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FUZZY VECTOR SPACES UNDER NORMS

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ABSTRACT. In this paper, by using t-norm T, we study purely algebraic properties of fuzzy vector spaces. Next we introduce sum, intersection, direct product and normality of them. Later we prove that sum, intersection, direct product and normality of them is also fuzzy vector space under t-norm T. Finally, we investigate them under linear transformations and we obtian that image and pre image of them is also fuzzy vector space under t-norms.

1. Introduction

Vector spaces stem from affine geometry via the introduction of coordinates in the plane or three-dimensional space. Around 1636, Descartes and Fermat founded analytic geometry by equating solutions to an equation of two variables with points on a plane curve. In 1804, to achieve geometric solutions without using coordinates, Bolzano introduced certain operations on points, lines and planes, which are predecessors of vectors. His work was then used in the conception of barycentric coordinates by Möbius in 1827. In 1828 Mourey suggested the existence of an algebra surpassing not only ordinary algebra but also two-dimensional algebra created by him searching a geometrical interpretation of complex numbers. The definition of vectors was founded on Bellavitis' notion of the bipoint, an oriented segment of which one end is the origin and the other a target, then further elaborated with the presentation of complex numbers by Argand and Hamilton and the introduction of quaternions and biquaternions by the latter. They are elements in $\mathbb{R}^2, \mathbb{R}^4$, and $\mathbb{R}^8$; their treatment as linear combinations can be traced back to Laguerre in 1867, who also defined systems of linear equations. In 1857, Cayley introduced matrix notation, which allows for a harmonization and simplification of linear maps. Around the same time, Grassmann studied the barycentric calculus initiated by Möbius. He envisaged sets of abstract objects endowed with operations. There is a growing interest in extensions of fuzzy set theory which can be model not only vague information, but also uncertainty. Fuzzy algebraic structures play an important role in mathematics

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with wide applications in many other branches such as theoretical physics, computer science, information sciences, coding theory, topological spaces, logic, set theory, group theory, real analysis, measure theory etc. The concept of a fuzzy subset of a nonempty set was introduced by Zadeh in 1965 [21] as a function from a nonempty set X into the unit real interval $I = [0, 1]$. Rosenfeld [19] applied this to the theory of groupoids and groups and then many researchers developed it in all the fields of algebra. The concepts of a fuzzy field and a fuzzy linear space over a fuzzy field were introduced and discussed by Nanda [12]. In 1977, Katsaras and Liu [7] formulated and studied the notion of fuzzy vector subspaces over the field of real or complex numbers. After Katsaras and Liu [7], many scholars investigated its properties and characteristics [1, 8, 9, 10]. The triangular norm, T-norm, originated from the studies of probabilistic metric spaces in which triangular inequalities were extended using the theory of T-norm. Later, Hohle [6], Alsina et al. [3] introduced the T-norm into fuzzy set theory and suggested that the T-norm be used for the intersection of fuzzy sets. Since then, many other researchers have presented various types of T-norms for particular purposes [5, 20]. The author by using norms, investigated some properties of fuzzy algebraic structures [14]- [18]. This work is an attempt to study purely algebraic properties of fuzzy vector spaces under t-norms. In the second section (preliminaries) we give all the definitions with some examples required to develop the paper. In Section 3, we have introduced and discussed about fuzzy vector spaces under t-norms. In this section, we show relainship between fuzzy vector spaces under t-norms and vector subspaces. Next we investigate conditions about fuzzy vector spaces under t-norms. In Section 4, we introduce sum, intersection, direct product of fuzzy vector spaces under t-norms and we prove that sum, intersection, direct product of them is also fuzzy vector spaces under t-norm. Also we define normal fuzzy vector spaces under t-norm and obtain some results about them. In Section 5, we investigate linear transformations over fuzzy vector spaces under t-norms. We show that image and pre image of them is also fuzzy vector space under t-norms. Also we prove that image and pre image of normal fuzzy vector spaces under t-norms is also normal fuzzy vector space under t-norms.

2. PRELIMINARIES

In this section, we review some elementary aspects that are necessary for this paper.

Definition 2.1. (See [4]) A vector space or a linear space consists of the following:
(1) a field $\mathbb{F}$ of scalars.
(2) a set V of objects called vectors.
(3) a rule (or operation) called vector addition; which associates with each pair of vectors $\alpha, \beta \in V; \alpha + \beta \in V$, called the sum of α and β in such a way that
(a) addition is commutative $\alpha + \beta = \beta + \alpha$.
(b) addition is associative $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
(c) there is a unique vector 0 in V, called the zero vector, such that $\alpha + 0 = \alpha$ for all $\alpha \in V$.
(d) for each vector α in V there is a unique vector$(-\alpha)$ in V such that $\alpha + (-\alpha) =$

0.
(e) a rule (or operation), called scalar multiplication, which associates with each scalar c in $\mathbb{F}$ and a vector α in V, a vector $c \bullet \alpha$ in V, called the product of c and α, in such a way that $1 \bullet \alpha = \alpha$, $(c_1 \bullet c_2) \bullet \alpha = c_1 \bullet (c_2 \bullet \alpha)$, $c \bullet (\alpha + \beta) = c \bullet \alpha + c \bullet \beta$, $(c_1 + c_2) \bullet \alpha = (c_1 \bullet \alpha) + (c_2 \bullet \alpha)$ for $\alpha, \beta \in V$ and $c, c_1, c_2 \in F$. It is important to note as the definition states that a vector space is a composite object consisting of a field, a set of 'vectors' and two operations with certain special properties. The same set of vectors may be part of a number of distinct vectors. We simply by default of notation just say V a vector space over the field $\mathbb{F}$ and call elements of V as vectors only as matter of convenience for the vectors in V.

Throughout this section, $\mathbb{F}$ is any field of characteristic zero.

Example 2.2. Let $V = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Then V is a vector space over $\mathbb{R}$ or $\mathbb{Q}$ but V is not a vector space over the complex field $\mathbb{C}$.

Definition 2.3. (See [4]) Let V be a vector space over the field $\mathbb{F}$. A subspace of V is a subset W of V which is itself a vector space over $\mathbb{F}$ with the operations of vector addition and scalar multiplication on V.

We have the following nice characterization theorem for subspaces.

