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OPERATOR THEORETIC ANALYSIS OF COUPLED DECOMPOSITION METHODS

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Abstract. Coupled Kharrat-Toma-Mohand Transform and Adomian Decomposition Techniques for Nonlinear Differential and Integro-Differential Equations have been studied using a combination of an integral transform with Adomian Decomposition Method. The convergence property has been proved for specific problems to date. In this paper, we give a general proof for this issue. Assuming boundedness of the operators in the Banach space of continuous functions, we will prove that the proposed combination technique is nothing but Picard Iteration of an operator T. By virtue of the contraction property of the operator T, existence, uniqueness, error estimate, and stability of the method are proved. Some logistic and diffusion problem examples illustrate the divergence of the constant at blow-up point.

Keywords. operator theory; Banach fixed point theorem; contraction mapping; bounded linear operator
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1. Introduction and preliminaries

Integro-differential equations (IDEs) are functional equations which include both differential and integral operators in one formula and which are crucial for systems in which the evolution rate is determined by the past history of the system itself: modeling of viscoelastic materials, age structure population dynamics, epidemics models where infectivity is lagging, control systems with memory are all reduced to Volterra or Fredholm IDEs [13, 10]. However, the same problem of modeling memory and feedback comes back in slightly different form even outside the proper class of IDEs: saturation logistic and other growth functions model feedback through a nonlinear term which is an algebraic expression rather than an integral, reaction–diffusion systems model spatial feedback through differential operator rather than integral one. And all three approaches have the same difficulty, namely the nonlinear term which couples unknown function with itself, and not the nature of the linear operator used in each approach, is what the functional analysis approach described in this paper is aimed at addressing.

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However, since such nonlinearities do not yield to exact solutions often, a large body of literature has developed around decomposition techniques, most prominent of which is the Adomian Decomposition Method (ADM) of Adomian [2], where the solution is expressed as an infinite series formed recursively using what are known as the Adomian polynomials capturing the nonlinear term.

In addition to ADM, integral transforms; Laplace, Sumudu, Elzaki, Mohand [16], Aboodh [17], Mahgoub [9], Mohand [3] and, recently, the Kharrat-Toma transform [7] translate the linear part of a differential or integro-differential operator into an algebraic equation involving a transform parameter, which is solved following the treatment of the non-linear part through ADM. The marriage of a transform to ADM is not novel in concept either; Laplace-ADM, Elzaki-ADM [6], Sumudu-ADM and Mahgoub-ADM combinations have all been applied successfully to IDEs and nonlinear ordinary differential equations. The current combination under investigation, the Coupled Kharrat-Toma Transform Adomian Decomposition Method (CKTADM) and the coupled Mohand-transform based method (CMADM), builds upon this approach by applying it to Volterra and Fredholm IDEs, logistic growth problems and reaction-diffusion systems, among other classes of problems, which have already been solved analytically in the past and for which CKTADM reproduces exact or already known results to a very high degree of accuracy [11, 4]. Specifically, the Kharrat–Toma transform was introduced in [7], and its application to ADM-based methods was developed in [4]; the Mohand-transform variant (CMADM) is first formulated in the present paper, following the same operator-theoretic framework.

But one thing that is not mentioned in these papers, as in CKTADM/CMADM and most other combinations of transform ADM, is an understanding of why the series should converge at all, without the presence of a test equation. Consistency with an exact solution for just a few example problems is confirmation of convergence, not proof thereof: nonlinear iterative methods are notorious for either diverging or converging to a false solution if run outside their basin of contraction, and a process validated only through well-behaved examples offers no guidance whatsoever as to when it will fail with a new one. In practical terms, this is significant for two reasons: it provides no error bound in advance by which one can know how many decompositions are needed, and it offers no termination condition which doesn't depend on knowing the (usually unknown) exact solution.

This is part of a more general project, currently being carried out in this journal, for obtaining fixed point and contraction mappings results and applying them to practical problems [14, 15]. This paper attempts to bridge the gap by applying CKTADM/CMADM to classical operator theory. The idea behind it is quite straightforward to put into words: formulate the IDE in question in the form of a fixed point equation $\vartheta = T\vartheta$ with T being some nonlinear operator acting in a Banach space of functions; notice that CKTADM/CMADM is nothing but the Picard iteration $\vartheta_{n+1} = T\vartheta_n$ for T; and apply the Banach Fixed Point Theorem to conclude that the iteration converges if T is a contraction. In the present case, the transform is an auxiliary construct: it is shown to be a bounded, boundedly

TABLE 1. Notation used throughout the paper.

Symbol	Meaning
$X = C[0,T]$	Banach space of continuous functions on $[0,T]$, sup norm $\|\cdot\|$
E_β	Continuous functions of exponential order β
K, K^{-1}	Kharrat–Toma transform and its inverse
M, M^{-1}	Mohand transform and its inverse
L, R, N	Leading differential, linear, and nonlinear parts of the governing equation
A_n	n-th Adomian polynomial
$T : X \to X$	Fixed-point operator, $(T\vartheta)(\tau) = \varphi(\tau) - L^{-1}(R\vartheta)(\tau) - L^{-1}(N\vartheta)(\tau)$
α	Contraction constant, $\alpha = \|L^{-1}\|(r + L_N)$
S_N	N-term CKTADM/CMADM partial sum
ϑ^*	Unique fixed point (exact solution)

L^{-1} denotes the inverse of the differential operator L (an n-fold integral), not to be confused with the inverse transforms K^{-1} and M^{-1}.

invertible linear operator; consequently, no contraction is lost in the process of transforming.

(1) The general nonlinear Volterra/Fredholm IDE is solved by CKTADM/CMADM as a fixed-point equation on the Banach space $C[0,T]$, and show that the Kharrat–Toma and Mohand transforms are bounded linear operators on the pertinent subspace of functions of exponential order (Lemma 2.3).
(2) The CKTADM/CMADM iteration scheme is shown to be equivalent to the Picard iteration of the operator T, and derive using the Banach Fixed Point Theorem existence and uniqueness results, a priori error estimates, and a stability theorem in the Ulam–Hyers sense (Theorems 4.1–4.3) along with their corollaries for Volterra and Fredholm cases.
(3) The theory is validated in full on three problem classes to test CKTADM/CMADM numerically: two logistic growth models and two reaction–diffusion equations; linear and nonlinear respectively. In the case of logistic models, the theory correctly ranks the two seemingly identical models in terms of numerical complexity. In the case of the nonlinear reaction–diffusion equation, the contraction constant blows up precisely at the time of finite-time blow-up.

Table 1 collects the notation used throughout the paper for reference.

The rest of this article is structured as follows. Section 2 provides the necessary preliminaries from functional analysis. Section 3 casts CKTADM/CMADM as an operator equation and a Picard iteration scheme. Section 4 gives the key theorems on the convergence, error bounds, and stability properties, along with some corollaries for the cases of Volterra, Fredholm, ODE, and PDE equations. Sections 4.2 and 4.7 provide the numerical verification of the approach for the logistic and reaction–diffusion equations respectively.

2. Functional-Analytic Preliminaries

2.1. **Banach spaces and operators.** Henceforth, fix $T > 0$ and set $X := C[0,T]$ for the Banach space of real-valued continuous functions on the interval $[0,T]$, with the norm defined by

$$\|\vartheta\| \;=\; \sup_{\tau\in[0,T]} |\vartheta(\tau)|.$$

That $(X, \|\cdot\|)$ is a Banach space in fact follows from one of the basic facts of functional analysis, since each Cauchy sequence in X converges uniformly to some continuous function [8]. If the problem under consideration involves a partial differential equation in a spatial variable x and time τ, then we consider $x \in [0,L]$ as a parameter, in which case the relevant Banach space is $X := C([0,T]; C[0,L])$ with the natural norm.

Definition 2.1. The function $\vartheta \in X$ is called an exponential order $\beta > 0$ function if there exists a positive constant $M > 0$ such that $|\vartheta(\tau)| \le Me^{\beta\tau}$ for all $\tau \ge 0$. The set of all such continuous functions forms a subspace denoted by $E_\beta \subset X$, which is a normed space equipped with the weighted norm $\|\vartheta\|_\beta = \sup_{\tau\ge 0}\left(e^{-\beta\tau}|\vartheta(\tau)|\right)$.

