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RADIAL DISTANCE FUNCTIONS FOR PARTITIONING SOFT SPACES

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Abstract. In this paper, we introduce a scalar measure for soft sets and define reference-based radial distances from a fixed reference soft set to every soft set in a soft space. We establish their fundamental properties, including non-negativity, identity of indiscernibles, and conditional monotonicity. The normalized radial distances induce a partition of the soft space into clusters determined by prescribed distance intervals. We formulate the partitioning task as a computational problem, develop an algorithm for constructing the induced partition, and establish its worst-case time and space complexities. The proposed framework is illustrated through a course outcomes assessment example

*Keywords.*Soft sets, Soft space, Measure of a soft set, Reference soft set, Radial distance.
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1. Introduction

Molodtsov [1] introduced soft set theory as a general mathematical framework for representing uncertainty, vagueness, and incomplete information. Building upon this foundation, Maji et al. [2] formalized the fundamental operations and properties of soft sets and demonstrated their utility in decision-making problems. M.I.Ali et al. [3] introduced some restricted union, restricted intersection, restricted difference, restricted complement, and proved De Morgan's laws based on these operations. Owing to this ability to handle uncertainty in a flexible manner, soft set theory has attracted considerable attention in mathematical modeling, decision-making, and several related fields. Rasuli [4] presented normal soft int-groups together with homomorphisms and isomorphisms of soft int-groups. Aykut et al. [5] formalized the concept of the soft int-subgroup generated by an α-subset of a group. Recent work by Alcantud et al. [6] provides a comprehensive review of major developments in soft set theory, including new theoretical extensions, diversified applications, and hybrid models that integrate soft sets with

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other uncertainty-handling frameworks.

Distance measures play an important role in data-mining tasks such as clustering, classification, and similarity search. They provide a useful framework for identifying similarities and dissimilarities in patterns across various domains. Comprehensive reviews of classical distance and similarity measures can be found in Subbarayan et al. [7] and Blanco-Mallo et al. [8]. Motivated by their wide applicability, several researchers have extended these measures to the soft set framework. Majumdar and Samanta [9] introduced Euclidean and Hamming distances, along with their normal forms, for pairs of soft sets. They defined similarity measures using matrix representations. Kharal [10] later observed that matrix-based formulations may lead to inconsistencies in certain cases and developed set-theoretic distance and similarity measures that more effectively handle both disjoint and overlapping soft sets. Sulaiman and Mohamad [11] proposed a Jaccard-inspired similarity measure that makes use of intersection and union operations, offering a more interpretable and computationally efficient alternative.

Researchers applied the concept of distance between pair of soft sets to soft-set extensions like fuzzy soft sets [12], complex intuitionistic fuzzy soft sets [13], intuitionistic fuzzy soft sets [14], generalized intuitionistic fuzzy soft sets [15], Pythagorean fuzzy soft sets [16], hesitant fuzzy soft sets [17], complex fuzzy soft sets [18], interval-valued intuitionistic fuzzy soft sets [19], Fermatean fuzzy-type soft sets [20], interval-valued Q-neutrosophic soft sets [21] and many more.

Much of the recent work in this area has focused on developing distance and similarity measures for these extensions rather than for classical soft sets. Many studies propose normalized Hamming, Euclidean, Hausdorff, Chebyshev, and related distance-based similarity measures for soft-set extensions, with applications largely centered on decision-making problems.

However, even without these extensions, distance measures can be applied effectively to classical soft sets, particularly in the analysis of survey data. Moreover, in most existing studies, distance and similarity measures for classical soft sets as well as their extensions are computed pairwise. This pairwise framework supports clustering of soft sets with similar characteristics and forms the basis for many machine learning and pattern recognition techniques. In some decision-making contexts, however, it is more meaningful to compare each soft set with a reference or ideal soft set that serves as a benchmark or target state rather than with other members of the collection. Such a perspective enables clustering and analysis to be performed according to the distances of soft sets from a predefined reference rather than through pairwise comparisons.

To address this gap, we propose a reference-based distance function, referred to as the *radial distance* for soft sets, and establish its theoretical properties. Using this distance, we partition the soft space into clusters, each consisting of soft sets whose distances from the reference soft set lie within a prescribed interval. This partitioning provides a new clustering framework for grouping soft sets according to their similarity to a chosen reference..

This paper is structured as follows: Section 2 provides basic definitions from soft set theory and reviews existing distance-based measures. Section 3 introduces a measure of a soft set and the corresponding measure-induced partition of the soft space. Section 4 introduces the proposed reference-based distance measure and establishes its key properties. