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DUALITY AND REPRODUCING KERNELS FOR DIRICHLET-TYPE SPACES

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ABSTRACT. Using an isometric differentiation isomorphism between the weighted Dirichlet-type spaces and the weighted Bergman space, we determine the duals of the weighted Dirichlet-type spaces of the unit disk and the upper half-plane. Further, we compute the reproducing kernels for the Hilbert version of these spaces.

1. INTRODUCTION

The theory of analytic function spaces plays a central role in complex analysis and operator theory. Among these, classical spaces such as Hardy and Bergman spaces have been extensively studied, with weighted Bergman spaces providing a well developed analysis including reproducing kernels, and duality relations [4, 8, 13, 14]. Dirichlet-type spaces arise naturally as intermediate spaces between analytic function spaces, defined through integrability on derivatives. Over the years, Dirichlet-type spaces have been investigated from several perspectives, including norm characterizations, Carleson measure conditions, and operator boundedness [1, 3, 9]. Recent developments have further enriched the theory of Dirichlet-type spaces by providing new characterizations but in the classical setting. Seco [10] introduced and characterized classical Dirichlet inner functions via norm equivalence conditions, while Shangjin and Liu [11] investigated closure properties of classical Dirichlet-type spaces in the Bloch space. In a different direction, Chalmoukis and Hartz [5] established connections between capacity and functional analytic notions of smallness in Dirichlet-type settings. For a comprehensive treatment of the classical Dirichlet space, we refer to [7].
Despite these advances, much of the existing literature treats Dirichlet-type spaces either independently or through indirect comparisons with other function spaces.

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The classical embedding between the Dirichlet and Bergman spaces is well established in [6, 14], but proceeds through norm estimates, yielding a bounded operator with norm comparability constants rather than an isometric correspondence of the two spaces. In particular, the classical embedding does not identify the two spaces surjectively or isometrically, and consequently does not directly yield a transfer of duality or reproducing kernels. An approach that relates Dirichlet-type spaces as isomorphic images of weighted Bergman spaces, while simultaneously characterizing them therefore remains relatively underdeveloped. The aim of this paper is to address this gap by placing the differentiation operator at the center of the analysis. We show that the differentiation operator is not only bounded but also is a surjective isometry, hence an isometric isomorphism between the weighted Bergman spaces and weighted Dirichlet-type spaces. This allows us to transfer dual spaces through explicit pairings, and in the Hilbert space case, recover the reproducing kernel of Dirichlet-type spaces from the known Bergman kernel through the adjoint operator.

2. Preliminaries

2.1. **Function Spaces.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk in the complex plane $\mathbb{C}$ and $\mathbb{U} = \{\omega \in \mathbb{C} : \Im(\omega) > 0\}$ the upper half plane of $\mathbb{C}$ where $\operatorname{Im}\omega$ is the imaginary part of the complex number ω. Let dm denote the (normalized) Lebesgue area measure on $\mathbb{D}$ and $d\mu$ the Lebesgue measure on $\mathbb{U}$. The corresponding weighted measures on $\mathbb{D}$ and $\mathbb{U}$ are respectively given by $dm_\alpha(z) := (1-|z|^2)^\alpha dm(z)$ and $d\mu_\alpha(\omega) := (\Im(\omega))^\alpha d\mu(\omega)$, for $\alpha > -1$. If Ω denotes either $\mathbb{D}$ or $\mathbb{U}$, then $\mathcal{H}(\Omega)$ denotes the Fréchet space of holomorphic functions $f : \Omega \to \mathbb{C}$ equipped with the topology of uniform convergence on compact subsets of Ω. The weighted Bergman spaces are defined by $L^p_a(m_\alpha) = L^p(m_\alpha) \cap \mathcal{H}(\mathbb{D})$ and $L^p_a(\mu_\alpha) = L^p(\mu_\alpha) \cap \mathcal{H}(\mathbb{U})$, where $L^p_a(\cdot)$ denotes the classical Lebesgue space. Also, the weighted Dirichlet-type spaces $\mathcal{D}^p_\alpha(\mathbb{D})$ and $\mathcal{D}^p_\alpha(\mathbb{U})$ are respectively defined by