Theorem 2.4. *(See [4]) Let W be a non-empty subset of a vector V over the field $\mathbb{F}$. Then W is a subspace of V if and only if for each pair $\alpha, \beta \in W$ and each scalar $c \in \mathbb{F}$ the vector $c\alpha + \beta \in W$.*

Example 2.5. Let $M_{n\times n} = \{(a_{ij}) \mid a_{ij} \in \mathbb{Q}\}$ be the vector space over $\mathbb{Q}$. Let $D_{n\times n} = \{(a_{ii}) \mid a_{ii} \in \mathbb{Q}\}$ be the set of all diagonal matrices with entries from $\mathbb{Q}$. Then $D_{n\times n}$ is a subspace of $M_{n\times n}$.

Definition 2.6. (See [4]) Let V and W be two vector spaces over the field of $\mathbb{F}$. A map $f : V \to W$ is called a linear transformation if $f(ax + y) = af(x) + f(y)$ for all $x, y \in V$ and $a \in \mathbb{F}$.

Definition 2.7. (See [11]) Let X a non-empty sets. A fuzzy subset μ of X is a function $\mu : X \to [0, 1]$. Denote by $[0, 1]^X$, the set of all fuzzy subset of X.

Definition 2.8. (See [11]) Let $f : V \to W$ be a linear transformation over the field $\mathbb{F}$. Let $\mu \in [0, 1]^V$ and $\nu \in [0, 1]^W$. Define $f(\mu) \in [0, 1]^W$ and $f^{-1}(\nu) \in [0, 1]^V$ as

$$f(\mu)(w) = \begin{cases} \sup\{\mu(v) \mid v \in V, f(v) = w\} & \text{if } f^{-1}(w) \neq \emptyset \\ 0 & \text{if } f^{-1}(w) = \emptyset \end{cases}$$

Also $f^{-1}(\nu)(v) = \nu(f(v))$.

Definition 2.9. (See [2]) A t-norm T is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:
(T1) $T(x, 1) = x$ (neutral element),
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),
(T3) $T(x, y) = T(y, x)$ (commutativity),

(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),
for all $x, y, z \in [0, 1]$.

Recall that T is idempotent if for all $x \in [0, 1]$, $T(x, x) = x$. Note that if $x \leq y$, and T be idempotent, then $T(x, y) = x$.

Example 2.10. (1) Standard intersection T-norm $T_m(x, y) = \min\{x, y\}$.
(2) Bounded sum T-norm $T_b(x, y) = \max\{0, x + y - 1\}$.
(3) algebraic product T-norm $T_p(x, y) = xy$.
(4) Drastic T-norm

$$T_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum T-norm

$$T_{nM}(x, y) = \begin{cases} \min\{x, y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x, y) \leq T(x, y) \leq T_{\min}(x, y)$ for all $x, y \in [0, 1]$.

Lemma 2.11. *(See [2]) Let T be a t-norm. Then*

$$T(T(x, y), T(w, z)) = T(T(x, w), T(y, z)),$$

for all $x, y, w, z \in [0, 1]$.

Definition 2.12. (See [13]) The intersection of fuzzy subsets μ_1 and μ_1 in a set X with respect to a t-norm T we mean the fuzzy subset $\mu = \mu_1 \cap \mu_2$ in the set X such that for any $x \in X$

$$\mu(x) = (\mu_1 \cap \mu_2)(x) = T(\mu_1(x), \mu_2(x)).$$

Definition 2.13. (See [14]) Let $\mu_1, \mu_2 \in FST(V)$. The sum of μ_1 and μ_2 is defined as follows:

$$(\mu_1 + \mu_2)(x) = \sup\{T(\mu_1(y), \mu_2(z)) \mid x = y + z \in V\}.$$

3. FUZZY VECTOR SPACES UNDER t-NORMS

In what follows, V is a vector space on a field $\mathbb{F}$, unless otherwise specified.

Definition 3.1. A fuzzy set $\mu : V \to [0, 1]$ is called a fuzzy vector space of V under t-norm T, if for all $x, y \in V$ and $a \in \mathbb{F}$ the following conditions hold:
(1) $\mu(x + y) \geq T(\mu(x), \mu(y))$,
(2) $\mu(-x) \geq \mu(x)$,
(3) $\mu(ax) \geq \mu(x)$.
Denote by $FST(V)$ the set of all fuzzy vector spaces of V under t-norm T.

Corollary 3.2. *Let $\mu \in FST(V)$. Then $\mu(-x) = \mu(x)$ for all $x \in V$.*

Proof. Let $x \in V$ then $\mu(-x) \geq \mu(x) = \mu(-(-x)) \geq \mu(-x)$ and so $\mu(-x) = \mu(x)$. □

Example 3.3. Let $V = \mathbb{R} \times \mathbb{R}$ be a vectorspace over a field $\mathbb{F} = \mathbb{R}$. Define $\mu : \mathbb{R} \times \mathbb{R} \to [0, 1]$ as

$$\mu(x, y) = \begin{cases} 0.95 & (x, y) \in \{(0, y) \mid y \in \mathbb{R}\} \\ 0.65 & \text{otherwise.} \end{cases}$$

Let $T(a, b) = T_p(a, b) = ab$ for all $a, b \in [0, 1]$. Then $\mu \in FST(V)$.

Theorem 3.4. *Let V be a subspace over field $\mathbb{F}$ and W be a subset of V and $\mu : W \to \{0, 1\}$ be the characteristic function. Then $\mu \in FST(V)$ if and only if W is a subspace of V.*

