proves its totality and establishes weak holomorphy. Finally, Section 7 applies the eigenvector field criterion and proves Devaney chaos.

2. Preliminaries

We recall in this section only the standard notions and results that will be used in the construction of the phase space and in the proof of chaos for the proportional Caputo-type operator.

Since fractional powers are multi-valued in the complex plane, we work on the slit domain

$$\Omega = \mathbb{C} \setminus (-\infty, 0].$$

The principal branch of the logarithm on Ω is denoted by $\operatorname{Log} z$, and we define

$$z^\gamma = \exp(\gamma \operatorname{Log} z), \qquad z \in \Omega, \quad \gamma \in \mathbb{R}.$$

All fractional powers appearing below are understood in this sense.

Definition 2.1. [16, 15, 6] Let $0 < \alpha < 1$. For a holomorphic function f for which the integral is well defined, the complex Caputo derivative of order α is

$$(D_C^\alpha f)(z) = \frac{1}{\Gamma(1-\alpha)} \int_0^z (z-\zeta)^{-\alpha} f'(\zeta)\, d\zeta, \qquad z \in \Omega.$$

The integration is taken along the segment joining 0 to z.

Lemma 2.2. [16, 15, 6] *Let $0 < \alpha < 1$. For $p > 0$,*

$$D_C^\alpha z^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} z^{p-\alpha}.$$

Moreover,

$$D_C^\alpha 1 = 0.$$

Definition 2.3. [6, 15] For $0 < \alpha < 1$, the Mittag-Leffler function is

$$E_\alpha(w) = \sum_{n=0}^{\infty} \frac{w^n}{\Gamma(\alpha n + 1)}, \qquad w \in \mathbb{C}.$$

Proposition 2.4. [6, 15] *The function E_α is entire. Moreover, for every $\lambda \in \mathbb{C}$,*

$$D_C^\alpha \left(E_\alpha(\lambda z^\alpha) \right) = \lambda E_\alpha(\lambda z^\alpha).$$

We shall also use a few standard notions from linear dynamics. We recall them in the form needed later; see [3, 7] for a general presentation.

Definition 2.5. [3, 7] Let X be a complex topological vector space and let $T : X \to X$ be a continuous linear operator. The operator T is hypercyclic if there exists $x \in X$ such that

$$\{T^n x : n \geq 0\}$$

is dense in X.

Definition 2.6. [3, 7] Let X be a topological space and $T : X \to X$ be continuous. The map T is topologically transitive if, for every pair of non-empty open sets $U, V \subset X$, there exists $n \geq 1$ such that

$$T^n(U) \cap V \neq \varnothing.$$

Definition 2.7. [7] Let $T : X \to X$ be a map. A point $x \in X$ is periodic if there exists $n \geq 1$ such that

$$T^n x = x.$$

The set of all periodic points is denoted by $\mathrm{Per}(T)$.

Definition 2.8. [5, 1] Let X be a metric space and $T : X \to X$ be continuous. The map T is chaotic in the sense of Devaney if it is topologically transitive, if $\mathrm{Per}(T)$ is dense in X, and if it has sensitive dependence on initial conditions.

We shall use the following consequence of the eigenvector field method. This criterion is standard in linear dynamics and is particularly effective for operators whose eigenvectors depend holomorphically on the eigenvalue; see [2, 3, 7].

Definition 2.9. Let X be a complex locally convex space and let $U \subset \mathbb{C}$ be open. A map $G : U \to X$ is weakly holomorphic if, for every $\Phi \in X'$, the scalar function

$$\lambda \mapsto \Phi(G(\lambda))$$

is holomorphic on U.

Theorem 2.10. [2, 3, 7] *Let X be a complex Fréchet space and let $T : X \to X$ be a continuous linear operator. Assume that there exist a non-empty connected open set $U \subset \mathbb{C}$, with*

$$U \cap \mathbb{T} \neq \varnothing, \qquad \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\},$$

and a weakly holomorphic map $G : U \to X$ such that

$$TG(\lambda) = \lambda G(\lambda), \qquad \lambda \in U,$$

and

$$\overline{\mathrm{span}\{G(\lambda) : \lambda \in U\}} = X.$$

Then T is chaotic in the sense of Devaney.

Finally, we recall the Köthe sequence spaces that will be used to define the functional model associated with our operator.

Definition 2.11. [9, 7] A Köthe matrix is a matrix

$$A = (a_{j,k})_{j,k\geq 0}$$

of non-negative real numbers such that, for every $j \geq 0$, at least one $a_{j,k}$ is positive, and

$$a_{j,k} \leq a_{j,k+1}, \qquad j,k \geq 0.$$

Definition 2.12. [9, 7] Let $A = (a_{j,k})_{j,k\geq 0}$ be a Köthe matrix. The space $\lambda^2(A)$ consists of all scalar sequences $b = (b_j)_{j\geq 0}$ such that

$$\|b\|_k^2 = \sum_{j=0}^{\infty} |b_j|^2 a_{j,k}^2 < \infty, \qquad k \geq 0.$$

Proposition 2.13. [9] *For every Köthe matrix A, the space $\lambda^2(A)$, endowed with the seminorms*

$$\|b\|_k = \left(\sum_{j=0}^{\infty} |b_j|^2 a_{j,k}^2\right)^{1/2}, \qquad k \geq 0,$$

is a Fréchet space.

Proposition 2.14. [9] *Let A be a Köthe matrix. If $Y \in (\lambda^2(A))'$, then there exists a scalar sequence $y = (y_j)_{j\geq 0}$ such that*

$$Y(b) = \sum_{j=0}^{\infty} b_j y_j, \qquad b = (b_j)_{j\geq 0} \in \lambda^2(A).$$

The Köthe matrix used in the sequel will be chosen from the weights of the backward shift induced by the proportional Caputo-type derivative. This is the only reason Köthe spaces are recalled here: they provide the precise locally convex framework in which the operator is continuous and the eigenvector field is total.