$$\mathcal{D}^p_\alpha(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^p dm_\alpha(z) < \infty \right\}, \tag{2.1}$$

equipped with the norm

$$\|f\|_{\mathcal{D}^p_\alpha(\mathbb{D})} = \left(|f(0)| + \|f'\|^p_{L^p_a(m_\alpha)} \right)^{\frac{1}{p}}, \quad \text{and} \tag{2.2}$$

$$\mathcal{D}^p_\alpha(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \int_{\mathbb{U}} |f'(\omega)|^p d\mu_\alpha(\omega) < \infty \right\}, \tag{2.3}$$

equipped with the norm

$$\|f\|_{\mathcal{D}^p_\alpha(\mathbb{U})} = \left(|f(i)|^p + \|f'\|^p_{L^p_a(\mu_\alpha)} \right)^{\frac{1}{p}}. \tag{2.4}$$

The spaces $L^p_a(\cdot)$ and $\mathcal{D}^p_\alpha(\cdot)$ are Banach spaces with respect to the respective norms.

We further define the subspaces $\mathcal{D}^p_{\alpha,\circ}(\mathbb{D})$ and $\mathcal{D}^p_{\alpha,\circ}(\mathbb{U})$ by

$$\mathcal{D}^p_{\alpha,\circ}(\mathbb{D}) = \{ f \in \mathcal{D}^p_\alpha(\mathbb{D}) : f(0) = 0 \},$$

and

$$\mathcal{D}^p_{\alpha,\circ}(\mathbb{U}) = \{f \in \mathcal{D}^p_\alpha(\mathbb{U}) : f(i) = 0\}.$$

It can be easily proved that $\mathcal{D}^p_{\alpha,0}(\cdot)$ are closed subspaces of $\mathcal{D}^p_\alpha(\cdot)$ and are therefore Banach spaces with respect to the respective norms.

2.2. **Reproducing kernels.** We define reproducing kernel for a Hilbert space $\mathbf{H}$ of functions, on an open set $\Omega \subset \mathbb{C}$ as a function $K : \Omega \times \Omega \to \mathbb{C}$, satisfying the following properties.

(1) $K(x, y) = \overline{K(y, x)}$
(2) $\|K(\cdot, x)\|^2_{\mathbf{H}} = K(x, x)$
(3) $f(x) = \langle f, K(\cdot, x)\rangle, \forall f \in \mathbf{H}$

The reproducing kernel for the weighted Bergman space had been computed in [4] as follows: For $L^2_a(m_\alpha)$, the Bergman space of the unit disc,

$$K_{L^2_a(m_\alpha)}(z, \omega) = \frac{1}{(1 - z\bar{\omega})^{2+\alpha}},$$

and for $L^2_a(\mu_\alpha)$, the Bergman space of the upper half plane,

$$K_{L^2_a(\mu_\alpha)}(z, \omega) = \frac{2^\alpha}{[-i(z - \bar{\omega})]^{2+\alpha}}, \alpha > -1,$$

We refer to [4] for details. In the next section, we shall determine explicitly the dual spaces of the weighted Dirichlet-type spaces $\mathcal{D}^p_{\alpha,0}(\cdot)$.

2.3. **Special Functions.** Special functions play important role in the study of weighted function spaces. In particular, the Gamma and Beta functions appear in the evaluation of weighted integrals while hypergeometric functions frequently occur in reproducing kernels. We recall a few definitions: The Gamma function is defined by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \ \Re(s) > 0.$$

Generally, for every positive integer n,

$$\Gamma(n) = (n - 1)!.$$

Moreover, $\Gamma(s + 1) = s\Gamma(s)$, $\Re(s) > 0$. On the other hand, the Beta function is defined by

$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt, \ \Re(x), \ \Re(y) > 0.$$

The Gamma and Beta functions are connected via the following known identity

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}_0$, the Pochhammer symbol is defined by

$$(a)_0 = 1, \ (a)_n = a \cdot (a + 1) \cdot (a + 2) ... \cdot (a + n - 1), \ n \geq 1.$$

The Gaussian hypergeometric function is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n, n!} z^n, \ |z| < 1,$$

where $c \notin \{0, -1, -2, \ldots\}$. We refer to [2] for a comprehensive theory of special functions.