Proof. Let $\mu \in FST(V)$ and we prove that W is a subspace of V. Let $x, y \in W$ and $a \in \mathbb{F}$ then $\mu(x) = \mu(y) = 1$ and

$$\mu(ax + y) \geq T(\mu(ax), \mu(y)) \geq T(\mu(x), \mu(y)) = T(1, 1) = 1$$

so $\mu(ax + y) = 1$ and from Theorem 2.4 we get W is a subspace of V.
Conversely, let W is a subspace of V and we prove that $\mu \in FST(V)$. Let $x, y \in V$ and we investigate the following conditions:
(1) If $x, y \in W$, then $x + y \in W$ and we have $\mu(x + y) = 1 \geq 1 = T(1, 1) = T(\mu(x), \mu(y))$.
(2) If $x \notin W$ and $y \in W$, then $x + y \notin W$ and then $\mu(x + y) = 0 \geq 0 = T(0, 1) = T(\mu(x), \mu(y))$.
(3) Finally, if $x, y \notin W$, then $\mu(x + y) \geq 0 = T(0, 0) = T(\mu(x), \mu(y))$.
Thus from (1)-(3) we have that $\mu(x + y) \geq T(\mu(x), \mu(y))$.
Also we have that the following conditions:
(1) If $x \in W$, then $-x \in W$ and then $\mu(-x) = 1 \geq 1 = \mu(x)$.
(2) If $x \notin W$, then $-x \notin W$ and then $\mu(-x) = 0 \geq 0 = \mu(x)$.
Thus from (1)-(2) we have that $\mu(-x) \geq \mu(x)$.
Now let $a \in \mathbb{F}$:
(1) If $x \in W$, then $ax \in W$ and so $\mu(ax) = 1 \geq 1 = \mu(x)$.
(2) If $x \notin W$, then $ax \notin W$ so $\mu(ax) = 0 \geq 0 = \mu(x)$.
Then from (1) and (2) we obtain $\mu(ax) \geq \mu(x)$.
Therefore from the above conditions we get that $\mu \in FST(V)$. □

Theorem 3.5. *Let $\mu \in FST(V)$. Then $W = \{x \mid x \in V : \mu(x) = 1\}$ is either empty or is a subspace of V.*

Proof. Let $x, y \in W$ and $a \in \mathbb{F}$. Now

$$\mu(ax + y) \geq T(\mu(ax), \mu(y)) \geq T(\mu(x), \mu(y)) = T(1, 1) = 1$$

and then $\mu(ax + y) = 1$ which means that $ax + y \in W$ and by Theorem 2.4 we get W is a subspace of V. □

Theorem 3.6. *If $\mu \in FST(V)$ and T be an idempotent t-norm, then $\mu(0) \geq \mu(x)$ for all $x \in V$.*

Proof. As $\mu \in FST(V)$ so

$$\mu(0) = \mu(x - x) = \mu(x + (-x)) \geq T(\mu(x), \mu(-x)) \geq T(\mu(x), \mu(x)) = \mu(x).$$

Thus $\mu(0) \geq \mu(x)$. □

Theorem 3.7. *Let $\mu \in FST(V)$ and T be an idempotent t-norm. Then $W = \{x \mid x \in V : \mu(x) = \mu(0)\}$ is either empty or is a subspace of V.*

Proof. Let $x, y \in W$ and $a \in \mathbb{F}$. Now

$$\begin{aligned}
\mu(ax + y) &\geq T(\mu(ax), \mu(y)) \\
&\geq T(\mu(x), \mu(y)) \\
&= T(\mu(0), \mu(0)) \\
&= \mu(0) \\
&\geq \mu(ax + y) \text{ (by Theorem 3.6)}.
\end{aligned}$$

Thus $\mu(ax + y) = \mu(0)$ and $ax + y \in W$ and Theorem 2.4 gives us that W is a subspace of V. □

Theorem 3.8. *Let $\mu \in FST(V)$ and T be an idempotent t-norm. If $\mu(x - y) = \mu(0)$, then $\mu(x) = \mu(y)$ for all $x, y \in V$.*

Proof. Let $\mu \in FST(V)$ then

$$\begin{aligned}
\mu(x) &= \mu(x - y + y) \\
&\geq T(\mu(x - y), \mu(y)) \\
&\geq T(\mu(0), \mu(y)) \\
&\geq T(\mu(y), \mu(y)) \\
&= \mu(y) \\
&= \mu(y + x - x) \\
&= \mu(-(x - y) + x) \\
&\geq T(\mu(-(x - y)), \mu(x)) \\
&\geq T(\mu(x - y), \mu(x)) \\
&= T(\mu(0), \mu(x)) \\
&\geq T(\mu(x), \mu(x)) \\
&= \mu(x).
\end{aligned}$$

Therefore $\mu(x) = \mu(y)$. □

Theorem 3.9. *Let T be an idempotent t-norm. Then $\mu \in FST(V)$ If and only if $\mu(x - y) \geq T(\mu(x), \mu(y))$ and $\mu(ax) \geq \mu(x)$ for all $x, y \in V$ and $a \in \mathbb{F}$.*

Proof. Let $\mu \in FST(V)$. Then

$$\mu(x - y) = \mu(x + (-y)) \geq T(\mu(x), \mu(-y)) \geq T(\mu(x), \mu(y)).$$

Conversely, let $\mu(x-y) \geq T(\mu(x), \mu(x))$ and $\mu(ax) \geq \mu(x)$. Then $\mu(0) = \mu(x-x) \geq T(\mu(x), \mu(x)) = \mu(x)$. Now

$$\mu(-x) = \mu(0-x) \geq T(\mu(0), \mu(x)) \geq T(\mu(x), \mu(x)) = \mu(x).$$

Also

$$\mu(x+y) = \mu(x-(-y)) \geq T(\mu(x), \mu(-y)) \geq T(\mu(x), \mu(y)).$$

Then $\mu \in FST(V)$. □

Theorem 3.10. *Let $\mu : V \to [0,1]$ be a fuzzy set. If $\mu(0) = 1$ and $\mu(x-y) \geq T(\mu(x), \mu(y))$ and $\mu(ax) \geq \mu(x)$, then $\mu \in FST(V)$ for all $x, y \in V$ and $a \in \mathbb{F}$.*

Proof. We obtain that

$$\mu(-x) = \mu(0-x) \geq T(\mu(0), \mu(x)) = T(1, \mu(x)) = \mu(x)$$

and so $\mu(-x) \geq \mu(x)$. Also

$$\mu(x+y) = \mu(x-(-y)) \geq T(\mu(x), \mu(-y)) \geq T(\mu(x), \mu(y)).$$

Thus $\mu \in FST(V)$. □

Theorem 3.11. *Let $\mu \in FST(V)$ and $\mu(x-y) = 1$. Then $\mu(x) = \mu(y)$ for all $x, y \in V$.*