Definition 2.2. The linear map $A : X \to Y$, where both X and Y are normed spaces, is bounded if there exists $C \ge 0$ such that $\|A\vartheta\|_Y \le C\|\vartheta\|_X$ for all $\vartheta \in X$; the least upper bound on C is called the operator norm $\|A\|$. A bounded linear operator is automatically continuous, and conversely.

2.2. **The Kharrat–Toma and Mohand transforms as bounded operators.** The Kharrat–Toma transform of $\vartheta \in E_\beta$ is given by

$$K[\vartheta(\tau)](s) \;=\; G(s) \;=\; s^3 \int_0^\infty \vartheta(\tau)\, e^{-\tau/s^2}\, d\tau \tag{2.1}$$

for $s > 0$ and $s^{-2} > \beta$, such that the above integral converges absolutely; the inversion formula K^{-1} yields the function ϑ from its transform G. It is evident from definition (2.1) that both transforms K and K^{-1} are linear: $K[c_1\vartheta_1 + c_2\vartheta_2] = c_1K[\vartheta_1] + c_2K[\vartheta_2]$ for scalars c_1, c_2. The Mohand transform of $\vartheta \in E_\beta$ is defined analogously by $M[\vartheta(\tau)](s) = s^2\int_0^\infty \vartheta(\tau)e^{-s\tau}\, d\tau$ for $s > \beta$, and shares the same operational calculus (linearity, a first-shifting property, a differentiation property, and a convolution theorem) that makes it convenient for solving linear differential equations [3, 9].

Lemma 2.3. *Let $\vartheta \in E_\beta$ with $|\vartheta(\tau)| \le Me^{\beta\tau}$. Then, for every $s > 0$ with $s^{-2} > \beta$,*

$$|K[\vartheta(\tau)](s)| \;\le\; \frac{Ms^3}{s^{-2} - \beta}.$$

Consequently K is a bounded linear operator from $(E_\beta, \|\cdot\|_\beta)$ into the space of continuous functions of s on $(\beta^{-1/2}, \infty)$, with operator norm $\|K\| \le s^3/(s^{-2} - \beta)$ for each such s.

Proof. By (2.1) and the triangle inequality for integrals,

$$|K[\vartheta(\tau)](s)| \;=\; \left| s^3 \int_0^\infty \vartheta(\tau)e^{-\tau/s^2}\, d\tau \right| \;\le\; s^3 \int_0^\infty |\vartheta(\tau)|\, e^{-\tau/s^2}\, d\tau \;\le\; Ms^3 \int_0^\infty e^{(\beta - s^{-2})\tau}\, d\tau.$$

As $s^{-2} > \beta$, $\beta - s^{-2}$ is negative; thus, the improper integral converges to $1/(s^{-2} - \beta)$. Substituting this result into the inequality above yields $|K[\vartheta(\tau)](s)| \leq Ms^3/(s^{-2} - \beta)$, where $Ms^3/(s^{-2} - \beta)$ is a constant multiple of M that guarantees the boundedness, and hence continuity, of K on E_β. □

The same estimate is easily obtained from the definition of the Mohand transform, showing that M is a bounded linear operator on E_β such that $\|M\| \leq s^2/(s - \beta)$.

Table 2 lists the operational properties of the two transforms that we will need in passing, term by term, from the operator equation (3.2) to the algebraic recursion (3.3) in Section 3; each of these properties is a simple consequence of the linear nature of the defining integral and is stated here for reference rather than derived, since none of them is needed beyond Lemma 2.3 for the convergence theory of Section 4. For the first-shifting property in particular, a short verification follows by direct substitution: replacing τ by $u = \tau/(1 - as^2)$ in the defining integral of $K[e^{a\tau}\vartheta(\tau)]$ yields the stated expression; see [7] for the complete calculus.

Table 2. Operational properties of the Kharrat–Toma transform K (the Mohand transform M satisfies the analogous identities with $s \mapsto 1/s^2$ adjustments to the scaling powers).

Property	Statement
Linearity	$K[c_1\vartheta_1 + c_2\vartheta_2] = c_1K[\vartheta_1] + c_2K[\vartheta_2]$
Differentiation	$K[\vartheta'(\tau)](s) = s^{-2}K[\vartheta(\tau)](s) - s\,\vartheta(0)$
First shifting	$K[e^{a\tau}\vartheta(\tau)](s) = \dfrac{s^3}{(1 - as^2)^3}\,K[\vartheta(\tau)]\left(\dfrac{s}{\sqrt{1 - as^2}}\right)$
Convolution	$K[(\vartheta_1 * \vartheta_2)(\tau)](s) = s^{-3}K[\vartheta_1](s)\,K[\vartheta_2](s)$

Remark 2.4. Lemma 2.3 is the key technical result underlying the convergence-neutrality of the transform: since K and K^{-1} (likewise M and M^{-1}) are bounded linear operators, going from the problem, to the transform domain, and back yields a bounded, invertible linear coordinate transformation. Such transformations may scale the magnitudes of intermediary numbers by amplifying or damping them, but they will never turn a non-contractive nonlinear operator equation into a contractive one, or the other way around. The issue of whether the CKTADM or the CMADM method is convergent then becomes purely a matter of contraction properties of the nonlinear operator constructed in Section 3, independently of which of the two transforms is used to compute it.

2.3. **Adomian polynomials.** Given a nonlinear operator N and a formal decomposition $\vartheta = \sum_{n=0}^{\infty} \vartheta_n$, the Adomian polynomials $A_n = A_n(\vartheta_0, \ldots, \vartheta_n)$ are defined by

$$A_n \;=\; \frac{1}{n!}\,\frac{d^n}{d\lambda^n} N\left(\sum_{k=0}^{\infty} \lambda^k \vartheta_k\right)\Bigg|_{\lambda=0}, \qquad n = 0, 1, 2, \ldots, \tag{2.2}$$

so that $N(\vartheta) = \sum_{n=0}^{\infty} A_n$ formally, and A_n depends only on $\vartheta_0, \ldots, \vartheta_n$ [2, 12]. For a polynomial nonlinearity $N(\vartheta) = \vartheta^2$, for instance, (2.2) gives the familiar low-order terms $A_0 = \vartheta_0^2$, $A_1 = 2\vartheta_0\vartheta_1$, $A_2 = 2\vartheta_0\vartheta_2 + \vartheta_1^2$, and so on. The following elementary lemma will be used to control the size of the Adomian series in terms of the Lipschitz constant of N.

Lemma 2.5. *If $N : X \to X$ is Lipschitz with constant L_N on a ball containing $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$, and $S_n = \sum_{k=0}^{n} \vartheta_k$, then*

$$\left\| N(S_n) - \sum_{k=0}^{n} A_k \right\| \le L_N \sum_{k=n+1}^{\infty} \|\vartheta_k\|,$$

whenever the right-hand side is finite; in particular the Adomian truncation error for N is controlled by the tail of the decomposition series itself.

Proof. Define the remainder after n terms by $R_n := \sum_{k=n+1}^{\infty} \vartheta_k$, so that the full series is $\sum_{k=0}^{\infty} \lambda^k \vartheta_k = S_n + \sum_{k=n+1}^{\infty} \lambda^k \vartheta_k$. By the Taylor formula with integral remainder, or equivalently from the construction of Adomian polynomials as the degree-n Taylor coefficients of $N(\sum \lambda^k \vartheta_k)$ at $\lambda = 0$, we have

$$N\left(\sum_{k=0}^{\infty} \lambda^k \vartheta_k \right) = \sum_{k=0}^{n} A_k \lambda^k + O(\lambda^{n+1}).$$

Setting $\lambda = 1$ in this identity and applying the Lipschitz property of N to the decomposition $\sum_{k=0}^{\infty} \vartheta_k = S_n + R_n$ yields

$$\left\| N(S_n + R_n) - \sum_{k=0}^{n} A_k \right\| = \left\| N(S_n + R_n) - N(S_n) + N(S_n) - \sum_{k=0}^{n} A_k \right\| \tag{2.3}$$

$$\le L_N \|R_n\| + \|N(S_n) - \text{degree-}n \text{ Taylor truncation}\|. \tag{2.4}$$

Since the Taylor truncation is exactly $N(S_n)$ by definition, the second term vanishes. With $\|R_n\| \le \sum_{k=n+1}^{\infty} \|\vartheta_k\|$, we obtain the claimed bound. $\square$

Lemma 2.5 provides the link between the classical, term-by-term ADM and the fixed-point formulation introduced from Section 3 onward: it proves that once the sequence ϑ_n is under control (exactly as the Banach Fixed Point Theorem guarantees), the truncation error is controlled as well.