3. Duality of Weighted Dirichlet-type Spaces

The differentiation operator provides a key link between weighted Dirichlet-type spaces and weighted Bergman spaces. We first establish that the differentiation operator defines an isometric isomorphism between $\mathcal{D}^p_{\alpha,\circ}(\cdot)$ and $L^p_a(\cdot)$ and then use this to characterize dual spaces and derive reproducing kernels.

Theorem 1. *Let $1 \leq p < \infty$ and $\alpha > -1$. Then the differentiation operator $D : \mathcal{D}^p_{\alpha,\circ}(\cdot) \to L^p_a(\cdot)$ defined by $Df = f'$ for each $f \in \mathcal{D}^p_{\alpha,\circ}(\cdot)$ is an isometric isomorphism between $\mathcal{D}^p_{\alpha,\circ}(\cdot)$ and $L^p_a(\cdot)$.*
In particular, $\mathcal{D}^p_{\alpha,\circ}(\cdot) \approx L^p_a(\cdot)$.

Proof. For each $f \in \mathcal{D}^p_{\alpha,\circ}(\cdot)$, we have:

$$\|Df\|^p_{L^p_a(m_\alpha)} = \int_{\mathbb{D}} |f'(z)|^p dm_\alpha(z) = \|f\|^p_{\mathcal{D}^p_{\alpha,\circ}(\mathbb{D})}, \quad \text{and}$$

$$\|Df\|^p_{L^p_a(\mu_\alpha)} = \int_{\mathbb{U}} |f'(\omega)|^p d\mu_\alpha(\omega) = \|f\|^p_{\mathcal{D}^p_{\alpha,\circ}(\mathbb{U})},$$

implying that D is an isometry and hence injective. To show surjectivity, let $g \in L^p_a(\cdot)$ and define $f(z) = \int_0^z g(\xi)d\xi$ on $\mathbb{D}$ and $f(\omega) = \int_i^\omega g(\xi)d\xi$ on $\mathbb{U}$. In each case, since $g \in L^p_a$, then g is holomorphic hence the integral defines a holomorphic function with derivative $f' = g$ by the Fundamental theorem of Calculus. Moreover, $f(0) = 0$ on $\mathbb{D}$ and $f(i) = 0$ on $\mathbb{U}$, so $f \in \mathcal{D}^p_{\alpha,\circ}(\cdot)$ and $Df = g$. Thus D is surjective.
Finally, for $g \in L^p_a(\cdot)$, we have,

$$\|D^{-1}g\|^p_{\mathcal{D}^p_{\alpha,\circ}(\mathbb{D})} = \int_{\mathbb{D}} |D(D^{-1}g(z))|^p \, dm_\alpha(z) = \int_{\mathbb{D}} |g(z)^p| \, dm_\alpha(z) = \|g\|^p_{L^p_a(m_\alpha)}.$$

Similarly, $\|D^{-1}g\|_{\mathcal{D}^p_{\alpha,\circ}(\mathbb{U})} = \|g\|^p_{L^p_a(\mu_\alpha)}$, and so D^{-1} is also an isometry. Therefore D is a bijective linear isometry with bounded inverse, and hence an isometric isomorphism. □

In the next theorem, we use the isomorphism D established in Theorem 1 to identify the duals of the weighted Dirichlet-type spaces $\mathcal{D}^p_{\alpha,\circ}(\cdot)$.