Proof. Let $x, y \in V$. Then

$$\begin{aligned}
\mu(x) &= \mu(x-y+y) \\
&\geq T(\mu(x-y), \mu(y)) \\
&= T(1, \mu(y)) \\
&= \mu(y) \\
&= \mu(-y) \text{ (by Corollary 3.2)} \\
&= \mu(x-x-y) \\
&= \mu(x-y-x) \\
&\geq T(\mu(x-y), \mu(-x)) \\
&= T(1, \mu(-x)) \\
&= \mu(-x) \\
&= \mu(x) \text{ (by Corollary 3.2)}.
\end{aligned}$$

Therefore $\mu(x) = \mu(y)$. □

Theorem 3.12. *Let $\mu \in FST(V)$ and $\mu(x-y) = 0$. Then either $\mu(x) = 0$ or $\mu(y) = 0$ for all $x, y \in V$.*

Proof. Let $x, y \in V$ then

$$0 = \mu(x-y) = \mu(x+(-y)) \geq T(\mu(x), \mu(-y)) \geq T(\mu(x), \mu(y))$$

which implies that either $\mu(x) = 0$ or $\mu(y) = 0$. □

Theorem 3.13. *Let $\mu \in FST(V)$ and $\mu(x) \neq \mu(y)$. Then $\mu(x+y) = T(\mu(x), \mu(y))$ for all $x, y \in V$.*

Proof. Let $\mu(x) > \mu(y)$ for all $x, y \in V$ which means that $\mu(x) > \mu(x+y)$. Then

$$\begin{aligned}
\mu(y) &= \mu(x - x + y) \\
&\geq T(\mu(x+y), \mu(-x)) \\
&\geq T(\mu(x+y), \mu(x)) \\
&\geq T(T(\mu(x), \mu(y)), \mu(x)) \\
&= T(T(\mu(x), \mu(x)), \mu(y)) \\
&= T(\mu(x), \mu(y)) \\
&\geq T(\mu(y), \mu(y)) \\
&\geq T(\mu(x-y), \mu(-x)) \\
&= \mu(y).
\end{aligned}$$

Then we obtain that $\mu(y) = T(\mu(x), \mu(y))$. Also

$$\begin{aligned}
\mu(y) &= T(\mu(x+y), \mu(x)) \\
&= T(\mu(x), \mu(x+y)) \\
&\geq T(\mu(x+y), \mu(x+y)) \\
&= \mu(x+y) \\
&\geq T(\mu(x), \mu(y)) \\
&\geq T(\mu(y), \mu(y)) \\
&= \mu(y)
\end{aligned}$$

which means that $\mu(y) = \mu(x+y)$.
Therefore $\mu(y) = \mu(x+y) = T(\mu(x), \mu(y))$. □

Theorem 3.14. *Let $\mu \in FST(V)$ and $\mu(x) < \mu(y)$ for some $x, y \in V$. If T be an idempotent t-norm, then $\mu(x+y) = \mu(x)$ for all $x, y \in V$.*

Proof. Let $\mu \in FST(V)$ and $\mu(x) < \mu(y)$ which means that $\mu(y) > \mu(x+y)$ for all $x, y \in V$. Then

$$\begin{aligned}
\mu(x+y) &\geq T(\mu(x), \mu(y)) \\
&\geq T(\mu(x), \mu(x)) \\
&= \mu(x) \\
&= \mu(x + y - y) \\
&\geq T(\mu(x+y), \mu(-y)) \\
&\geq T(\mu(x+y), \mu(y)) \\
&\geq T(\mu(x+y), \mu(x+y)) \\
&= \mu(x+y).
\end{aligned}$$

Thus $\mu(x+y) = \mu(x)$. □

Theorem 3.15. *Let $\mu \in FST(V)$ and $\mu(x) > \mu(y)$ for some $x, y \in V$. If T be an idempotent t-norm, then $\mu(x+y) = \mu(y)$ for all $x, y \in V$.*

Proof. It is trivial. □

Theorem 3.16. *Let $\mu \in FST(V)$ and T be an idempotent t-norm. Then $\mu(x-y)=\mu(y)$ if and only if $\mu(x)=\mu(0)$ for all $x,y \in V$.*

Proof. Let $\mu(x-y)=\mu(y)$ then from letting $y=0$ we have that $\mu(x)=\mu(0)$. Conversely, suppose that $\mu(x)=\mu(0)$ and from Theorem 3.6 we get that $\mu(x)=\mu(0)\geq \mu(x-y)$ and $\mu(x)=\mu(0)\geq\mu(-y)$. Then

$$\begin{aligned}
\mu(x-y) &= \mu(x+(-y)) \\
&\geq T(\mu(x),\mu(-y)) \\
&= T(\mu(0),\mu(-y)) \\
&\geq T(\mu(-y),\mu(-y)) \\
&= \mu(-y) \\
&= \mu(x-y-x) \\
&= \mu(x-y+(-x)) \\
&\geq T(\mu(x-y),\mu(-x)) \\
&\geq T(\mu(x-y),\mu(x)) \\
&= T(\mu(x-y),\mu(0)) \\
&\geq T(\mu(x-y),\mu(x-y)) \\
&= \mu(x-y)
\end{aligned}$$

and thus $\mu(x-y)=\mu(-y)=\mu(y)$. □

Theorem 3.17. *Let $\mu \in FST(V)$ and $f:[0,\mu(0)]\to[0,1]$ be an increasing map. Define a fuzzy set $\mu^f : V \to [0,1]$ by $\mu^f(x)=f(\mu(x))$. Then $\mu^f \in FST(V)$.*

Proof. Let $x,y\in V$ and $a\in\mathbb{F}$. Then
(1)

$$\mu^f(x+y)=f(\mu(x+y))\geq f(T(\mu(x),\mu(y)))=T(f(\mu(x)),f(\mu(y)))=T(\mu^f(x),\mu^f(y)).$$

(2)

$$\mu^f(-x)=f(\mu(-x))\geq f(\mu(x))=\mu^f(x).$$

(3)

$$\mu^f(ax)=f(\mu(ax))\geq f(\mu(x))=\mu^f(x).$$

Therefore $\mu^f\in FST(V)$. □

4. SUM, INTERSECTION, DIRECT PRODUCT AND NOMALITY OF FUZZY VECTOR SPACES UNDER t-NORMS

Proposition 4.1. *Let $\mu_1,\mu_2\in FST(V)$ and T be idempotent. Then $(\mu_1+\mu_2)\in FST(V)$.*