3. The proportional Caputo-type derivative

In this section, we introduce the proportional Caputo-type fractional derivative which will be the main operator studied in the paper, and we identify the simple structural relation that connects it with the classical Caputo derivative. Throughout this section, we fix

$$0 < \alpha < 1, \qquad \rho > 0,$$

and we set

$$\beta = \frac{\rho - 1}{\rho}.$$

The proportional derivative is defined by

$$\mathfrak{D}_\rho f = \rho f' + (1-\rho) f.$$

For functions for which the following integral is well defined, we consider the proportional Caputo-type fractional derivative

$$(\mathfrak{D}^{\alpha,\rho} f)(z) = \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_0^z \mathrm{e}^{\beta(z-\zeta)}(z-\zeta)^{-\alpha}(\mathfrak{D}_\rho f)(\zeta)\ d\zeta. \tag{3.1}$$

The integral is taken along the segment from 0 to z, and fractional powers are computed using the principal branch on

$$\Omega = \mathbb{C} \setminus (-\infty, 0].$$

We denote by M_β the multiplication operator

$$(M_\beta u)(z) = \mathrm{e}^{\beta z} u(z).$$

The following identity is the basic structural fact of the paper.

Lemma 3.1 (Exponential conjugacy)**.** *Let u be a function for which all the terms below are well defined. Then*

$$\mathfrak{D}^{\alpha,\rho}(M_\beta u) = \rho^\alpha M_\beta D_C^\alpha u.$$

Equivalently,

$$\mathfrak{D}^{\alpha,\rho} M_\beta = \rho^\alpha M_\beta D_C^\alpha.$$

Proof. Put

$$f = M_\beta u.$$

Then

$$f(z) = \mathrm{e}^{\beta z} u(z).$$

By the product rule,

$$f'(z) = \beta \mathrm{e}^{\beta z} u(z) + \mathrm{e}^{\beta z} u'(z) = \mathrm{e}^{\beta z}\big(u'(z) + \beta u(z)\big).$$

Hence

$$f'(z) - \beta f(z) = \mathrm{e}^{\beta z}\big(u'(z) + \beta u(z)\big) - \beta \mathrm{e}^{\beta z} u(z).$$

The terms containing $\beta u(z)$ cancel. Therefore

$$f'(z) - \beta f(z) = \mathrm{e}^{\beta z} u'(z).$$

We obtain

$$(\mathfrak{D}_\rho f)(z) = \rho \mathrm{e}^{\beta z} u'(z). \tag{3.2}$$

Substituting (3.2) into (3.1) gives

$$(\mathfrak{D}^{\alpha,\rho} f)(z) = \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_0^z \mathrm{e}^{\beta(z-\zeta)}(z-\zeta)^{-\alpha} \rho \mathrm{e}^{\beta\zeta} u'(\zeta)\ d\zeta.$$

We take the constant ρ outside the integral:

$$(\mathfrak{D}^{\alpha,\rho} f)(z) = \frac{\rho}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_0^z \mathrm{e}^{\beta(z-\zeta)} \mathrm{e}^{\beta\zeta}(z-\zeta)^{-\alpha} u'(\zeta)\ d\zeta.$$

Since

$$\mathrm{e}^{\beta(z-\zeta)}\mathrm{e}^{\beta\zeta} = \mathrm{e}^{\beta z},$$

the exponential factor no longer depends on ζ. Thus

$$(\mathfrak{D}^{\alpha,\rho} f)(z) = \frac{\rho \mathrm{e}^{\beta z}}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_0^z (z-\zeta)^{-\alpha} u'(\zeta)\ d\zeta.$$

Finally,

$$(\mathfrak{D}^{\alpha,\rho} f)(z) = \rho^\alpha e^{\beta z} \frac{1}{\Gamma(1-\alpha)} \int_0^z (z-\zeta)^{-\alpha} u'(\zeta)\, d\zeta.$$

The last integral is exactly $D_C^\alpha u(z)$. Hence

$$\mathfrak{D}^{\alpha,\rho}(M_\beta u)(z) = \rho^\alpha e^{\beta z} D_C^\alpha u(z) = \rho^\alpha M_\beta D_C^\alpha u(z).$$

The proof is complete. □

The conjugacy reduces the eigenvalue problem for $\mathfrak{D}^{\alpha,\rho}$ to the usual Caputo eigenvalue problem.

Corollary 3.2 (Eigenvalue equation)**.** *Let $\lambda \in \mathbb{C}$. In the Mittag-Leffler class considered in this paper, the solution of*

$$\mathfrak{D}^{\alpha,\rho} f = \lambda f, \qquad f(0) = c,$$

is

$$f_\lambda(z) = c\, e^{\beta z} E_\alpha\left(\frac{\lambda}{\rho^\alpha} z^\alpha\right). \tag{3.3}$$

Proof. Write

$$f = M_\beta u, \qquad f(z) = e^{\beta z} u(z).$$

By Lemma 3.1,

$$\mathfrak{D}^{\alpha,\rho} f = \rho^\alpha M_\beta D_C^\alpha u.$$

The equation

$$\mathfrak{D}^{\alpha,\rho} f = \lambda f$$

therefore becomes

$$\rho^\alpha M_\beta D_C^\alpha u = \lambda M_\beta u.$$

the operator M_β is injective. Hence

$$\rho^\alpha D_C^\alpha u = \lambda u.$$

Equivalently,

$$D_C^\alpha u = \frac{\lambda}{\rho^\alpha} u. \tag{3.4}$$

Set

$$\mu = \frac{\lambda}{\rho^\alpha}.$$

Then (3.4) becomes

$$D_C^\alpha u = \mu u.$$

In the analytic fractional-power class, the normalized solution is

$$u(z) = u(0) E_\alpha(\mu z^\alpha).$$

Therefore

$$u(z) = u(0) E_\alpha\left(\frac{\lambda}{\rho^\alpha} z^\alpha\right).$$

Since

$$f(0) = e^0 u(0) = u(0) = c.$$

Consequently,

$$f_\lambda(z) = c\,\mathrm{e}^{\beta z} E_\alpha\left(\frac{\lambda}{\rho^\alpha} z^\alpha\right).$$

This proves (3.3). □

Remark 3.3. The previous result shows that the proportional term does not destroy the Mittag-Leffler eigenstructure. It changes it by an exponential conjugacy and by the spectral rescaling

$$\lambda \longmapsto \frac{\lambda}{\rho^\alpha}.$$

This is the mechanism behind the functional model constructed below.