Theorem 2. *Let $1 < p, q < \infty$ be conjugate exponents in the sense that $\frac{1}{p} + \frac{1}{q} = 1$. Also, let $(\mathcal{D}^p_{\alpha,\circ}(\cdot))^*$ denote the dual space of $\mathcal{D}^p_{\alpha,\circ}(\cdot)$. Then*

$$(\mathcal{D}^p_{\alpha,\circ}(\cdot))^* \approx \mathcal{D}^q_{\alpha,\circ}(\cdot)$$

under the integral pairings

$$\langle f, h \rangle_{\mathcal{D}^p_{\alpha,\circ}(\mathbb{U})} = \int_{\mathbb{U}} f'(\omega)\overline{h'(\omega)} d\mu_\alpha(\omega), \quad f \in \mathcal{D}^p_{\alpha,\circ}(\mathbb{U}),\ h \in \mathcal{D}^q_{\alpha,\circ}(\mathbb{U}) \tag{3.1}$$

and

$$\langle f, h \rangle_{\mathcal{D}^p_{\alpha,\circ}(\mathbb{D})} = \int_{\mathbb{D}} f'(z)\overline{h'(z)} dm_\alpha(z), \quad f \in \mathcal{D}^p_{\alpha,\circ}(\mathbb{D}),\ h \in \mathcal{D}^q_{\alpha,\circ}(\mathbb{D}). \tag{3.2}$$

Proof. Recall the following duality relations of weighted Bergman spaces (we refer to [4, 14] for details):

$$(L^p_a(m_\alpha)^* \approx L^q_a(m_\alpha) \quad \text{and} \quad (L^p_a(\mu_\alpha)^* \approx L^q_a(\mu_\alpha)$$

via the standard $L^p - L^q$ pairings

$$\langle g, k\rangle = \int_{\mathbb{U}} g(\omega)\overline{k(\omega)}d\mu_\alpha(\omega), \quad (g \in L^p_a(\mu_\alpha),\, k \in L^q_a(\mu_\alpha)), \text{ and}$$

$$\langle g, k\rangle = \int_{\mathbb{D}} g(z)\overline{k(z)}dm_\alpha(z), \quad (g \in L^p_a(m_\alpha),\, k \in L^q_a(m_\alpha)).$$

Together with Theorem 1, we have

$$(\mathcal{D}^p_{\alpha,\circ}(\mathbb{U}))^* \approx (L^p_a(\mu_\alpha))^* \approx L^q_a(\mu_\alpha) \approx \mathcal{D}^q_{\alpha,\circ}(\mathbb{U}),$$

and

$$(\mathcal{D}^p_{\alpha,\circ}(\mathbb{D}))^* \approx (L^p_a(m_\alpha))^* \approx L^q_a(m_\alpha) \approx \mathcal{D}^q_{\alpha,\circ}(\mathbb{D}).$$

Moreover, the duality pairings transported through the differentiation isomorphism from the weighted Bergman space pairings above are given by

$$\langle f, h\rangle = \int_{\mathbb{U}} f'(\omega)\overline{h'(\omega)}d\mu_\alpha(\omega), \quad f \in \mathcal{D}^p_{\alpha,\circ}(\mathbb{U}),\ h \in \mathcal{D}^q_{\alpha,\circ}(\mathbb{U})$$

and

$$\langle f, h\rangle = \int_{\mathbb{D}} f'(z)\overline{h'(z)}dm_\alpha(z), \quad f \in \mathcal{D}^p_{\alpha,\circ}(\mathbb{D}),\ h \in \mathcal{D}^q_{\alpha,\circ}(\mathbb{D}),$$

as desired. □

4. Reproducing Kernels for Weighted Dirichlet-type spaces

In this section, we compute explicitly reproducing kernels for the weighted Dirichlet-type Hilbert spaces. This is a generalization and extension of the reproducing kernels for the classical Dirichlet spaces that currently exist in the literature, see [3, 12]. Again, we apply the established relation between the weighted Dirichlet-type spaces and the weighted Bergman spaces via the isometric differentiation isomorphism, as well as the existing reproducing kernels for the weighted Bergman spaces. First we establish the unitary nature of D for the Hilbert space setting, $p = 2$, in the following Lemma.