Proof. (1) Let $x_1,x_2,y_1,y_2,z_1,z_2\in V$. Then

$$(\mu_1+\mu_2)(x_1+x_2)=\sup\{T(\mu_1(y_1+y_2),\mu_2(z_1+z_2)) \mid x_1+x_2=y_1+y_2+z_1+z_2\}$$

$$\geq \sup\{T(T(\mu_1(y_1),\mu_1(y_2)),T(\mu_2(z_1),\mu_2(z_2))) \mid x_1+x_2=y_1+z_1+(y_2+z_2)\}$$
$$(\text{ from Lemma 2.11 })$$
$$= \sup\{T(T(\mu_1(y_1),\mu_2(z_1)),T(\mu_1(y_2),\mu_2(z_2))) \mid x_1+x_2=y_1+z_1+(y_2+z_2)\}$$
$$= T(\sup\{T(\mu_1(y_1),\mu_2(z_1)) \mid x_1=y_1+z_1\},\sup\{T(\mu_1(y_2),\mu_2(z_2)) \mid x_2=y_2+z_2\})$$
$$= T((\mu_1+\mu_2)(x_1),(\mu_1+\mu_2)(x_2)).$$

(2) Let $x,y,z \in V$ then

$$\begin{aligned}(\mu_1+\mu_2)(-x) &= \sup\{T(\mu_1(-y),\mu_2(-z)) \mid -x=-y+(-z)\}\\ &\geq \sup\{T(\mu_1(y),\mu_2(z)) \mid x=y+z\}\\ &= (\mu_1+\mu_2)(x).\end{aligned}$$

(3) Let $x,y,z \in V$ and $a \in \mathbb{F}$. Then

$$\begin{aligned}(\mu_1+\mu_2)(ax) &= \sup\{T(\mu_1(ay),\mu_2(az)) \mid ax=ay+az\}\\ &\geq \sup\{T(\mu_1(y),\mu_2(z)) \mid x=y+z\}\\ &= (\mu_1+\mu_2)(x).\end{aligned}$$

Thus $(\mu_1+\mu_2) \in FST(V)$. □

Theorem 4.2. *If $\mu_1,\mu_2 \in FST(V)$, then $\mu_1 \cap \mu_2 \in FST(V)$.*

Proof. Let $x,y \in V$ and $a \in \mathbb{F}$. Then
(1)

$$\begin{aligned}(\mu_1\cap\mu_2)(x+y) &= T(\mu_1(x+y),\mu_2)(x+y)\\ &\geq T(T(\mu_1(x),\mu_1(y)),T(\mu_2(x),\mu_2(y)))\\ &= T(T(\mu_1(x),\mu_2(x)),T(\mu_1(y),\mu_2(y)))\ \textit{(by Lemma 2.11)}\\ &= T((\mu_1\cap\mu_2)(x),(\mu_1\cap\mu_2)(y))\end{aligned}$$

thus

$$(\mu_1\cap\mu_2)(x+y) \geq T((\mu_1\cap\mu_2)(x),(\mu_1\cap\mu_2)(y)).$$

(2)

$$(\mu_1\cap\mu_2)(-x) = T(\mu_1(-x),\mu_2(-x)) \geq T(\mu_1(x),\mu_2(x)) = (\mu_1\cap\mu_2)(x).$$

(3)

$$(\mu_1\cap\mu_2)(ax) = T(\mu_1(ax),\mu_2(ax)) \geq T(\mu_1(x),\mu_2(x)) = (\mu_1\cap\mu_2)(x).$$

Thus $\mu_1 \cap \mu_2 \in FST(V)$. □

Definition 4.3. Let $\{\mu_i\}_{i\in I}$ be a family fuzzy subspaces of V under t-norm T. Then their intersection $\mu = \cap_{i\in I}\mu_i : V \to [0,1]$ defined by $\mu(x) = \inf_{i\in I}\mu_i(x)$ for all $x \in V$.

Theorem 4.4. *Let $\{\mu_i\}_{i\in I} \subseteq FST(V)$. Then $\mu = \cap_{i\in I}\mu_i \in FST(V)$.*

Proof. Let $x, y \in V$ and $a \in \mathbb{F}$. Then
(1)

$$\begin{aligned}\mu(x+y) &= \inf_{i\in I} \mu_i(x+y)\\ &\geq \inf_{i\in I} T(\mu_i(x), \mu_i(y))\\ &= T(\inf_{i\in I} \mu_i(x), \inf_{i\in I} \mu_i(y))\\ &= T(\mu(x), \mu(y))\end{aligned}$$

so

$$\mu(x+y) \geq T(\mu(x), \mu(y)).$$

(2)

$$\mu(-x) = \inf_{i\in I} \mu_i(-x) \geq \inf_{i\in I} \mu_i(x) = \mu(x).$$

(3)

$$\mu(ax) = \inf_{i\in I} \mu_i(ax) \geq \inf_{i\in I} \mu_i(x) = \mu(x).$$

Therefore $\mu = \cap_{i\in I}\mu_i \in FST(V)$. □

Definition 4.5. For any two $\mu_1 \in FST(V_1)$ and $\mu_2 \in FST(V_2)$. The direct product of μ_1 and μ_2, denoted by $\mu_1 \times \mu_2 : V_1 \times V_2 \to [0, 1]$, is defined as $(\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2))$ for all $x_1 \in V_1$ and $x_2 \in V_2$.

Theorem 4.6. *Let $\mu_1 \in FST(V_1)$ and $\mu_2 \in FST(V_2)$. Then $\mu_1 \times \mu_2 \in FST(V_1 \times V_2)$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in V_1 \times V_2$ and $a \in \mathbb{F}$. Then
(1)

$$\begin{aligned}(\mu_1 \times \mu_2)((x_1, y_1) + (x_2, y_2)) &= (\mu_1 \times \mu_2)(x_1 + x_2, y_1 + y_2)\\ &= T(\mu_1(x_1 + x_2), \mu_2(y_1 + y_2))\\ &\geq T(T(\mu_1(x_1), \mu_1(x_2)), T(\mu_2(y_1), \mu_2(y_2)))\\ &= T(T(\mu_1(x_1), \mu_2(y_1)), T(\mu_1(x_2), \mu_2(y_2))) \textit{ (by Lemma 2.11)}\\ &= T((\mu_1 \times \mu_2)(x_1, y_1), (\mu_1 \times \mu_2)(x_2, y_2))\end{aligned}$$

therefore

$$(\mu_1 \times \mu_2)((x_1, y_1) + (x_2, y_2)) \geq T((\mu_1 \times \mu_2)(x_1, y_1), (\mu_1 \times \mu_2)(x_2, y_2)).$$