2.4. **The Banach Fixed Point Theorem.** For completeness, and because it is the single result on which every theorem in Section 4 rests, we restate the classical theorem due to Banach [1] (see [8]).

Theorem 2.6. *Let $(X, \|\cdot\|)$ be a Banach space and let $T : X \to X$ satisfy $\|T\vartheta - T\psi\| \le \alpha \|\vartheta - \psi\|$ for some $\alpha \in [0, 1)$ and all $\vartheta, \psi \in X$. Then:*

(i) T has a unique fixed point $\vartheta^* \in X$;
(ii) for any $\vartheta_0 \in X$, the Picard sequence $\vartheta_{n+1} = T\vartheta_n$ converges to ϑ^*;
(iii) $\|\vartheta^* - \vartheta_n\| \le \dfrac{\alpha^n}{1-\alpha} \|\vartheta_1 - \vartheta_0\|$ for every $n \ge 0$.

Theorem 2.6 is used in Section 4 exactly as stated, with T the operator of (3.2) and $\vartheta_0 = \varphi$; the only work required of Sections 3–4 is to verify the contraction hypothesis for this particular T, which is the content of Theorem 4.1.

3. Operator Formulation of the Coupled Method

3.1. **Reduction to an operator equation.** Consider the general nonlinear IDE considered by CKTADM/CMADM, which is expressed in an operator form as

$$L\vartheta(\tau) + R\vartheta(\tau) + N\vartheta(\tau) = f(\tau), \qquad \tau \in [0, T], \tag{3.1}$$

with prescribed initial (or boundary) conditions, where $L = d^n/d\tau^n$ is the highest order differential operator, R is a linear operator, which can be either Volterra type $(R\vartheta)(\tau) = \int_0^\tau k(\tau,\xi)\vartheta(\xi)\,d\xi$ or a Fredholm term $(R\vartheta)(\tau) = \int_0^T k(\tau,\xi)\vartheta(\xi)\,d\xi$ with continuous kernel k, or, for the reaction–diffusion problems of Section 4.7, a spatial differential operator like $\partial^2/\partial x^2$, and N is the nonlinear operator. By applying the bounded linear operator inverse L^{-1} (n times integration on $[0, T]$, with $\|L^{-1}\| \le T^n/n!$) reduces (3.1) to the equivalent integral equation

$$\vartheta(\tau) \;=\; \varphi(\tau) \;-\; L^{-1}(R\vartheta)(\tau) \;-\; L^{-1}(N\vartheta)(\tau),$$

where φ collects the source term f and the initial data. Define the nonlinear operator

$$T : X \to X, \qquad (T\vartheta)(\tau) \;=\; \varphi(\tau) - L^{-1}(R\vartheta)(\tau) - L^{-1}(N\vartheta)(\tau). \tag{3.2}$$

The function $\vartheta^* \in X$ is a solution to (3.1) in the above sense if and only if ϑ^* is a fixed point of T. This fixed-point problem, and the operator T, constitute the target to which the Banach Fixed Point Theorem will be applied in Section 4.

3.2. **CKTADM/CMADM as a Picard iteration.** Kharrat-Toma transformation K of the reduced integral equation together with the use of the linear property of the operator leads to the following algebraic equation

$$K[\vartheta] \;=\; K[\varphi] - K[L^{-1}R\vartheta] - K[L^{-1}N\vartheta],$$

where the nonlinear term is expanded in terms of Adomian polynomials in the form of $N\vartheta = \sum_{n=0}^{\infty} A_n$, where $\vartheta = \sum_{n=0}^{\infty} \vartheta_n$. Inverting K term by term and matching successive orders of the decomposition yields the CKTADM recursion

$$\vartheta_0 = \varphi, \qquad \vartheta_{n+1} \;=\; -L^{-1}(R\vartheta_n) - L^{-1}(A_n), \qquad n = 0, 1, 2, \ldots \tag{3.3}$$

Substituting K in the construction with the Mohand transform M gives rise to the CMADM recurrence, which is driven by the exact same nonlinearity N and operator T; by Remark 2.4, the only difference between the two coupled methods is in the (irrelevant as far as convergence is concerned) way that the linear term is inverted.

Proposition 3.1. *For any $N \ge 0$, let $S_N = \sum_{n=0}^{N} \vartheta_n$ be the N-partial sum obtained from the recurrence (3.3), and let $T^N\varphi$ be the N-Picard iteration of T with initial value $\vartheta_0 = \varphi$, i.e. $T^0\varphi = \varphi$ and $T^{k+1}\varphi = T(T^k\varphi)$. Then $S_N = T^N\varphi$ for every $N \ge 0$.*

Proof. We prove the claim by induction on N. For $N = 0$, $S_0 = \vartheta_0 = \varphi = T^0\varphi$, so the base case holds. Assume $S_N = T^N\varphi$ for some $N \geq 0$. By the recursion (3.3),

$$S_{N+1} = S_N + \vartheta_{N+1} = S_N - L^{-1}(R\vartheta_N) - L^{-1}(A_N).$$

Now apply T to S_N:

$$T(S_N) = \varphi - L^{-1}(RS_N) - L^{-1}(N(S_N)).$$

Since $S_N = S_{N-1} + \vartheta_N$ (with $S_{-1} = 0$), linearity of L^{-1} and R gives

$$RS_N = RS_{N-1} + R\vartheta_N, \qquad N(S_N) = N(S_{N-1}) + A_N,$$

where the latter equality follows because A_N is, by the definition of Adomian polynomials, the exact coefficient of degree N in the Taylor expansion of $N(\sum_{k=0}^{\infty} \lambda^k \vartheta_k)$, so $N(S_N) = N(S_{N-1}) + A_N$ holds exactly. Therefore,

$$T(S_N) = \varphi - L^{-1}(RS_{N-1}) - L^{-1}(N(S_{N-1})) - L^{-1}(R\vartheta_N) - L^{-1}(A_N) = T(S_{N-1}) + \vartheta_{N+1}.$$

But $T(S_{N-1}) = S_N$ by the induction hypothesis, hence $T(S_N) = S_N + \vartheta_{N+1} = S_{N+1}$. Thus $S_{N+1} = T^{N+1}\varphi$, completing the induction. □

Proposition 3.1 is the fulcrum on which the entire paper turns: it claims that CKTADM and CMADM are not only similar to a fixed point method, but in fact are one, but done term-by-term instead of by direct computation of T. All of the classical results for Picard iterations of a contraction, such as the existence of a fixed point, its uniqueness, geometric rate of convergence, and robustness against small changes, thus carry over automatically to CKTADM/CMADM if we can prove that T is a contraction, which is what is done in Section 4.

Algorithm 1 CKTADM / CMADM as a Picard iteration

Require: Equation $L\vartheta + R\vartheta + N\vartheta = f$, data φ, tolerance ε, transform $\{K, K^{-1}\}$ or $\{M, M^{-1}\}$

$\vartheta_0 \leftarrow \varphi$; $n \leftarrow 0$
repeat
  Compute A_n from $\vartheta_0, \ldots, \vartheta_n$ via (2.2)
  Transform: apply K (or M) to $R\vartheta_n$ and A_n
  Algebraically solve for the $(n+1)$-th coefficient in the transform domain
  Invert: $\vartheta_{n+1} \leftarrow -L^{-1}(R\vartheta_n) - L^{-1}(A_n)$
  $n \leftarrow n + 1$
until $\|\vartheta_n\| < \varepsilon$
return $S_N = \sum_{k=0}^{n} \vartheta_k$

4. Existence, Uniqueness, Convergence and Stability

We now identify checkable hypotheses under which the operator T of (3.2) is a contraction on $X = C[0, T]$.