The case $\rho = 1$ recovers the classical Caputo setting, since $\beta = 0$, $\Theta_n(z) = z^{\alpha n}$, and $G_1(\lambda)(z) = E_\alpha(\lambda z^\alpha)$. For $0 < \rho < 1$, one has $\beta = (\rho - 1)/\rho < 0$, so that along the positive real axis the exponential weight $e^{\beta t}$ is decaying. In contrast, for $\rho > 1$, $\beta > 0$ and the same weight is increasing. Thus, varying ρ modifies the spatial weighting of the eigenvectors, in addition to changing the factor ρ^α in the coefficients of the associated weighted backward shift. Consequently, although the conjugacy preserves the underlying topological mechanism responsible for chaos, the proportional parameter changes the concrete realization and finite-time amplification of the dynamics. This parameter-dependent effect is illustrated numerically in Section 8.

4. The weighted Mittag-Leffler–Caputo space

We now construct the phase space on which the proportional Caputo-type operator will act continuously. The conjugacy formula suggests the weighted fractional monomials

$$\Theta_n(z) = \mathrm{e}^{\beta z} z^{\alpha n}, \qquad n \geq 0.$$

They are holomorphic on

$$\Omega = \mathbb{C} \setminus (-\infty, 0],$$

where all fractional powers are defined by the principal branch of the logarithm.

From Lemma 3.1 and the Caputo power rule, one gets

$$\mathfrak{D}^{\alpha,\rho}\Theta_0 = 0,$$

and, for $n \geq 1$,

$$\mathfrak{D}^{\alpha,\rho}\Theta_n = \rho^\alpha \frac{\Gamma(\alpha n + 1)}{\Gamma(\alpha(n-1)+1)}\Theta_{n-1}. \tag{4.1}$$

Thus the operator is represented, in the basis $(\Theta_n)_{n\geq 0}$, by a weighted backward shift.

The choice of the coefficient space is dictated by the iterates of this shift. Indeed, for $k \geq 1$ and $n \geq k$, repeated use of (4.1) gives

$$(\mathfrak{D}^{\alpha,\rho})^k\Theta_n = \rho^{k\alpha} \frac{\Gamma(\alpha n + 1)}{\Gamma(\alpha(n-k)+1)}\Theta_{n-k},$$

whereas

$$(\mathfrak{D}^{\alpha,\rho})^k\Theta_n = 0, \qquad n < k.$$

Consequently, if

$$f = \sum_{n=0}^{\infty} b_n \Theta_n,$$

then the k-th iterate has the formal coefficient representation

$$(\mathfrak{D}^{\alpha,\rho})^k f = \rho^{k\alpha} \sum_{j=0}^{\infty} b_{j+k} \frac{\Gamma(\alpha(j+k)+1)}{\Gamma(\alpha j+1)} \Theta_j. \tag{4.2}$$

The weights below are chosen precisely to control the coefficients appearing in (4.2).

For $j, k \in \mathbb{N}_0$, define

$$a_{j,k} = \begin{cases} \dfrac{\Gamma(\alpha(j+k-1)+1)}{\sqrt{\Gamma(\alpha j+1)}}, & 0 \le j < k, \\ \dfrac{\Gamma(\alpha j+1)}{\sqrt{\Gamma(\alpha(j-k)+1)}}, & j \ge k. \end{cases}$$

All these numbers are positive, since every argument of the Gamma function is positive. In particular, for $j \ge k$,

$$a_{j,k}^2 = \frac{\Gamma(\alpha j+1)^2}{\Gamma(\alpha(j-k)+1)}.$$

Writing $j = \ell + k$, we obtain

$$a_{\ell+k,k}^2 = \frac{\Gamma(\alpha(\ell+k)+1)^2}{\Gamma(\alpha\ell+1)}.$$

This is exactly the weight needed to estimate the coefficient

$$b_{\ell+k} \frac{\Gamma(\alpha(\ell+k)+1)}{\Gamma(\alpha\ell+1)}$$

in (4.2).

The family $(a_{j,k})$ is not used directly, because monotonicity in k is not automatic. We therefore regularize it by setting

$$\widetilde{a}_{j,k} = \max_{0 \le r \le k} a_{j,r}.$$

Then

$$\widetilde{a}_{j,k} \le \widetilde{a}_{j,k+1}, \qquad j, k \ge 0,$$

and

$$\widetilde{a}_{j,k} > 0.$$

Hence

$$\widetilde{A} = (\widetilde{a}_{j,k})_{j,k \ge 0}$$

is a Köthe matrix.

Let

$$\lambda^2(\widetilde{A}) = \left\{ b = (b_j)_{j \ge 0} : \|b\|_k^2 = \sum_{j=0}^{\infty} |b_j|^2 \widetilde{a}_{j,k}^2 < \infty \quad \text{for every } k \ge 0 \right\}.$$

This is a Fréchet space with the increasing family of seminorms

$$\|b\|_k = \left(\sum_{j=0}^{\infty} |b_j|^2 \widetilde{a}_{j,k}^2\right)^{1/2}.$$

For $b \in \lambda^2(\widetilde{A})$, define

$$J_\rho b(z) = \sum_{j=0}^{\infty} b_j \Theta_j(z) = \mathrm{e}^{\beta z} \sum_{j=0}^{\infty} b_j z^{\alpha j}.$$

We first show that the series is meaningful. Since

$$a_{j,0} = \frac{\Gamma(\alpha j+1)}{\sqrt{\Gamma(\alpha j+1)}} = \sqrt{\Gamma(\alpha j+1)},$$

we have

$$\widetilde{a}_{j,0} = a_{j,0} = \sqrt{\Gamma(\alpha j+1)}.$$

Thus

$$\sum_{j=0}^{\infty} |b_j|^2 \Gamma(\alpha j+1) < \infty. \tag{4.3}$$

Let $K \subset \Omega$ be compact. Choose $R > 0$ such that

$$|z| \le R, \qquad z \in K.$$

Then, for $z \in K$,

$$\sum_{j=0}^{\infty} |b_j||z|^{\alpha j} \le \sum_{j=0}^{\infty} |b_j| R^{\alpha j}.$$

By Cauchy–Schwarz and (4.3),

$$\sum_{j=0}^{\infty} |b_j| R^{\alpha j} \le \left(\sum_{j=0}^{\infty} |b_j|^2 \Gamma(\alpha j+1)\right)^{1/2} \left(\sum_{j=0}^{\infty} \frac{R^{2\alpha j}}{\Gamma(\alpha j+1)}\right)^{1/2}.$$

The second factor is finite, since

$$\sum_{j=0}^{\infty} \frac{R^{2\alpha j}}{\Gamma(\alpha j+1)} = E_\alpha(R^{2\alpha}).$$

Hence

$$\sum_{j=0}^{\infty} b_j z^{\alpha j}$$

converges normally on every compact subset of Ω. Therefore $J_\rho b$ is holomorphic on Ω.