(2)

$$\begin{aligned}(\mu_1 \times \mu_2)(-(x_1, y_1)) &= (\mu_1 \times \mu_2)(-x_1, -y_1)\\ &= T(\mu_1(-x_1), \mu_2(-y_1))\\ &\geq T(\mu_1(x_1), \mu_2(y_1))\\ &= (\mu_1 \times \mu_2)(x_1, y_1)\end{aligned}$$

so

$$(\mu_1 \times \mu_2)(-(x_1, y_1)) \geq (\mu_1 \times \mu_2)(x_1, y_1).$$

(3)

$$\begin{aligned}(\mu_1 \times \mu_2)(a(x_1, y_1)) &= (\mu_1 \times \mu_2)(ax_1, ay_1)\\ &= T(\mu_1(ax_1), \mu_2(ay_1))\\ &\geq T(\mu_1(x_1), \mu_2(y_1))\\ &= (\mu_1 \times \mu_2)(x_1, y_1)\end{aligned}$$

then

$$(\mu_1 \times \mu_2)(a(x_1, y_1)) \geq (\mu_1 \times \mu_2)(x_1, y_1).$$

Then $\mu_1 \times \mu_2 \in FST(V_1 \times V_2)$. □

Theorem 4.7. *Let T be an idempotent t-norm and suppose that 0_{V_1} and 0_{V_2} are the identity elements of V_1 and V_2, respectively. If $\mu_1 \times \mu_2 \in FST(V_1 \times V_2)$, then*

$$(\mu_1 \times \mu_2)(0_{V_1}, 0_{V_2}) \geq (\mu_1 \times \mu_2)(x, y)$$

for all $(x, y) \in V_1 \times V_2$.

Proof. Let $(x, y) \in V_1 \times V_2$. Then

$$\begin{aligned}(\mu_1 \times \mu_2)(0_{V_1}, 0_{V_2}) &= T(\mu_1(0_{V_1}), \mu_2(0_{V_2}))\\ &\geq T(\mu_1(x), \mu_2(y)) \textit{ (by Theorem 3.6)}\\ &= (\mu_1 \times \mu_2)(x, y).\end{aligned}$$

□

Theorem 4.8. *Let $\mu_1 : V_1 \to [0, 1]$ and $\mu_2 : V_2 \to [0, 1]$ be two fuzzy subsets of the vector spaces V_1 and V_2, respectively. Suppose that 0_{V_1} and 0_{V_2} be the identity elements of V_1 and V_2, respectively. Let T be an idempotent t-norm and $\mu_1 \times \mu_2 \in FST(V_1 \times V_2)$ then at least one of the following two statements must hold.*
(1) $\mu_2(0_{V_2}) \geq \mu_1(x)$ for all $x \in V_1$.
(2) $\mu_1(0_{V_1}) \geq \mu_2(y)$ for all $y \in V_2$.

Proof. Let $\mu_1 \times \mu_2 \in FST(V_1 \times V_2)$. By contra positive, suppose that none of the statements (1) and (2) holds. Then we can find $z \in V_1$ and $t \in V_2$ such that $\mu_1(z) > \mu_2(0_{V_2})$ and $\mu_2(t) > \mu_1(0_{V_1})$. Then

$$\begin{aligned}(\mu_1 \times \mu_2)(z, t) &= T(\mu_1(z), \mu_2(t))\\ &> T(\mu_2(0_{V_2}), \mu_1(0_{V_1}))\\ &= T(\mu_1(0_{V_1}), \mu_2(0_{V_2}))\\ &= (\mu_1 \times \mu_2)(0_{V_1}, 0_{V_2}).\end{aligned}$$

Now from Theorem 4.7 we get that $\mu_1 \times \mu_2 \notin FST(V_1 \times V_2)$ and this is a contradiction and then either $\mu_2(0_{V_2}) \geq \mu_1(x)$ for all $x \in V_1$ or $\mu_1(0_{V_1}) \geq \mu_2(y)$ for all $y \in V_2$. □

Theorem 4.9. *Let $\mu_1 : V_1 \to [0, 1]$ and $\mu_2 : V_2 \to [0, 1]$ be two fuzzy subsets of the vector spaces V_1 and V_2, respectively. Suppose that 0_{V_1} and 0_{V_2} be the identity elements of V_1 and V_2, respectively. Let T be an idempotent t-norm and $\mu_1 \times \mu_2 \in FST(V_1 \times V_2)$ then the following statements are hold.*

(1) If $\mu_1(x) \leq \mu_2(0_{V_2})$ for all $x \in V_1$, then $\mu_1 \in FST(V_1)$.
(2) If $\mu_2(x) \leq \mu_1(0_{V_1})$ for all $x \in V_2$, then $\mu_2 \in FST(V_2)$.
(3) Either $\mu_1 \in FST(V_1)$ or $\mu_2 \in FST(V_2)$.

Proof. (1) Let $x, y \in V_1$ and $a \in \mathbb{F}$ and $\mu_1 \times \mu_2 \in FST(V_1 \times V_2)$. We prove that $\mu_1 \in FST(V_1)$. As $\mu_1(x) \leq \mu_2(0_{V_2})$ so $\mu_1(x+y) \leq \mu_2(0_{V_2})$ for all $x, y \in V_1$. Now

$$\begin{aligned}
\mu_1(x+y) &= T(\mu_1(x+y), \mu_2(0_{V_2})) \\
&= T(\mu_1(x+y), \mu_2(0_{V_2} + 0_{V_2})) \\
&= (\mu_1 \times \mu_2)(x+y, 0_{V_2} + 0_{V_2}) \\
&= (\mu_1 \times \mu_2)((x, 0_{V_2}) + (y, 0_{V_2})) \\
&\geq T((\mu_1 \times \mu_2)(x, 0_{V_2}), (\mu_1 \times \mu_2)(y, 0_{V_2})) \\
&= T(T(\mu_1(x), \mu_2(0_{V_2})), (T(\mu_1(y), \mu_2(0_{V_2}))) \\
&= T(\mu_1(x), \mu_1(y))
\end{aligned}$$