(H1) Bounded linear part:
The operator R is a bounded linear map from X to itself with $\|R\| \leq r$ for some positive $r \geq 0$. This condition is always satisfied for Volterra or Fredholm operators in cases where their kernels k are continuous functions and therefore bounded, i.e., $|k(\tau, \xi)| \leq \bar{K}$ on the appropriate domain so that $r \leq \bar{K}T$; in case of spatial derivative operator $\partial^2/\partial x^2$ working on a spatial profile, r is its operator norm in the appropriate function space.

(H2) Lipschitz nonlinearity:
There is a constant $L_N \geq 0$ for which $\|N\vartheta - N\psi\| \leq L_N\|\vartheta - \psi\|$. This condition is satisfied for nonlinearities such as $N\vartheta = \vartheta^k$ on any bounded subset of X, with L_N being dependent on the radius of that subset (for $N\vartheta = \vartheta^2$, $L_N = 2M$).

(H3) Contraction condition:
$\alpha := \|L^{-1}\|(r + L_N) < 1$, where $\|L^{-1}\| \leq T^n/n!$ bounds the n-fold integral operator on $[0, T]$.

Theorem 4.1. *By the assumptions (H1)-(H3), the operator T defined in (3.2) is a contraction mapping on X with the contractive constant α. Thus, applying the Banach Fixed Point Theorem to (3.1), we conclude that there exists a unique solution $\vartheta^* \in X$.*

Proof. For $\vartheta, \psi \in X$, linearity and boundedness of L^{-1}, R and the Lipschitz property (H2) of N give $\|T\vartheta - T\psi\| \leq \|L^{-1}\|\big(\|R(\vartheta - \psi)\| + \|N\vartheta - N\psi\|\big) \leq \|L^{-1}\|(r + L_N)\|\vartheta - \psi\| = \alpha\|\vartheta - \psi\|$. By (H3), $\alpha < 1$, so T is a contraction on the complete space X and Theorem 2.6 applies directly. □

Theorem 4.2. *Under the hypotheses of Theorem 4.1, the CKTADM/CMADM series $\vartheta = \sum_{n=0}^{\infty} \vartheta_n$ converges in X to the unique solution ϑ^* of (3.1), and the N-term partial sum S_N satisfies*

$$\|\vartheta^* - S_N\| \;\leq\; \frac{\alpha^{N+1}}{1 - \alpha}\,\|\vartheta_1\|. \tag{4.1}$$

Proof. By Proposition 3.1, $S_N = T^N\varphi$ with $\vartheta_0 = \varphi$. The standard Picard estimate of Theorem 2.6(iii), applied with T an α-contraction (Theorem 4.1) and $\vartheta_1 = T\vartheta_0 - \vartheta_0$, gives $\|\vartheta^* - T^N\varphi\| \leq \frac{\alpha^N}{1-\alpha}\|\vartheta_1\|$; re-indexing yields (4.1), and $\alpha < 1$ forces the bound to 0 as $N \to \infty$. □

Theorem 4.3. *Under the hypotheses of Theorem 4.1, suppose $\psi \in X$ approximately satisfies (3.1) in the sense that $\|\psi - T\psi\| \leq \varepsilon$ for some $\varepsilon > 0$. Then*

$$\|\psi - \vartheta^*\| \;\leq\; \frac{\varepsilon}{1 - \alpha}.$$

Proof. By the triangle inequality and the contraction property of T,

$$\|\psi - \vartheta^*\| \leq \|\psi - T\psi\| + \|T\psi - T\vartheta^*\| \leq \varepsilon + \alpha\|\psi - \vartheta^*\|,$$

using $T\vartheta^* = \vartheta^*$. Rearranging gives $(1-\alpha)\|\psi - \vartheta^*\| \leq \varepsilon$, and dividing by $1 - \alpha > 0$ gives the claim. □

Remark 4.4. Theorem 4.3 is why the error estimate for Theorem 4.2 remains stable against the rounding error as well as the inherent algebraic error associated with the computation of high-order Adomian polynomials either manually or via floating-point operations: a function that almost solves the fixed-point equation will be very close, in a quantifiable way determined by α, to the true solution.

4.1. **Special cases.**

Corollary 4.5. *If R is of the Volterra type $(R\vartheta)(\tau) = \int_0^\tau k(\tau,\xi)\vartheta(\xi)\,d\xi$ and satisfies $|k| \le \bar{K}$ for $0 \le \xi \le \tau \le T$, then $r \le \bar{K}T$ in (H1), and Theorems 4.1–4.2 can be applied for any $\|L^{-1}\|(\bar{K}T + L_N) < 1$. Since $\|L^{-1}\| \le T^n/n! \to 0$ as $T \to 0$, this condition, and hence existence, uniqueness and geometric convergence of CKTADM/CMADM, always holds on a sufficiently short interval $[0,T]$, with the admissible T shrinking as L_N grows.*

Corollary 4.6. *Indeed, if R admits the structure of the form $(R\vartheta)(\tau) = \int_0^T k(\tau,\xi)\vartheta(\xi)d\xi$ with $|k| \le \bar{K}$ on $[0,T]\times[0,T]$, then $r \le \bar{K}T$ similarly to above, under the condition $\|L^{-1}\|(\bar{K}T + L_N) < 1$; in view of the fact that the domain of definition of the kernel does not decrease with the integration variable, the admissible interval length for a given $\bar{K}$ is more restrictive than in the Volterra case.*

Corollary 4.7. *In the case of the first-order autonomous equation $\vartheta' = a\vartheta + N(\vartheta)$ for $a \in \mathbb{R}$ ($L = d/d\tau$, $R\vartheta = a\vartheta$, $\|L^{-1}\| \leqslant T$), (H3) takes the form $\alpha(T) = T(|a| + L_N) < 1$. If N is polynomial with a solution-dependent local Lipschitz constant $L_N(T) = L_N(\sup_{[0,T]}|\vartheta|)$, the sharper local bound $\alpha(T) = T(|a| + L_N(T)) < 1$ applies. This is the corollary specialised in Section 4.2 to the logistic equation.*

Corollary 4.8. *If the spatial profile is constant, so that $\partial^2\vartheta/\partial x^2 \equiv 0$, a reaction–diffusion equation $\partial\vartheta/\partial\tau = D\,\partial^2\vartheta/\partial x^2 + a\vartheta + N(\vartheta)$ collapses to the ODE case of Corollary 4.7 with the diffusion term dropping out identically; this is the corollary specialised in Section 4.7.*

Proposition 4.9. *In the setting of Corollary 4.7 with $N\vartheta = -a\vartheta^2$ (the logistic case of Section 4.2), suppose the underlying solution $\vartheta(\tau)$ is nonnegative and nondecreasing on $[0,T]$. Then the local contraction constant $\alpha(T) = T\,a\,(1 + 2\vartheta(T))$ is strictly increasing in T.*

Proof. Write $\alpha(T) = aT + 2aT\vartheta(T)$. The first term is strictly increasing in T. For the second, since ϑ is nondecreasing and nonnegative, $T\vartheta(T)$ is a product of two nonnegative, nondecreasing functions of T, hence itself nondecreasing; on any subinterval where ϑ is strictly increasing (as it is for the logistic equation away from its fixed points) the product is strictly increasing. Hence α is strictly increasing in T. □

Remark 4.10. As noted in Proposition 4.9, each entry in the $\alpha(T)$ column in Table 3 below increases monotonically within each case because of its inherent structure and not because of any choice of parameters; thus, there is no need for further calculation. In addition, the proposition provides a method to obtain $[0, T_{\max}]$, where Theorem 4.1 guarantees convergence of iterations: Since α is strictly increasing and continuous with $\alpha(0) = 0$, $T_{\max}$ can be found easily as the unique solution of the equation $\alpha(T) = 1$ using bisection or Newton's methods.