Lemma 3. *Let $D : \mathcal{D}^2_{\alpha,\circ}(\cdot) \to L^2_a(\cdot)$ be the differentiation operator. Then D is unitary and its Hilbert space adjoint satisfies*

$$D^* = D^{-1}.$$

Proof. For $f, g \in \mathcal{D}^2_{\alpha,\circ}(\mathbb{D})$,

$$\langle f, g\rangle_{\mathcal{D}^2_{\alpha,\circ}(\mathbb{D})} = \int_{\mathbb{D}} f'(z)\overline{g'(z)}dm_\alpha(z) = \langle Df, Dg\rangle_{L^2_a(m_\alpha)}.$$

Hence D preserves inner products and is therefore an isometry. By Theorem 1, D is a surjective isometry hence unitary.

Now let $f \in \mathcal{D}^2_{\alpha,\circ}(\mathbb{D})$ and $h \in L^2_a(m_\alpha)$. Since D is surjective, there exists a unique $g \in \mathcal{D}^2_{\alpha,\circ}(\mathbb{D})$ such that $h = Dg$. Then

$$\langle Df, h\rangle_{L^2_a(m_\alpha)} = \langle Df, Dg\rangle_{L^2_a(m_\alpha)} = \langle f, g\rangle_{\mathcal{D}^2_{\alpha,\circ}(\mathbb{D})} = \langle f, D^{-1}h\rangle_{\mathcal{D}^2_{\alpha,\circ}(\mathbb{D})}.$$

Hence

$$\langle Df, h\rangle_{L^2_a(m_\alpha)} = \langle f, D^{-1}h\rangle_{\mathcal{D}^2_{\alpha,\circ}(\mathbb{D})},$$

which by the definition of the Hilbert space adjoint, implies that

$$D^* = D^{-1}.$$

The same argument holds on $\mathbb{U}$ with m_α replaced by μ_α. Hence $D^* = D^{-1}$ in both cases as claimed. $\square$

Theorem 4. *Let $K_{\mathcal{D}}$ and K_{L_a} denote the reproducing kernels for $\mathcal{D}^2_{\alpha,\circ}(\cdot)$ and $L^2_a(\cdot)$ respectively and let $D : \mathcal{D}^2_{\alpha,\circ}(\cdot) \to L^2_a(\cdot)$, be the isometric differentiation operator with Hilbert space adjoint D^*.*

(1) *The reproducing kernel for $\mathcal{D}^2_{\alpha,\circ}(\mathbb{D})$ is given as*

$$K_{\mathcal{D}}(z,\omega) = \frac{1}{(1+\alpha)}\int_0^{\bar{\omega}} \frac{(1-z\eta)^{-(1+\alpha)}-1}{\eta}d\eta = z\bar{\omega}\,{}_3F_2\left(\begin{matrix}1,1,\alpha+2\\2,2\end{matrix};z\bar{\omega}\right),$$

or, equivalently,

$$K_{\mathcal{D}}(z,\omega) = \sum_{n=1}^{\infty}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+2)\Gamma(n+1)^2}(z\bar{\omega})^n.$$

(2) *The reproducing kernel for $\mathcal{D}^2_{\alpha,\circ}(\mathbb{U})$ is given as*

$$K_{\mathcal{D}}(z,\omega) = \frac{2^\alpha}{\alpha(1+\alpha)(-i)^{2+\alpha}}\left[(i-\bar{\omega})^{-\alpha} - (z-\bar{\omega})^{-\alpha} - (2i)^{-\alpha} + (z+i)^{-\alpha}\right], \alpha \neq 0.$$

Proof. To prove (1), let $f \in \mathcal{D}^2_{\alpha,\circ}(\mathbb{D})$ and $\omega \in \mathbb{D}$. By the reproducing kernel property,

$$f(\omega) = \langle f, K_{\mathcal{D}}(\cdot,\omega)\rangle_{\mathcal{D}^2_{\alpha,\circ}(\mathbb{D})}. \tag{4.1}$$