We now prove uniqueness of the coefficients. Suppose that

$$J_\rho b = 0.$$

Since $\mathrm{e}^{\beta z} \neq 0$, we get

$$\sum_{j=0}^{\infty} b_j z^{\alpha j} = 0, \qquad z \in \Omega.$$

Choose a sector $S \Subset \Omega$ on which the map

$$z \mapsto w = z^{\alpha}$$

is biholomorphic onto an open sector S_{α}. Then

$$\sum_{j=0}^{\infty} b_j w^j = 0, \qquad w \in S_{\alpha}.$$

This is an ordinary holomorphic power series in w. Since it vanishes on the non-empty open set S_{α}, the identity theorem gives

$$b_j = 0, \qquad j \geq 0.$$

Therefore J_{ρ} is injective. We define

$$MLC^2_{\rho}(\Omega; \alpha) = J_{\rho}\big(\lambda^2(\widetilde{A})\big).$$

Equivalently,

$$MLC^2_{\rho}(\Omega; \alpha) = \left\{ f(z) = \mathrm{e}^{\beta z} \sum_{j=0}^{\infty} b_j z^{\alpha j} : (b_j)_{j \geq 0} \in \lambda^2(\widetilde{A}) \right\}.$$

If

$$f = J_{\rho} b,$$

we set

$$\|f\|_{k,\rho} = \|b\|_k, \qquad k \geq 0.$$

This is well defined because J_{ρ} is injective. Moreover,

$$\|J_{\rho} b\|_{k,\rho} = \|b\|_k, \qquad k \geq 0.$$

Proposition 4.1. *The space $MLC^2_{\rho}(\Omega; \alpha)$, endowed with the family $(\| \cdot \|_{k,\rho})_{k \geq 0}$, is a Fréchet space.*

Proof. The Köthe space $\lambda^2(\widetilde{A})$ is Fréchet. The map

$$J_{\rho} : \lambda^2(\widetilde{A}) \to MLC^2_{\rho}(\Omega; \alpha)$$

is linear, bijective by definition of the range, and injective by the uniqueness proved above. Also,

$$\|J_{\rho} b\|_{k,\rho} = \|b\|_k.$$

Hence J_{ρ} is a topological isomorphism. Completeness and metrizability are therefore transported from $\lambda^2(\widetilde{A})$ to $MLC^2_{\rho}(\Omega; \alpha)$. □

Proposition 4.2. *Every element of $MLC^2_{\rho}(\Omega; \alpha)$ is holomorphic on Ω.*

Proof. Let $f \in MLC^2_{\rho}(\Omega; \alpha)$. Then $f = J_{\rho} b$ for some $b \in \lambda^2(\widetilde{A})$. The normal convergence on compact subsets of Ω has already been proved. Hence

$$\sum_{j=0}^{\infty} b_j z^{\alpha j}$$

is holomorphic on Ω. Since $\mathrm{e}^{\beta z}$ is entire, the product

$$f(z) = \mathrm{e}^{\beta z} \sum_{j=0}^{\infty} b_j z^{\alpha j}$$

is holomorphic on Ω. $\square$

5. Continuity of the operator

We now prove that the weighted backward shift induced by $\mathfrak{D}^{\alpha,\rho}$ is continuous on the space constructed above.

For

$$f = J_\rho b = \sum_{j=0}^{\infty} b_j \Theta_j,$$

define the sequence $c = (c_j)_{j\geq 0}$ by

$$c_j = \rho^\alpha b_{j+1} \frac{\Gamma(\alpha(j+1)+1)}{\Gamma(\alpha j+1)}, \qquad j \geq 0. \tag{5.1}$$

Then

$$\mathfrak{D}^{\alpha,\rho} f = J_\rho c$$

whenever $c \in \lambda^2(\widetilde{A})$.

Proposition 5.1. *The operator*

$$\mathfrak{D}^{\alpha,\rho} : MLC^2_\rho(\Omega;\alpha) \longrightarrow MLC^2_\rho(\Omega;\alpha)$$

is well defined and continuous. More precisely, for every $k \geq 0$, there exists $C_k > 0$ such that

$$\|\mathfrak{D}^{\alpha,\rho} f\|_{k,\rho} \leq C_k \|f\|_{k+1,\rho}, \qquad f \in MLC^2_\rho(\Omega;\alpha).$$

Proof. Let $f = J_\rho b$ and let c be defined by (5.1). Since $\mathfrak{D}^{\alpha,\rho} f = J_\rho c$, it suffices to estimate the coefficient seminorms. Fix $k \geq 0$. For the auxiliary weights $A = (a_{j,k})$, write

$$\|c\|^2_{k,A} = \sum_{j=0}^{k-1} |c_j|^2 a^2_{j,k} + \sum_{j=k}^{\infty} |c_j|^2 a^2_{j,k},$$

where the first sum is understood to be zero when $k = 0$.

For $j \geq k$, the definitions of c_j and $a_{j,k}$ give

$$\sum_{j=k}^{\infty} |c_j|^2 a^2_{j,k} = \rho^{2\alpha} \sum_{\ell=0}^{\infty} |b_{\ell+k+1}|^2 a^2_{\ell+k+1,k+1} \leq \rho^{2\alpha} \|b\|^2_{k+1,A}.$$

The finite part involves only $b_1, \ldots, b_k$; hence there exists $C'_k > 0$ such that

$$\sum_{j=0}^{k-1} |c_j|^2 a^2_{j,k} \leq C'_k \|b\|^2_{k+1,A},$$

since the finitely many weights $a_{m,k+1}$, $1 \leq m \leq k$, are strictly positive. Consequently,

$$\|c\|_{k,A} \leq C_{k,A} \|b\|_{k+1,A}$$

for some $C_{k,A} > 0$.