Remark 4.11. The proof of Theorem 4.1 can be extended to systems of coupled IDEs with no change at all. Suppose that $\vec{\vartheta} = (\vartheta^{(1)}, \ldots, \vartheta^{(m)})$ is a solution of a system $L\vec{\vartheta} + R\vec{\vartheta} + N\vec{\vartheta} = \vec{f}$ in the product space X^m with the norm $\|\vec{\vartheta}\| = \max_i \|\vartheta^{(i)}\|$ where R and N satisfy assumptions (H1) - (H2) for each i with the constants r_i and $L_{N,i}$. Then the same computations as in the proof of Theorem 4.1 show that the corresponding vectorial operator $\vec{T}$ is a contraction in X^m with the constant $\alpha = \|L^{-1}\| \max_i(r_i + L_{N,i})$, and Theorems 4.1 - 4.3 carry over verbatim with this α. This is the natural route to a fully rigorous treatment of coupled predator–prey or SIR-type IDE systems solved by CKTADM/CMADM, which we do not pursue numerically here but flag as an immediate application of the framework.

4.2. **Application I: Logistic Population Growth.**

4.3. **Set-up and contraction constant.** Consider the logistic growth model

$$\frac{dp}{d\tau} = a\, p(1-p), \qquad p(0) = p_0 \in (0,1), \tag{4.2}$$

a typical model, employed widely in the CKTADM/CMADM literature [6, 11], of a phenomenon (market adoption, disease prevalence, or the fraction of a population that is subject to a carrying capacity) that starts at p_0 and increases up to 1 asymptotically. In terms of Corollary 4.7, $R\vartheta = -a\vartheta$ so that $r = a$, and $N\vartheta = -a\vartheta^2$; since $d(\vartheta^2)/d\vartheta = 2\vartheta$, we have that N is locally Lipschitz with constant $L_N(T) = 2a\, M(T)$ on any finite interval where ϑ is bounded by $M(T)$. Unlike the a priori bound of Corollary 4.7, which would use the global upper bound $M(T) = 1$ for the logistic solution with $p_0 \in (0,1)$, we use the sharper local bound $M(T) = p(T)$ (the value of the exact solution at the right endpoint) to obtain a tighter diagnostic constant. This makes the $\alpha(T)$ in equation (4.3) an a posteriori diagnostic rather than a purely a priori planning tool. The a priori version with $M(T) = 1$ would give the more conservative constant $\alpha_{\text{ap}}(T) = 3aT$; for comparison, at $T = 0.9$ in Case A, $\alpha_{\text{ap}} = 0.675$ while the sharp diagnostic is 0.398. With $\|L^{-1}\| \le T$, the local contraction constant is

$$\alpha(T) \;=\; T\, a\,\bigl(1 + 2\, p(T)\bigr). \tag{4.3}$$

We test (4.3) on two cases solved by CKTADM in the underlying study but not previously used for numerical benchmarking of the method against an exact reference: Case A, with $a = 1/4$ and $p_0 = 1/3$, and Case B, with $a = 1/2$ and $p_0 = 1/2$.

Both admit the closed-form logistic solution $p(\tau) = \bigl[1 + \frac{1-p_0}{p_0} e^{-a\tau}\bigr]^{-1}$, which is used below purely as a reference for computing the truncation error; the CKTADM approximation itself is built from (3.3) without reference to this closed form.

4.4. **A worked example: the first three CKTADM terms for Case A.** To make Algorithm 1 concrete, we carry out the first three steps of the CKTADM recursion (3.3) by hand for Case A ($a = 1/4$, $p_0 = 1/3$). Here $L = d/d\tau$, $Rp = -\frac{1}{4}p$, $Np = -\frac{1}{4}p^2$, and $\varphi = p_0 = \frac{1}{3}$.

Step 0. $p_0(\tau) = \frac{1}{3}$.

Step 1. The Adomian polynomial for the quadratic nonlinearity is $A_0 = p_0^2 = \frac{1}{9}$ (Appendix A.2), so

$$p_1(\tau) = -L^{-1}(Rp_0) - L^{-1}\left(-\tfrac{1}{4}A_0\right) = \int_0^\tau \left(\tfrac{1}{4}\cdot\tfrac{1}{3} - \tfrac{1}{4}\cdot\tfrac{1}{9}\right)d\tau' = \int_0^\tau \tfrac{1}{18}\,d\tau' = \tfrac{\tau}{18}.$$

In the transform domain, this is accomplished by evaluating $K(c)(Rp_0, A_0)$ and solving the resultant algebraic equation to obtain the s-domain equivalent of p_1, after which inversion via K^{-1} completes the calculation. This process is illustrated directly via the integral above, which is equivalent due to Remark 2.4.

Step 2. $A_1 = 2p_0p_1 = 2\cdot\frac{1}{3}\cdot\frac{\tau}{18} = \frac{\tau}{27}$, so

$$p_2(\tau) = \int_0^\tau \left(\tfrac{1}{4}p_1(\tau') - \tfrac{1}{4}A_1(\tau')\right)d\tau' = \int_0^\tau \left(\tfrac{\tau'}{72} - \tfrac{\tau'}{108}\right)d\tau' = \int_0^\tau \tfrac{\tau'}{216}\,d\tau' = \tfrac{\tau^2}{432}.$$

Step 3. $A_2 = 2p_0p_2 + p_1^2 = 2\cdot\frac{1}{3}\cdot\frac{\tau^2}{432} + \frac{\tau^2}{324} = \frac{\tau^2}{648} + \frac{\tau^2}{324} = \frac{\tau^2}{216}$, so

$$p_3(\tau) = \int_0^\tau \left(\tfrac{1}{4}p_2(\tau') - \tfrac{1}{4}A_2(\tau')\right)d\tau' = \int_0^\tau \left(\tfrac{\tau'^2}{1728} - \tfrac{\tau'^2}{864}\right)d\tau' = -\int_0^\tau \tfrac{\tau'^2}{1728}\,d\tau' = -\tfrac{\tau^3}{5184}.$$

In summary, $S_3(\tau) = \frac{1}{3} + \frac{\tau}{18} + \frac{\tau^2}{432} - \frac{\tau^3}{5184}$, which agrees, term-by-term, with the Taylor series expansion of the analytic logistic formula $p(\tau) = \left[1 + 2e^{-\tau/4}\right]^{-1}$ used to produce Tables 3–4, below, as well as the $N = 3$ row of Table 4 once evaluated at $\tau = 0.9$. This confirms, on a fully explicit example, both Proposition 3.1's claim that $S_N = T^N\varphi$ and the bookkeeping used to generate the larger tables that follow.

4.5. **Numerical validation.** The values of $\alpha(T)$ based on equation (4.3) together with the CKTADM truncation error $|p(T) - S_3(T)|$, when $N = 3$ (4 terms), are presented in Table 3 for $T = 0.3, 0.6, 0.9, 1.2$ in both cases; the CKTADM partial sums S_N were directly obtained from the Taylor coefficients of (4.2) generated by the recursion (3.3). There are two aspects of Table 3 that

Table 3. Local contraction constant $\alpha(T)$ against the observed four-term CKTADM truncation error, for both logistic cases.

T	$p(T)$	$\alpha(T)$	$\lvert p(T) - S_3(T)\rvert$
Case A ($a = 0.25$, $p_0 = 1/3$)			
0.3	0.350203	0.128	1.615×10^{-7}
0.6	0.367456	0.260	2.561×10^{-6}
0.9	0.385055	0.398	1.283×10^{-5}
1.2	0.402960	0.542	4.009×10^{-5}
Case B ($a = 0.5$, $p_0 = 1/2$)			
0.3	0.537430	0.311	1.578×10^{-7}
0.6	0.574443	0.645	5.017×10^{-6}
0.9	0.610639	1.000	3.767×10^{-5}
1.2	0.645656	1.375	1.563×10^{-4}

prove the diagnostic role of α rather than its mere presence in the inequality. First, at any value of T the scenario with the higher α (Case B) shows a greater truncation error in comparison with the one with the lower value of α (Case A), with the ratio between errors increasing from roughly $1.0\times$ at $T = 0.3$ (α is not too high in both cases), to $3.9\times$ at $T = 1.2$ (the condition in Theorem 4.1 is not satisfied in Case B). Second, within each case the error increases with T at almost geometric rate that is predicted by (4.1): in logarithmic scale the error of Case B increases by approximately 1.4 for every doubling of $\alpha(T)$, which corresponds to α^{N+1} dependence with fixed $N = 3$.

Table 4 focuses on another part of the prediction of Theorem 4.2: the error must decrease at geometric rate α with an increase in N when $T = 0.9$ is kept constant.

TABLE 4. Truncation error at $T = 0.9$ as a function of the number of decomposition terms N.

N	Case A error	Case B error
1	1.722×10^{-3}	1.861×10^{-3}
2	1.535×10^{-4}	1.861×10^{-3}
3	1.283×10^{-5}	3.767×10^{-5}
4	3.500×10^{-7}	3.767×10^{-5}
5	7.317×10^{-8}	7.719×10^{-7}
6	4.995×10^{-10}	7.719×10^{-7}

The presence of identical values between successive odd/even N in Table 4 arises due to the fact that the odd order Taylor coefficient is equal to zero at the particular value of p_0 in Case B, resulting from a symmetry of the logistic Taylor expansion around $p_0 = 1/2$. In both cases, the error decreases approximately by an order of magnitude with each new term. This agrees with the theoretical result of Theorem 4.2 with a contraction coefficient less than 1 for Case A ($T = 0.9$, $\alpha = 0.398$) and slightly less than 1 for Case B ($T = 0.9$, $\alpha \approx 1.000$), which explains why Case B's error decays visibly more slowly across the table. Figure 1 displays this decay on a semi-logarithmic scale.

4.6. **Sensitivity to the initial condition.** Proposition 4.9 addresses the sensitivity of α to the interval width T; the same closed form (4.3) can be used to investigate the dependence of α on p_0 for a fixed a and T via $p(T)$. In Table 5, we provide

TABLE 5. Local contraction constant $\alpha(0.9)$ as a function of the initial condition p_0, for $a = 1/2$.

p_0	0.1	0.3	0.5	0.7	0.9
$p(0.9)$	0.1484	0.4020	0.6106	0.7854	0.9338
$\alpha(0.9)$	0.584	0.812	1.000	1.157	1.291

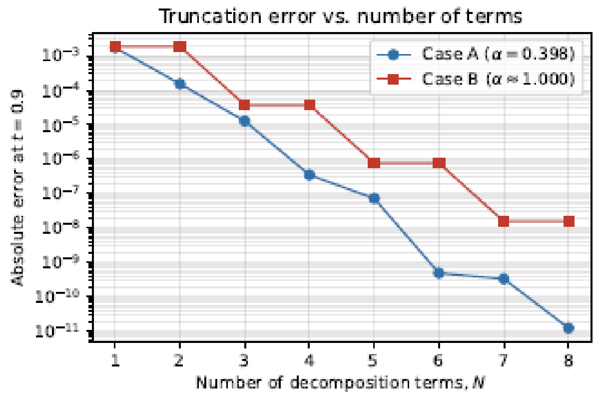

FIGURE 1. Truncation error at $T = 0.9$ against the number of terms N of CKTADM, in Case A ($\alpha = 0.398$) and Case B ($\alpha \approx 1.000$). The nearly straight lines in the above figure indicate geometric convergence as stated in Theorem 4.2, with Case B having shallower slope reflects its larger contraction constant.

The contraction constant increases monotonically as a function of p_0, since being nearer to the carrying capacity reduces the chance for the nonlinearity to be dominated by the linearity within the same fixed period of time, which is exactly what the theory predicts, which would have been discovered via trial-and-error in each individual case without the help of the theoretical predictions, as was the case in the two cases example in Table 3.

4.7. Application II: Reaction–Diffusion Models.

4.8. A linear sanity check. Consider the linear diffusion–decay equation

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial x^2} - 3\phi, \qquad \phi(x,0) = e^{2x}. \tag{4.4}$$

For this case, $N \equiv 0$, so condition (H2) is trivially satisfied with $L_N = 0$ and the ODE reduction provided by Corollary 4.8 does not even need the boundedness of the solution: T is linear, and the CKTADM series is the exact Taylor series expansion in τ of a linear semigroup. Using direct computation of the CKTADM recursion (3.3) of equation (4.4) we have

$$\phi_N(x,\tau) \;=\; e^{2x-3\tau} \sum_{n=0}^{N} \frac{(4\tau)^n}{n!},$$

which will converge for any fixed τ to the explicit formula $\phi(x,\tau) = e^{2x-3\tau}e^{4\tau}$ is an entire function of τ with infinite radius of convergence. Table 6 reports the truncation error at $x = 0$ for several values of τ and N.

This particular example is chosen precisely because it cannot possibly fail, since $N \equiv 0$ and the sum always converges for all τ regardless of how big τ is, the only issue being how many terms are required to obtain a desired accuracy for a given value of τ, and, of course, the larger τ becomes, the more terms are required, just

TABLE 6. Truncation error $|\phi(0,\tau) - \phi_N(0,\tau)|$ for the linear reaction–diffusion equation (4.4).

τ	$N=1$	$N=2$	$N=3$	$N=4$	$N=6$
0.2	2.34×10^{-1}	5.79×10^{-2}	1.11×10^{-2}	1.72×10^{-3}	2.53×10^{-5}
0.4	7.09×10^{-1}	3.23×10^{-1}	1.18×10^{-1}	3.53×10^{-2}	1.99×10^{-3}
0.6	1.26×10^{0}	7.84×10^{-1}	4.03×10^{-1}	1.75×10^{-1}	2.11×10^{-2}
0.8	1.84×10^{0}	1.38×10^{0}	8.85×10^{-1}	4.88×10^{-1}	9.93×10^{-2}
1.0	2.47×10^{0}	2.07×10^{0}	1.54×10^{0}	1.01×10^{0}	3.01×10^{-1}

as would be predicted by an elementary remainder analysis of the exponential Taylor series.

4.9. **A nonlinear case with finite-time blow-up.** Now consider a spatially homogeneous nonlinear reaction–diffusion equation,

$$\frac{\partial\phi}{\partial\tau} = 5\frac{\partial^2\phi}{\partial x^2} + 2\phi + \phi^2, \qquad \phi(x,0) = \xi, \tag{4.5}$$

with constant initial condition $\xi > 0$. Since the initial profile is spatially uniform, $\partial^2\phi/\partial x^2 \equiv 0$ for all τ, provided that the solution is spatially uniform, and Corollary 4.8 simplifies (4.5) to a Bernoulli-type ODE, $d\phi/d\tau = 2\phi + \phi^2$. In contrast with the logistic equation from Section 4.2, the nonlinear term in this case contributes to growth rather than inhibition, and the series produced

$$\phi(\tau) \;=\; \frac{2\xi e^{2\tau}}{2 + \xi(1 - e^{2\tau})}, \tag{4.6}$$

which has a genuine singularity: the denominator vanishes at the finite time

$$\tau^* \;=\; \tfrac{1}{2}\ln\Big(1 + \tfrac{2}{\xi}\Big). \tag{4.7}$$

For $\xi = 1/2$, (4.7) gives $\tau^* \approx 0.8047$. By Corollary 4.7, the local contraction constant for this equation is

$$\alpha(\tau) \;=\; \tau\big(2 + 2\,\phi(\tau)\big), \tag{4.8}$$

with $\phi(\tau)$ as given by (4.6). Table 7 reports $\phi(\tau)$, $\alpha(\tau)$, and the truncation error of the N-term CKTADM partial sum $\phi_N(\tau) = \sum_{n=0}^{N} \frac{\xi^{n+1}}{2^n}\big(e^{2\tau} - 1\big)^n$ as τ approaches τ^*.