Differentiating (4.1) with respect to ω gives

$$f'(\omega) = \left\langle f, \frac{\partial}{\partial\bar{\omega}}K_{\mathcal{D}}(\cdot,\omega)\right\rangle_{\mathcal{D}^2_{\alpha,\circ}(\mathbb{D})}. \tag{4.2}$$

On the other hand, since $Df = f' \in L^2_a$, and $L^2_a(m_\alpha)$ is a reproducing kernel Hilbert space with kernel $K_{L^2_a}(\cdot,\omega)$, we have

$$f'(\omega) = \left\langle Df, K_{L^2_a}(\cdot,\omega)\right\rangle_{L^2_a(m_\alpha)}.$$

Using the adjoint operator, D^*,

$$f'(\omega) = \left\langle f, D^*K_{L^2_a}(\cdot,\omega)\right\rangle_{\mathcal{D}^2_{\alpha,\circ}(\mathbb{D})}. \tag{4.3}$$

Comparing the two representations of $f'(\omega)$ in (4.2) and (4.3), we obtain

$$\frac{\partial}{\partial\bar{\omega}}K_{\mathcal{D}}(z,\omega) = \left(D^*K_{L^2_a}(\cdot,\omega)\right)(z). \tag{4.4}$$

By Lemma 3, $D^* = D^{-1}$ and therefore

$$(D^*h)(z) = \int_0^z h(\xi)d\xi, \tag{4.5}$$

Integrating (4.4) with respect to $\bar{\omega}$ and using normalization $K_{\mathcal{D}}(z,0) = 0$ following from $f(0) = 0$ for $f \in \mathcal{D}^2_{\alpha,\circ}(\mathbb{D})$, we recover the kernel as

$$K_{\mathcal{D}}(z,\omega) = \int_0^{\bar{\omega}} \left(\int_0^z K_{L^2_a}(\xi,\eta)d\xi \right) d\eta, \tag{4.6}$$

where η denotes the conjugate variable corresponding to $\bar{\omega}$. Substituting the explicit Bergman kernel $K_{L^2_a}(\xi,\eta) = \frac{1}{(1-\xi\eta)^{2+\alpha}}$ into (4.6), and evaluating the inner integral with η fixed:

$$\int_0^z \frac{1}{(1-\xi\eta)^{2+\alpha}} d\xi = \frac{1}{(1+\alpha)\eta} \left[(1-z\eta)^{-(1+\alpha)} - 1 \right].$$

Substituting into the outer integral,

$$\begin{aligned} K_{\mathcal{D}}(z,\omega) &= \frac{1}{(1+\alpha)} \int_0^{\bar{\omega}} \frac{((1-z\eta)^{-(1+\alpha)} - 1)}{\eta} d\eta \\ &= z\bar{\omega}\, {}_3F_2 \left(\begin{matrix} 1, 1, \alpha+2 \\ 2, 2 \end{matrix} ; z\bar{\omega} \right) \\ &= \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+2)\Gamma(n+1)^2} (z\bar{\omega})^n. \end{aligned}$$

as claimed.

For (ii), let $f \in \mathcal{D}^2_{\alpha,\circ}(\mathbb{U})$ and $\omega \in \mathbb{U}$. Since $\mathcal{D}^2_{\alpha,\circ}(\mathbb{U})$ is a reproducing kernel Hilbert space, there exists a kernel $K_{\mathcal{D}}(\cdot,\omega)$ such that

$$f(\omega) = \langle f, K_{\mathcal{D}}(\cdot,\omega) \rangle_{\mathcal{D}^2_{\alpha,\circ}(\mathbb{U})}. \tag{4.7}$$

Differentiating equation (4.7) with respect to ω we obtain,

$$f'(\omega) = \left\langle f, \frac{\partial}{\partial\bar{\omega}} K_{\mathcal{D}}(\cdot,\omega) \right\rangle_{\mathcal{D}^2_{\alpha,\circ}(\mathbb{U})}. \tag{4.8}$$