Finally, since $\widetilde{a}_{j,k}^2 \le \sum_{r=0}^{k} a_{j,r}^2$ and $a_{j,r+1} \le \widetilde{a}_{j,k+1}$ for $r \le k$, we obtain

$$\|c\|_k^2 \le \sum_{r=0}^{k} \|c\|_{r,A}^2 \le \Big(\sum_{r=0}^{k} C_{r,A}^2\Big) \|b\|_{k+1}^2.$$

Therefore, for some $C_k > 0$,

$$\|\mathfrak{D}^{\alpha,\rho} f\|_{k,\rho} = \|c\|_k \le C_k \|b\|_{k+1} = C_k \|f\|_{k+1,\rho}.$$

Thus $c \in \lambda^2(\widetilde{A})$ whenever $b \in \lambda^2(\widetilde{A})$, so $\mathfrak{D}^{\alpha,\rho}$ is well defined on $MLC_\rho^2(\Omega;\alpha)$, and the preceding seminorm estimate proves its continuity. □

6. The eigenvector field

We now construct the eigenvector field. For $\lambda \in \mathbb{C}$, define

$$G_\rho(\lambda) = \sum_{n=0}^{\infty} \frac{(\lambda \rho^{-\alpha})^n}{\Gamma(\alpha n + 1)} \Theta_n.$$

Since

$$\Theta_n(z) = \mathrm{e}^{\beta z} z^{\alpha n},$$

this can be written as

$$G_\rho(\lambda)(z) = \mathrm{e}^{\beta z} E_\alpha(\lambda \rho^{-\alpha} z^\alpha).$$

We first prove that this formula defines a vector of $MLC_\rho^2(\Omega;\alpha)$.

Lemma 6.1. *For every $\lambda \in \mathbb{C}$,*

$$G_\rho(\lambda) \in MLC_\rho^2(\Omega;\alpha).$$

Proof. Let

$$x(\lambda) = \big(x_n(\lambda)\big)_{n\ge 0} \frac{(\lambda \rho^{-\alpha})^n}{\Gamma(\alpha n + 1)}.$$

It is enough to show that

$$x(\lambda) \in \lambda^2(\widetilde{A}).$$

Fix $k \ge 0$. Since

$$\widetilde{a}_{n,k} = \max_{0 \le r \le k} a_{n,r},$$

we have

$$\widetilde{a}_{n,k}^2 \le \sum_{r=0}^{k} a_{n,r}^2. \tag{6.1}$$

Therefore it suffices to prove that, for every $r = 0, \dots, k$,

$$\sum_{n=0}^{\infty} |x_n(\lambda)|^2 a_{n,r}^2 < \infty.$$

Fix such an r. The finite part

$$\sum_{n=0}^{r-1} |x_n(\lambda)|^2 a_{n,r}^2$$

is finite. For the tail $n \geq r$, the definition of $a_{n,r}$ gives

$$a_{n,r}^2 = \frac{\Gamma(\alpha n+1)^2}{\Gamma(\alpha(n-r)+1)}.$$

Hence

$$|x_n(\lambda)|^2 a_{n,r}^2 = \frac{|\lambda\rho^{-\alpha}|^{2n}}{\Gamma(\alpha(n-r)+1)}.$$

Putting $m = n - r$, we get

$$\sum_{n=r}^{\infty} |x_n(\lambda)|^2 a_{n,r}^2 = |\lambda\rho^{-\alpha}|^{2r} \sum_{m=0}^{\infty} \frac{|\lambda\rho^{-\alpha}|^{2m}}{\Gamma(\alpha m+1)}.$$

The last series is

$$E_\alpha(|\lambda\rho^{-\alpha}|^2),$$

and it is finite because E_α is entire. Therefore

$$\sum_{n=0}^{\infty} |x_n(\lambda)|^2 a_{n,r}^2 < \infty$$

for all $r = 0, \ldots, k$. By (6.1),

$$\sum_{n=0}^{\infty} |x_n(\lambda)|^2 \widetilde{a}_{n,k}^2 < \infty.$$

Since k is arbitrary,

$$x(\lambda) \in \lambda^2(\widetilde{A}).$$

Thus

$$G_\rho(\lambda) = J_\rho x(\lambda) \in MLC_\rho^2(\Omega;\alpha).$$

$\square$

We next check the eigenvalue equation.

Proposition 6.2. *For every $\lambda \in \mathbb{C}$,*

$$\mathfrak{D}^{\alpha,\rho} G_\rho(\lambda) = \lambda G_\rho(\lambda).$$

Proof. Using the action of $\mathfrak{D}^{\alpha,\rho}$ on the basis, the term $n = 0$ vanishes and

$$\mathfrak{D}^{\alpha,\rho} G_\rho(\lambda) = \sum_{n=1}^{\infty} \frac{(\lambda\rho^{-\alpha})^n}{\Gamma(\alpha n+1)} \rho^\alpha \frac{\Gamma(\alpha n+1)}{\Gamma(\alpha(n-1)+1)} \Theta_{n-1}.$$

After cancellation of $\Gamma(\alpha n+1)$, this becomes

$$\mathfrak{D}^{\alpha,\rho} G_\rho(\lambda) = \sum_{n=1}^{\infty} (\lambda\rho^{-\alpha})^n \rho^\alpha \frac{\Theta_{n-1}}{\Gamma(\alpha(n-1)+1)}.$$

We obtain

$$\mathfrak{D}^{\alpha,\rho} G_\rho(\lambda) = \lambda \sum_{n=1}^{\infty} \frac{(\lambda\rho^{-\alpha})^{n-1}}{\Gamma(\alpha(n-1)+1)} \Theta_{n-1}.$$

With $m = n - 1$,

$$\mathfrak{D}^{\alpha,\rho} G_\rho(\lambda) = \lambda \sum_{m=0}^{\infty} \frac{(\lambda\rho^{-\alpha})^m}{\Gamma(\alpha m + 1)} \Theta_m = \lambda G_\rho(\lambda).$$

□

We now prove that the eigenvectors are total.