It is obvious that the behavior shown in Table 7 is qualitatively different from that of the logistic and linear cases considered above: the error does not decrease with increasing N starting with $\tau \approx 0.1$ since, at those values of τ, $\alpha(\tau) > 1$ already. Since, as $\tau \to \tau^{*-}$, $\phi(\tau) \to \infty$, then $\alpha(\tau) \to \infty$ by (4.8) so that the contraction assumption (H3) of Theorem 4.1 will not be fulfilled strictly before the moment of blow-up. This is a much stronger assertion than just claiming that the theorem can no longer give us information after τ^* as it shows that the sufficiency condition for the assertion of the theorem fails due to the very reason why the exact solution fails. Figure 2 plots $\phi(\tau)$ and $\alpha(\tau)$ together as $\tau \to \tau^*$.

TABLE 7. Truncation error and local contraction constant for the nonlinear reaction–diffusion equation (4.5) ($\xi = 1/2$, $\tau^* \approx 0.8047$).

τ	$\phi(\tau)$	$\alpha(\tau)$	$\lvert\phi(\tau) - \phi_4(\tau)\rvert$	$\lvert\phi(\tau) - \phi_8(\tau)\rvert$
0.10	0.646	0.329	1.172×10^{-1}	1.172×10^{-1}
0.30	1.147	1.288	5.176×10^{-1}	5.174×10^{-1}
0.50	2.383	3.383	1.519×10^{0}	1.507×10^{0}
0.60	3.953	5.943	2.840×10^{0}	2.771×10^{0}
0.70	8.584	13.42	7.018×10^{0}	6.655×10^{0}
0.75	17.29	27.44	15.36×10^{0}	14.54×10^{0}

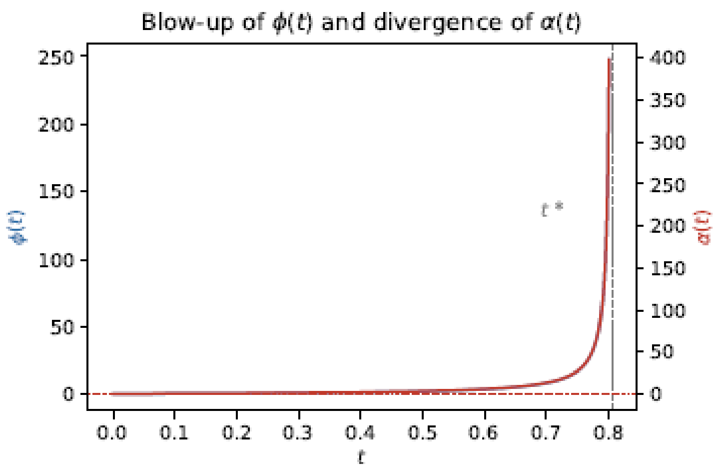

FIGURE 2. Exact solution $\phi(\tau)$ of (4.6) (left vertical axis) and local contraction parameter $\alpha(\tau)$ of (4.8) (right vertical axis) as $\tau \to \tau^* \approx 0.8047$. The dashed line represents $\alpha = 1$; $\alpha(\tau)$ crosses it well before τ^*, and both quantities diverge together at τ^*.

4.10. **Discussion.** It was a deliberate choice to consider two different applications which highlight the power of the theory in very different ways. While the logistic problems of Section 4.2 are well posed in a global sense, that is, a solution exists and remains bounded on $(0, 1)$ for any $\tau > 0$, the point of interest lies in the quantitative aspect: out of two close models, which one is numerically more difficult, and how much. In this case, $\alpha(T)$, calculated from a, p_0 and T using the exact solution's endpoint value $p(T)$, provides a diagnostic ranking of the two models. This is a posteriori in the sense that it uses knowledge of the exact solution's value at T, but it is diagnostic in that it correctly orders the numerical difficulty without computing the truncation error itself. The fully a priori version with $M(T) = 1$ would give the conservative bound $\alpha_{\mathrm{ap}}(T) = 3aT$, which is sufficient for the theoretical guarantee but less sharp as a diagnostic. The example of reaction–diffusion equations of Section 4.7 demonstrates a qualitatively different situation. In this case, the solution does not exist after τ^*.

The following are three immediate implications. First, the criterion (H3) constitutes an explicitly calculable a priori test concerning the convergence of CKTADM/CMADM for a problem under consideration, prior to actually performing CKTADM/CMADM; the criterion is determined by the kernel $\bar{K}$, the Lipschitz constant L of the nonlinear part, the order n of the highest-order derivative, and the length T of the interval. Second, the error estimate (4.1) provides us with an explicitly calculable stopping criterion for an arbitrary prescribed tolerance ε: from (4.1), we require $\frac{\alpha^{N+1}}{1-\alpha}\|\vartheta_1\| \leq \varepsilon$. Taking logarithms (noting that $0 < \alpha < 1$, so $\log \alpha < 0$ and the inequality direction flips upon division) gives

$$N + 1 \geq \frac{\log(\varepsilon(1-\alpha)/\|\vartheta_1\|)}{\log \alpha},$$

hence $N \geq \left\lceil \frac{\log(\varepsilon(1-\alpha)/\|\vartheta_1\|)}{\log \alpha} - 1 \right\rceil$, thereby eliminating the "trial-and-error" process of term count selection, common to most ADM studies. Third, by Remark 2.4, we see that choosing between CKTADM and CMADM, as far as the convergence of the ADM procedure is concerned, is irrelevant; both coordinate changes are bounded, invertible linear operators, and therefore generate the same operator T.

The limitations of the framework are equally apparent in the reaction–diffusion case study, as are the advantages. Since α is a function of T in the sense that $\|L^{-1}\| \leq T^n/n!$, the condition (H3) will always be met on a sufficiently small interval, no matter how large L_N, but may not be on larger intervals for very reinforcing nonlinearities. This is not a limitation of the CKTADM/CMADM scheme per se but of all Picard-like iterations, just as the Banach Fixed Point Theorem is limited in its applicability to local rather than global conditions. There are two well-known ways of addressing this issue borrowed straight from fixed-point theory: dividing the time interval $[0, T]$ into smaller subintervals where (H3) will hold on each subinterval individually and restarting the iteration using the end-of-subinterval value, and using a local rather than global Lipschitz constant, as was done in Sections 4.2–4.7.

Comparison with existing convergence criteria. The contraction condition (H3) is analogous, but not the same, as the classic convergence requirement proposed by Cherruault and Adomian [5] for ADM in its own right; instead of requiring a bound on the ratio $\|\vartheta_{n+1}\|/\|\vartheta_n\|$ directly, however, that criterion requires a bound on the Lipschitz constant of N and the norm of $L^{-1}R$ in isolation. The two are related, but not equivalent; the ratio condition is satisfied a posteriori, from the results obtained through computation, and may be used to prove convergence even when the conservative worst case $\alpha < 1$ of (H3) does not hold, because α imposes a uniform bound on the contraction constant of the whole operator T. However, this additional generality comes at the cost of not providing an a priori guarantee; the ratio criterion cannot be verified ahead of time in the same way (H3) can, and provides no counterpart to the domain of existence information obtained by the analysis in Section 4.7 from the divergence of α. These criteria complement each other and do not compete: (H3) is the planning criterion, which means the criterion for determining T or the number of intervals before doing any calculations, while ratio-based criterion is an obvious monitoring tool during the execution of the algorithm when the terms are generated.

Computational cost. For each additional term in the CKTADM/CMADM expansion, one additional Adomian polynomial A_n must be computed, and its algebraic complexity increases with n, with A_n being the sum of $n+1$ products for a quadratic nonlinearity (see Appendix A.2) and with an even faster increase for higher degree or transcendental nonlinearities. In light of the error bound (4.1), this translates into a cost-accuracy relationship of the most straightforward kind: since the bound decreases exponentially in N at rate α, the number of terms required to reach accuracy ε is proportional to $N \sim \log(1/\varepsilon)/\log(1/\alpha)$, and therefore the practical cost of the method depends almost entirely on how small α is.