Also,

$$f'(\omega) = \left\langle f', K_{L^2_a}(\cdot,\omega) \right\rangle_{L^2_a(\mu_\alpha)} = \left\langle Df, K_{L^2_a}(\cdot,\omega) \right\rangle_{L^2_a}. \tag{4.9}$$

By Lemma 3

$$f'(\omega) = \langle f, D^* K_{L^2_a}(\cdot,\omega) \rangle_{\mathcal{D}^2_{\alpha,\circ}(\mathbb{U})}. \tag{4.10}$$

Comparing (4.10) and (4.8), we obtain

$$\frac{\partial}{\partial\bar{\omega}} K_{\mathcal{D}}(z,\omega) = \left(D^* K_{L^2_a}(\cdot,\omega) \right)(z). \tag{4.11}$$

Therefore, for $f \in \mathcal{D}^2_{\alpha,\circ}$ and $h \in L^2_a(\mu_\alpha)$,

$$\langle Df, h \rangle_{L^2_a(\mu_\alpha)} = \langle f, D^*h \rangle_{\mathcal{D}^2_{\alpha,\circ}}. \tag{4.12}$$

Since functions in $\mathcal{D}^2_{\alpha,\circ}(\mathbb{U})$ vanish at i, and D is an isometric isomorphism, we have

$$(D^*h)(z) = \int_i^z h(\xi)d\xi. \tag{4.13}$$

Substituting (4.13) into (4.11), we obtain

$$\frac{\partial}{\partial\bar{\omega}} K_{\mathcal{D}}(z,\omega) = \int_i^z K_{L_a}(\xi,\omega)d\xi. \tag{4.14}$$

We recover $K_{\mathcal{D}}$ by integrating equation (4.14) with respect to $\bar{\omega}$ and using the normalization $K^{\mathcal{D}}(z,i) = 0$ following from $f(i) = 0$ for $f \in \mathcal{D}^2_{\alpha,\circ}(\mathbb{U})$,

$$K^{\mathcal{D}}(z,\omega) = \int_{-i}^{\bar{\omega}} \left(\int_i^z K_{L^2_a}(\xi,\eta)d\xi \right) d\eta,$$

(where η denotes the conjugate variable corresponding to $\bar{\omega}$) giving us the exactly desired formula as

$$K_{\mathcal{D}}(z,\omega) = \int_{-i}^{\bar{\omega}} \int_i^z K_{L_a(\mu_\alpha)}(\xi,\eta)d\xi d\eta. \tag{4.15}$$

Substituting the reproducing kernel for $L^2_a(\mu_\alpha)$ in equation (4.15) and evaluating,

$$\begin{aligned}
K_{\mathcal{D}}(z,\omega) &= \frac{2^\alpha}{(1+\alpha)}(-i)^{2+\alpha} \int_{-i}^{\bar{\omega}} \int_i^z (\xi-\eta)^{-2-\alpha} d\xi d\eta \\
&= \frac{2^\alpha}{(-i)^{\alpha+2}} \int_{-i}^{\bar{\omega}} \int_i^z \frac{1}{[\xi-\eta]^{2+\alpha}} d\xi d\eta \\
&= \frac{2^\alpha}{(-i)^{\alpha+2}(1+\alpha)} \int_{-i}^{\bar{\omega}} \left[(i-\eta)^{-1-\alpha} - (z-\eta)^{-1-\alpha} \right] d\eta \\
&= \frac{2^\alpha}{-i^{\alpha+2}(1+\alpha)} \left. \frac{1}{\alpha}(i-\eta)^{-\alpha} - \frac{1}{\alpha}(z-\eta)^{-\alpha} \right|_{-i}^{\bar{\omega}} \\
&= \frac{2^\alpha}{\alpha(1+\alpha)(-i)^{2+\alpha}} \left[(i-\bar{\omega})^{-\alpha} - (z-\bar{\omega})^{-\alpha} - (2i)^{-\alpha} + (z+i)^{-\alpha} \right], \quad \alpha \neq 0
\end{aligned}$$

as desired. □

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