Proposition 6.3. *The eigenvector field G_ρ is total:*

$$\overline{\operatorname{span}\{G_\rho(\lambda) : \lambda \in \mathbb{C}\}} = MLC^2_\rho(\Omega;\alpha).$$

Proof. Under the isomorphism

$$J_\rho : \lambda^2(\widetilde{A}) \to MLC^2_\rho(\Omega;\alpha),$$

the vector $G_\rho(\lambda)$ corresponds to

$$x(\lambda) = \left(\frac{(\lambda\rho^{-\alpha})^n}{\Gamma(\alpha n + 1)} \right)_{n \geq 0}.$$

It is therefore enough to prove that

$$\overline{\operatorname{span}\{x(\lambda) : \lambda \in \mathbb{C}\}} = \lambda^2(\widetilde{A}).$$

Let

$$Y \in (\lambda^2(\widetilde{A}))'$$

be such that

$$Y(x(\lambda)) = 0, \qquad \lambda \in \mathbb{C}.$$

By the dual representation of Köthe spaces, there exists a scalar sequence

$$y = (y_n)_{n \geq 0}$$

such that

$$Y(b) = \sum_{n=0}^{\infty} b_n y_n, \qquad b \in \lambda^2(\widetilde{A}).$$

Thus

$$0 = Y(x(\lambda)) = \sum_{n=0}^{\infty} y_n \frac{(\lambda\rho^{-\alpha})^n}{\Gamma(\alpha n + 1)}$$

for every $\lambda \in \mathbb{C}$. Equivalently,

$$\sum_{n=0}^{\infty} y_n \frac{\rho^{-\alpha n}}{\Gamma(\alpha n + 1)} \lambda^n = 0, \qquad \lambda \in \mathbb{C}.$$

The left-hand side is an entire function. Since it vanishes identically, all its Taylor coefficients are zero:

$$y_n \frac{\rho^{-\alpha n}}{\Gamma(\alpha n + 1)} = 0, \qquad n \geq 0.$$

Because

$$\rho^{-\alpha n} \neq 0 \qquad \text{and} \qquad \Gamma(\alpha n + 1) \neq 0,$$

we get

$$y_n = 0, \qquad n \geq 0.$$

Hence

$$Y = 0.$$

By the Hahn–Banach theorem, this span is dense in $\lambda^2(\widetilde{A})$. Applying J_ρ, we obtain

$$\overline{\operatorname{span}\{G_\rho(\lambda) : \lambda \in \mathbb{C}\}} = MLC_\rho^2(\Omega; \alpha).$$

□

It remains to prove the required holomorphic dependence.

Proposition 6.4. *The map*

$$G_\rho : \mathbb{C} \to MLC_\rho^2(\Omega; \alpha)$$

is holomorphic. In particular, it is weakly holomorphic.

Proof. We prove holomorphy in the Fréchet topology. It is enough to show local uniform convergence of the power series in each defining norm. Fix $k \geq 0$ and $R > 0$. For $|\lambda| \leq R$, consider the partial sums

$$S_N(\lambda) = \sum_{n=0}^{N} \frac{(\lambda\rho^{-\alpha})^n}{\Gamma(\alpha n + 1)} e_n$$

in $\lambda^2(\widetilde{A})$, where e_n denotes the n-th coordinate vector. We show that S_N converges uniformly on $|\lambda| \leq R$ with respect to $\| \cdot \|_k$. For $M > N$, using (6.1),

$$\|S_M(\lambda) - S_N(\lambda)\|_k^2 \leq \sum_{r=0}^{k} \sum_{n=N+1}^{M} \frac{|\lambda\rho^{-\alpha}|^{2n}}{\Gamma(\alpha n + 1)^2} a_{n,r}^2.$$

It suffices to estimate each fixed r. The finitely many terms $n < r$ do not matter for the tail. For $n \geq r$,

$$\frac{|\lambda\rho^{-\alpha}|^{2n}}{\Gamma(\alpha n + 1)^2} a_{n,r}^2 = \frac{|\lambda\rho^{-\alpha}|^{2n}}{\Gamma(\alpha(n-r) + 1)} \leq \frac{(R\rho^{-\alpha})^{2n}}{\Gamma(\alpha(n-r) + 1)}.$$

Therefore, with $m = n - r$,

$$\sum_{n \geq \max\{r, N+1\}} \frac{(R\rho^{-\alpha})^{2n}}{\Gamma(\alpha(n-r) + 1)} \leq (R\rho^{-\alpha})^{2r} \sum_{m \geq N+1-r} \frac{(R\rho^{-\alpha})^{2m}}{\Gamma(\alpha m + 1)}.$$

The Mittag-Leffler series

$$\sum_{m=0}^{\infty} \frac{(R\rho^{-\alpha})^{2m}}{\Gamma(\alpha m + 1)}$$

converges. Hence the tails tend to zero uniformly for $|\lambda| \leq R$. Thus

$$\lambda \mapsto x(\lambda)$$

is holomorphic from $\mathbb{C}$ into $\lambda^2(\widetilde{A})$. Since J_ρ is a continuous linear isomorphism,

$$G_\rho(\lambda) = J_\rho x(\lambda)$$

is holomorphic from $\mathbb{C}$ into $MLC_\rho^2(\Omega; \alpha)$. Strong holomorphy implies weak holomorphy. Hence G_ρ is weakly holomorphic. □

7. Devaney chaos

We now derive the main dynamical consequence of the eigenvector field constructed above. The proof is an application of the eigenvector field criterion recalled in Theorem 2.10.

Theorem 7.1. *Let*

$$0 < \alpha < 1, \qquad \rho > 0.$$

Then the operator

$$\mathfrak{D}^{\alpha,\rho} : MLC^2_\rho(\Omega;\alpha) \longrightarrow MLC^2_\rho(\Omega;\alpha)$$

is chaotic in the sense of Devaney.