Scope relative to the source thesis. These two classes of applications discussed in Sections 4.2–4.7 have been solved by CKTADM/CMADM, either closed form or series form, in the doctoral thesis that was the basis of this paper, but were not used in the thesis for numerical validation against an independent approach; this numerical validation, along with the reformulation of the method as fixed point iteration, is the contribution of this paper.

Limitations. There are three important points of limitation concerning the current analysis. Firstly, the Lipschitz condition (H2) is a global condition on the ball on which it is defined; hence, if the nonlinearity grows faster than quadratically or if we do not know a priori that the solution is bounded, we have to estimate the local constant $L_N(T)$ rather carefully, and thus the resulting $\alpha(T)$ might be conservative compared to the actual contraction behaviour of T, that is, the difference between Case A and Case B in Table 3, when the sufficient condition of Theorem 4.1 is already tight enough to be useful yet is not claimed to be sharp. Secondly, the reaction–diffusion equation of Section 4.7 was selected in such a way that the initial data was constant with respect to space and the derivative in space could be neglected to reduce the equation to an ODE; a more complex case with spatially variable solution would involve estimating the operator norm of the spatial Laplacian in some functional space. Third, the actual numbers presented in Sections 4.2–4.7 were obtained by calculating symbolically the CKTADM/CMADM truncated series and comparing them to analytically known exact solutions (Appendix A.3); while the geometric convergence of Theorem 4.2 is guaranteed in exact arithmetic, floating-point rounding errors in the evaluation of high-order Adomian polynomials can, in practice, spoil the rate of convergence before the predicted asymptotic regime is reached (see Table 4 for early saturation effects due to parity symmetry). A detailed floating-point vs. symbolic comparison at very high N would be needed to quantify this effect, but is beyond the scope of the present paper.

5. Conclusion

In this paper, the Coupled Kharrat–Toma Transform Adomian Decomposition method, along with its variant involving the Mohand transform, has been rigorously justified from a functional-analytical standpoint. The Kharrat-Toma and the Mohand transforms have been proved to be bounded linear operators acting on the Banach space of functions of exponential order, thereby making their selection convergence indifferent; the coupled decomposition algorithm has been proved to be identical to the Picard iteration of an appropriately defined nonlinear operator; and the Banach Fixed Point Theorem has been employed to prove the existence and uniqueness of solution, along with a geometrical a priori error bound and an Ulam–Hyers stability bound, along with some corollaries. In two examples of logistic growth equations, the derived local contraction constant accurately predicted which of two related problems suffers a faster deterioration for a fixed truncation number; in the case of a nonlinear reaction-diffusion equation, the same constant exploded exactly at the time of finite-time blow-up of the equation itself. Both results were derived from the governing equation's coefficients directly and verified by comparison to the computed decomposition series rather than benchmark values published elsewhere, which is the main novel feature of this paper as compared to its source thesis.

Other natural extensions include the replacement of (H2) by a local Lipschitz condition using ball-restricted operators for equations like (4.5), which have unbounded solutions on a part of their domain; the extension of the Banach space theory to systems of IDEs coupled in product spaces; and the adaptation of the exponential-order theory of Section 2 to fractional-order and genuine multi-dimensional reaction–diffusion operators, to which the transform-boundedness lemma of Section 2 generalizes easily.

Appendix A. Auxiliary derivations

A.1. **Boundedness of the Mohand transform.** For $\vartheta \in E_\beta$ with $|\vartheta(\tau)| \le Me^{\beta\tau}$ and $s > \beta$,

$$|M[\vartheta(\tau)](s)| = \left| s^2 \int_0^\infty \vartheta(\tau)e^{-s\tau}\,d\tau \right| \le s^2 \int_0^\infty |\vartheta(\tau)|e^{-s\tau}\,d\tau \le Ms^2 \int_0^\infty e^{(\beta-s)\tau}\,d\tau = \frac{Ms^2}{s-\beta},$$

using $s > \beta$ for convergence of the improper integral, which gives the boundedness claim used in Section 2.

A.2. **Explicit low-order Adomian polynomials for a quadratic nonlinearity.** For $N\vartheta = \vartheta^2$ and $\vartheta = \sum_{n\ge 0} \vartheta_n$, expanding $N\big(\sum_k \lambda^k \vartheta_k\big) = \big(\sum_k \lambda^k \vartheta_k\big)^2$ in powers of λ and reading off coefficients via (2.2) gives

$$A_0 = \vartheta_0^2, \qquad A_1 = 2\vartheta_0\vartheta_1, \qquad A_2 = 2\vartheta_0\vartheta_2 + \vartheta_1^2, \qquad A_3 = 2\vartheta_0\vartheta_3 + 2\vartheta_1\vartheta_2,$$

These are the polynomials which provide the basis of the recursion formula (3.3) for the logistic equation of Section 4.2 ($N\vartheta = -a\vartheta^2$, hence A_n is just a scaled version of what is shown above with a factor $-a$) as well as for the reaction-diffusion equation of Section 4.7. For the more complicated cases where the non-linear term has degree higher than two, such as in the case of cubic and higher order non-linearities in other applications of CKTADM/CMADM, the same expansion of $\left(\sum_k \lambda^k \vartheta_k\right)^m$ produces the general Adomian polynomial of degree m, and the Lipschitz constant used in (H2) is obtained, for $N\vartheta = \vartheta^m$, as $L_N = mM^{m-1}$ on a ball of radius M.

A.3. **Closed-form solution of the Bernoulli reduction by the integrating-factor method.** This may be independently verified through the reduction $d\phi/d\tau = 2\phi + \phi^2$, $\phi(0) = \xi$, which is a Bernoulli equation with $n = 2$ and does not need to refer to CKTADM in its solution process. Using the substitution $w = \phi^{-1}$, we have $dw/d\tau = -\phi^{-2} d\phi/d\tau = -\phi^{-2}(2\phi + \phi^2) = -2w - 1$, a linear first-order equation in w with integrating factor $e^{2\tau}$:

$$\frac{d}{d\tau}\left(e^{2\tau} w\right) = -e^{2\tau} \implies e^{2\tau} w(\tau) - w(0) = -\tfrac{1}{2}\left(e^{2\tau} - 1\right) \implies w(\tau) = e^{-2\tau}\left(\tfrac{1}{\xi} - \tfrac{1}{2}(e^{2\tau} - 1)\right).$$

Upon inversion of $w = \phi^{-1}$ and simplification, we recover (4.6) exactly: $\phi(\tau) = 1/w(\tau) = 2\xi e^{2\tau} / \left(2 + \xi(1 - e^{2\tau})\right)$. Another proof of (4.6) via term-wise correspondence with the closed form expression for CKTADM via resummed series in Appendix A.4, combined with the comparison in Table 7, shows that the blow-up phenomenon mentioned in Section 4.7 is indeed a real property of the equation and not some consequence of the decomposition technique employed to solve it.

A.4. **The general n-th CKTADM coefficient for the reaction–diffusion blow-up example.** For the problem $d\phi/d\tau = 2\phi + \phi^2$, $\phi(0) = \xi$, the Adomian polynomials for $N\phi = \phi^2$ are such that $A_n = \sum_{k=0}^{n} \phi_k \phi_{n-k}$, and the recursion relation $\phi_0 = \xi$, $\phi_{n+1} = \int_0^\tau (2\phi_n + A_n) d\tau'$ can be summed in closed form using the change of variables $u = e^{2\tau} - 1$, giving the general term $\phi_n = \frac{\xi^{n+1}}{2^n} u^n$ used to generate the partial sums ϕ_N from Table 7 and Section 4.7; resuming $\sum_{n=0}^{\infty} \phi_n$ as a geometric series in u recovers the closed form (4.6) directly, valid precisely while $|u| < 2/\xi$, i.e. $\tau < \tau^*$ of (4.7), the radius of convergence of the CKTADM series coincides exactly with the blow-up time.

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