thus $\mu_1(x+y) \geq T(\mu_1(x), \mu_1(y))$.
Also since $\mu_1(x) \leq \mu_2(0_{V_2})$ so $\mu_1(-x) \leq \mu_2(0_{V_2})$ for all $x \in V_1$ and then

$$\begin{aligned}
\mu_1(-x) &= T(\mu_1(-x), \mu_2(0_{V_2})) \\
&= (\mu_1 \times \mu_2)(-x, 0_{V_2}) \\
&= (\mu_1 \times \mu_2)(-(x, 0_{V_2})) \\
&\geq (\mu_1 \times \mu_2)(x, 0_{V_2}) \\
&= T(\mu_1(x), \mu_2(0_{V_2})) \\
&= \mu_1(x)
\end{aligned}$$

and then $\mu_1(-x) \geq \mu_1(x)$.
Also by using $\mu_1(x) \leq \mu_2(0_{V_2})$ we get that $\mu_1(ax) \leq \mu_2(0_{V_2})$ for all $x \in V_1$ and $a \in \mathbb{F}$. Now

$$\begin{aligned}
\mu_1(ax) &= T(\mu_1(ax), \mu_2(0_{V_2})) \\
&= (\mu_1 \times \mu_2)(ax, 0_{V_2}) \\
&= (\mu_1 \times \mu_2)(a(x, 0_{V_2})) \\
&\geq (\mu_1 \times \mu_2)(x, 0_{V_2}) \\
&= T(\mu_1(x), \mu_2(0_{V_2})) \\
&= \mu_1(x)
\end{aligned}$$

and thus $\mu_1(ax) \geq \mu_1(x)$.
Therefore $\mu_1 \in FST(V_1)$.
(2) The proof of (2) is similar to (1).
(3) It is clear. □

Definition 4.10. Let V be a subspace over field $\mathbb{F}$ and W be a subspace of V and $\mu_W : W \to [0, 1]$ be a fuzzy subspace. Define $\mu_{\frac{V}{W}} : \frac{V}{W} \to [0, 1]$ as

$$\mu_{\frac{V}{W}}(x + W) = \begin{cases} T(\mu_W(x), \mu_W(w)) & \text{if } x \neq w \\ 1 & \text{if } x = w \end{cases}$$

for all $x \in V$ and $w \in W$.

Theorem 4.11. *In Definition 4.10 if T be idempotent t-norm and $\mu_W \in FST(W)$, then $\mu_{\frac{V}{W}} \in FST(\frac{V}{W})$.*

Proof. Let $x+W, y+W \in \frac{V}{W}$ such that $x, y \neq w$ and $a \in \mathbb{F}$. Then
(1)

$$\begin{aligned}
\mu_{\frac{V}{W}}((x+W)+(y+W)) &= \mu_{\frac{V}{W}}(x+y+W)\\
&= T(\mu_W(x+y), \mu_W(w))\\
&\geq T(T(\mu_W(x), \mu_W(y)), \mu_W(w))\\
&= T(T(\mu_W(x), \mu_W(y)), T(\mu_W(w), \mu_W(w))) \text{ (as } T \text{ idempotent)}\\
&= T(T(\mu_W(x), \mu_W(w)), T(\mu_W(y), \mu_W(w))) \text{ (by Lemma 2.11)}\\
&= T(\mu_{\frac{V}{W}}(x+W), \mu_{\frac{V}{W}}(y+W))
\end{aligned}$$

thus

$$\mu_{\frac{V}{W}}((x+W)+(y+W)) \geq T(\mu_{\frac{V}{W}}(x+W), \mu_{\frac{V}{W}}(y+W)).$$

(2)

$$\mu_{\frac{V}{W}}(-x+W) = T(\mu_W(-x), \mu_W(w)) \geq T(\mu_W(x), \mu_W(w)) = \mu_{\frac{V}{W}}(x+W).$$

(3)

$$\mu_{\frac{V}{W}}(ax+W) = T(\mu_W(ax), \mu_W(w)) \geq T(\mu_W(x), \mu_W(w)) = \mu_{\frac{V}{W}}(x+W).$$

Then $\mu_{\frac{V}{W}} \in FST(\frac{V}{W})$. □

Definition 4.12. Let $\mu \in FST(V)$. We say that μ is normal if there exists $x \in V$ such that $\mu(x) = 1$. Note that if μ normal, then $\mu(0) = 1$. Hence μ is a normal if and only if $\mu(0) = 1$.
Denote by $NFST(V)$ the set of all normal fuzzy subspaces of V under t-norm T.

Theorem 4.13. *Let $\mu \in FST(V)$ and $\acute{\mu} : V \to [0,1]$ be a fuzzy set and defined by $\acute{\mu}(x) = \mu(x) + 1 - \mu(0)$ for all $x \in V$. Then $\acute{\mu} \in NFST(V)$.*

Proof. Let $x, y \in V$ and $a \in \mathbb{F}$. Then
(1)

$$\begin{aligned}
\acute{\mu}(x+y) &= \mu(x+y) + 1 - \mu(0)\\
&\geq T(\mu(x), \mu(y)) + 1 - \mu(0)\\
&= T(\mu(x) + 1 - \mu(0), \mu(y) + 1 - \mu(0))\\
&= T(\acute{\mu}(x), \acute{\mu}(y)).
\end{aligned}$$

(2)

$$\acute{\mu}(-x) = \mu(-x) + 1 - \mu(0) \geq \mu(x) + 1 - \mu(0) = \acute{\mu}(x).$$

(3)

$$\acute{\mu}(ax) = \mu(ax) + 1 - \mu(0) \geq \mu(x) + 1 - \mu(0) = \acute{\mu}(x).$$

Thus $\acute{\mu} \in FST(V)$.
Also

$$\acute{\mu}(0) = \mu(0) + 1 - \mu(0) = 1.$$

Therefore $\acute{\mu} \in NFST(V)$. ☐

Theorem 4.14. *Let $\mu \in FST(V)$. Then $\mu \in NFST(V)$ if and only if $\acute{\mu} = \mu$.*

Proof. Let $\mu \in NFST(V)$ then $\mu(0) = 1$ and then $\acute{\mu}(x) = \mu(x) + 1 - \mu(0) = \mu(x) + 1 - 1 = \mu(x)$ for all $x \in V$.
Conversely, let $\acute{\mu} = \mu$ and from Theorem 4.13 we have that $\acute{\mu} \in NFST(V)$ and so $\mu \in NFST(V)$. ☐