Proof. We verify the assumptions of Theorem 2.10. First, by Proposition 4.1,

$$MLC^2_\rho(\Omega;\alpha)$$

is a complex Fréchet space. By Proposition 5.1,

$$\mathfrak{D}^{\alpha,\rho}$$

is a continuous linear operator on this space.

Second, the field

$$G_\rho : \mathbb{C} \to MLC^2_\rho(\Omega;\alpha)$$

is well defined. For every $\lambda \in \mathbb{C}$, Proposition 6.2 gives

$$\mathfrak{D}^{\alpha,\rho} G_\rho(\lambda) = \lambda G_\rho(\lambda).$$

Thus $G_\rho(\lambda)$ is an eigenvector of $\mathfrak{D}^{\alpha,\rho}$ associated with the eigenvalue λ. Third, Proposition 6.4 shows that

$$G_\rho$$

is holomorphic as a map into the Fréchet space. In particular, it is weakly holomorphic. Fourth, Proposition 6.3 gives the totality condition

$$\overline{\operatorname{span}\{G_\rho(\lambda) : \lambda \in \mathbb{C}\}} = MLC^2_\rho(\Omega;\alpha).$$

We choose

$$U = \mathbb{C}.$$

Then U is non-empty, open and connected. Moreover,

$$U \cap \mathbb{T} = \mathbb{T} \neq \varnothing, \qquad \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

Hence the eigenvector field contains eigenvectors corresponding to unimodular eigenvalues.

All hypotheses of Theorem 2.10 are satisfied. Consequently,

$$\mathfrak{D}^{\alpha,\rho}$$

is chaotic in the sense of Devaney on

$$MLC^2_\rho(\Omega;\alpha).$$

$\square$

The theorem shows that the proportional Caputo-type derivative has genuinely chaotic dynamics on the weighted Mittag-Leffler–Caputo space. The role of the parameter ρ is absorbed into the weighted basis and into the rescaling of the eigenvalues, while the dense holomorphic eigenvector field provides the dynamical mechanism responsible for chaos.

8. Numerical illustration

To complement the infinite-dimensional proof of Devaney chaos, we include a finite-dimensional numerical illustration of a proportional L1 approximation. The purpose of the experiment is not to replace the analytical proof, but to visualize the influence of the proportional parameter ρ and of the discretization step τ.

Let B_N denote the backward shift on $\mathbb{C}^{N+1}$, truncated at level N. On the uniform grid $t_n = n\tau$, we use the finite proportional L1 operator

$$\mathcal{D}^{\alpha,\rho}_{\tau,N} = \frac{\rho^\alpha \tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{N} b_j q_\rho^j B_N^j, \qquad q_\rho = \exp\left(-\frac{\rho-1}{\rho}\tau\right), \tag{8.1}$$

where

$$b_0 = 1, \qquad b_j = (j+1)^{1-\alpha} - 2j^{1-\alpha} + (j-1)^{1-\alpha}, \quad j \geq 1.$$

The factor q_ρ^j is the discrete counterpart of the exponential conjugacy. When $\rho = 1$, one has $q_\rho = 1$ and (8.1) reduces to the usual L1 finite-section form used for the Caputo case. For all computations we take

$$\alpha = 0.7, \qquad N = 80, \qquad m_{\max} = 15,$$

and the normalized initial vector

$$x_0 = \frac{u}{\|u\|_2}, \qquad u_n = \frac{(-1)^n}{n+1}, \quad n = 0, \dots, N.$$

To visualize the response to a small perturbation, we set

$$y_0 = x_0 + 10^{-8} e_N.$$

Figure 1 displays

$$\log_{10} \left\| (\mathcal{D}^{\alpha,\rho}_{\tau,N})^m y_0 - (\mathcal{D}^{\alpha,\rho}_{\tau,N})^m x_0 \right\|_2$$

for $\rho = 0.7, 1, 1.3$ with $\tau = 0.1$. The three curves show that the magnitude of perturbation growth depends noticeably on ρ.

Figure 2 compares the orbit norms for the same three values of ρ. This gives a direct numerical view of the factor ρ^α and of the exponential weighting introduced by the proportional derivative.

The influence of the discretization step is shown in Figure 3, where $\rho = 1.3$ is fixed and τ varies. Finally, Figure 4 measures the difference between the proportional and classical Caputo finite-section dynamics at a fixed iterate $m_* = 10$,

$$E(\rho) = \left\| (\mathcal{D}^{\alpha,\rho}_{\tau,N})^{m_*} x_0 - (\mathcal{D}^{\alpha,1}_{\tau,N})^{m_*} x_0 \right\|_2 .$$

The minimum at $\rho = 1$ numerically confirms the expected recovery of the Caputo case.

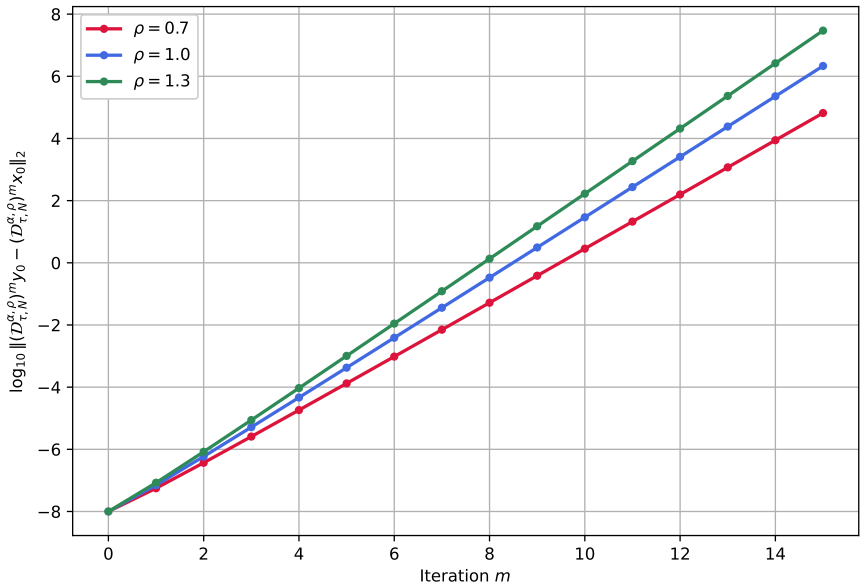

Figure 1. Separation of two nearby initial states for the proportional L1 finite-section approximation with $\alpha = 0.7$, $N = 80$, and $\tau = 0.1$.