Theorem 4.15. *Let $\mu \in FST(V)$ and $\bar{\mu} : V \to [0,1]$ be a fuzzy set defined by $\bar{\mu}(x) = \dfrac{\mu(x)}{\mu(0)}$ for all $x \in V$ with $\mu(0) \neq 0$. Then $\bar{\mu} \in NFST(V)$.*

Proof. Let $x, y \in V$ and $a \in \mathbb{F}$. Then
(1)

$$\begin{aligned} \bar{\mu}(x+y) &= \frac{\mu(x+y)}{\mu(0)} \\ &\geq \frac{T(\mu(x), \mu(y))}{\mu(0)} \\ &= T(\frac{\mu(x)}{\mu(0)}, \frac{\mu(y)}{\mu(0)}) \\ &= T(\bar{\mu}(x), \bar{\mu}(y)). \end{aligned}$$

(2)

$$\bar{\mu}(-x) = \frac{\mu(-x)}{\mu(0)} \geq \frac{\mu(x)}{\mu(0)} = \bar{\mu}(x).$$

(3)

$$\bar{\mu}(ax) = \frac{\mu(ax)}{\mu(0)} \geq \frac{\mu(x)}{\mu(0)} = \bar{\mu}(x).$$

Thus $\bar{\mu} \in FST(V)$. Also $\bar{\mu}(0) = \dfrac{\mu(0)}{\mu(0)} = 1$ and then $\bar{\mu} \in NFST(V)$. ☐

5. LINEAR TRANSFORMATIONS OVER FUZZY VECTOR SPACES UNDER t-NORMS

Theorem 5.1. *Let f be an epimorphism linear transformation from vector space V into vector space W over field $\mathbb{F}$. If $\mu \in FST(V)$, then $f(\mu) \in FST(W)$.*

Proof. (1) Let $w_1, w_2 \in W$. Then

$$\begin{aligned} f(\mu)(w_1 + w_2) &= \sup\{\mu(v_1 + v_2) \mid v_1, v_2 \in V, f(v_1) = w_1, f(v_2) = w_2\} \\ &\geq \sup\{T(\mu(v_1), \mu(v_2)) \mid v_1, v_2 \in V, f(v_1) = w_1, f(v_2) = w_2\} \\ &= T(\sup\{\mu(v_1) \mid f(v_1) = w_1\}, \sup\{\mu(v_2) \mid f(v_2) = w_2\}) \\ &= T(f(\mu)(w_1), f(\mu)(w_2)) \end{aligned}$$

thus

$$f(\mu)(w_1 + w_2) \geq T(f(\mu)(w_1), f(\mu)(w_2)).$$

(2)

$$\begin{aligned} f(\mu)(-w) &= \sup\{\mu(-v)) \mid v \in V, f(-v) = -w\} \\ &= \sup\{\mu(-v)) \mid v \in V, -f(v) = -w\} \\ &\geq \sup\{\mu(v) \mid v \in V, f(v) = w\} \\ &= f(\mu)(w) \end{aligned}$$

then $f(\mu)(-w) \geq f(\mu)(w)$.

(3) Let $v \in V$ and $a \in \mathbb{F}$.

$$\begin{aligned} f(\mu)(aw) &= \sup\{\mu(av)) \mid av \in V, f(av) = aw\} \\ &\geq \sup\{\mu(v)) \mid v \in V, af(v) = aw\} \\ &= \sup\{\mu(v)) \mid v \in V, f(v) = w\} \\ &= f(\mu)(w) \end{aligned}$$

so $f(\mu)(aw) \geq f(\mu)(w)$.
Therefore $f(\mu) \in FST(W)$. □

Theorem 5.2. *Let f be an epimorphism linear transformation from vector space V into vector space W over field $\mathbb{F}$. If $\mu \in NFST(V)$, then $f(\mu) \in NFST(W)$.*

Proof. As Theorem 5.1, we have that $f(\mu) \in FST(W)$. Suppose that 0_V and 0_W be the identity elements of V and W, respectively. Since $\mu \in NFST(V)$ so $\mu(0_V) = 1$ and then

$$f(\mu)(0_W) = \sup\{\mu(0_V) \mid 0_V \in V, f(0_V) = 0_W\} = \sup\{1 \mid 0_V \in V, f(0_V) = 0_W\} = 1$$

and then $f(\mu) \in NFST(W)$. □

Theorem 5.3. *Let f be a linear transformation from vector space V into vector space W over field $\mathbb{F}$. If $\nu \in FST(W)$, then $f^{-1}(\nu) \in FST(V)$.*

Proof. (1) Let $v_1, v_2 \in V$. Then

$$\begin{aligned} f^{-1}(\nu)(v_1 + v_2) &= \nu(f(v_1 + v_2)) \\ &= \nu(f(v_1) + f(v_2)) \\ &\geq T(\nu(f(v_1), \nu(f(v_2)) \\ &= T(f^{-1}(\nu)(v_1), f^{-1}(\nu)(v_2)). \end{aligned}$$

(2) Let $v \in V$. Then

$$f^{-1}(\nu)(-v) = \nu(f(-v)) = \nu(-f(v)) \geq \nu(f(v)) = f^{-1}(\nu)(v).$$

(3) Let $v \in V$ and $a \in \mathbb{F}$. Then

$$f^{-1}(\nu)(av) = \nu(f(av)) = \nu(af(v)) \geq \nu(f(v)) = f^{-1}(\nu)(v).$$

Thus $f^{-1}(\nu) \in FST(V)$. □

Theorem 5.4. *Let f be a linear transformation from vector space V into vector space W over field $\mathbb{F}$. If $\nu \in NFST(W)$, then $f^{-1}(\nu) \in NFST(V)$.*

Proof. Theorem 5.3 gives us that $f^{-1}(\nu) \in FST(V)$. Suppose that 0_V and 0_W be the identity elements of V and W, respectively. As $\nu \in NFST(W)$ so $\nu(0_W) = 1$ and then $f^{-1}(\nu)(0_V) = \nu(f(0_V)) = \nu(0_W) = 1$. Thus $f^{-1}(\nu) \in NFST(V)$. □

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