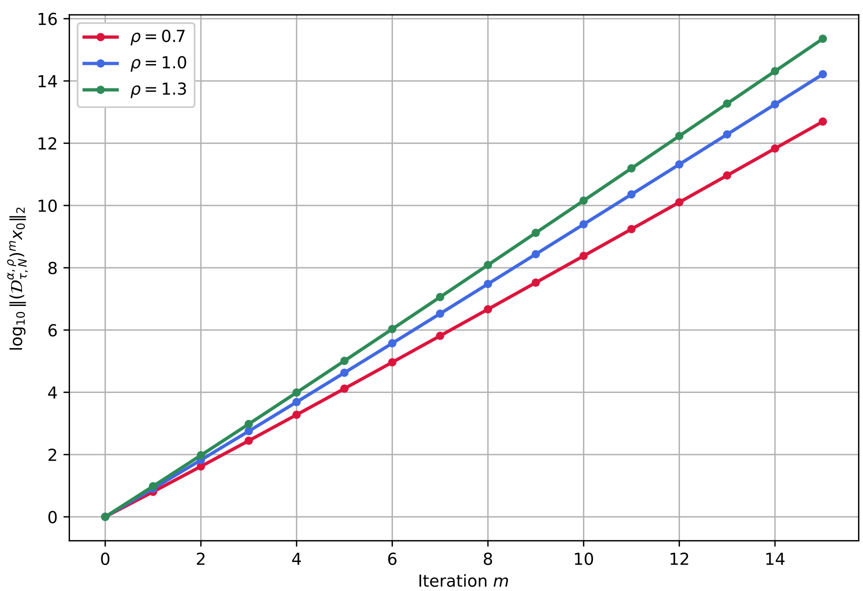

Figure 2. Finite-section orbit norms for $\rho = 0.7$, 1, and 1.3.

Because all four computations use finite sections, these plots should be interpreted only as illustrations of parameter-dependent transient dynamics. The Devaney-chaos statement itself remains the infinite-dimensional analytical result proved in Section 7.

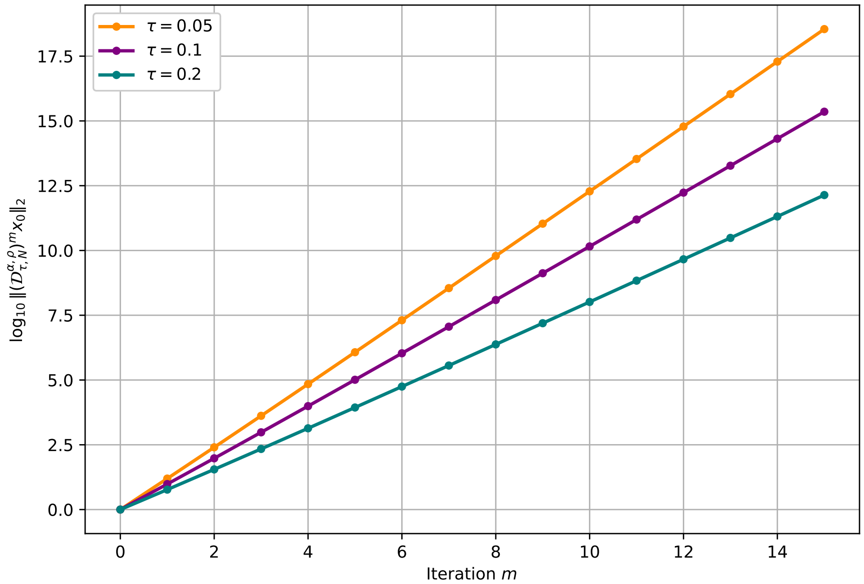

FIGURE 3. Effect of the step size τ on the finite-section orbit norms for fixed $\rho = 1.3$.

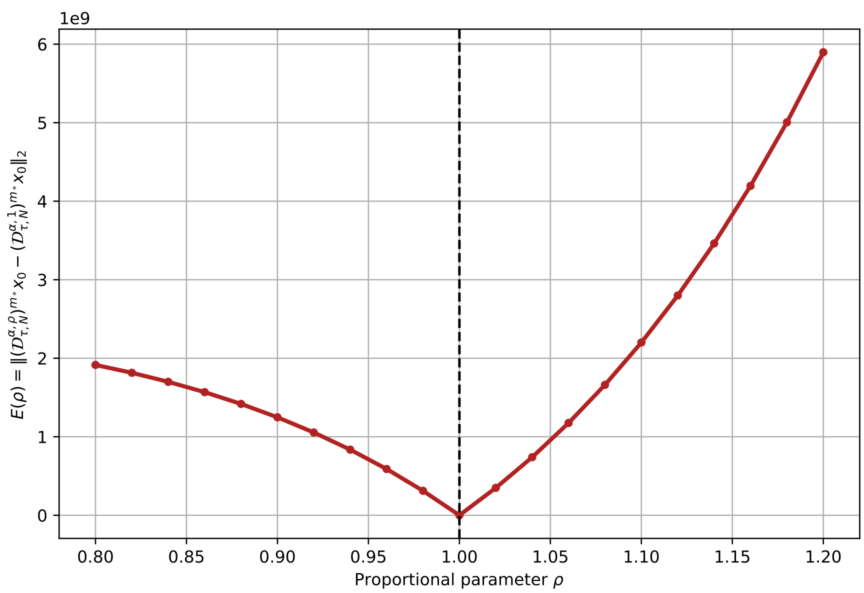

FIGURE 4. Difference from the Caputo finite-section dynamics as a function of ρ at $m_* = 10$.

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Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, PO Box 532, Beni Mellal 23000, Morocco

Email address: El-mahdi Nafia: nafia.el-mahdi@usms.ac.ma

Email address: M'hamed El Omari: m.elomari@usms.ma

Email address: Hasnaa Alatoune: rihablotfi2017@gmail.com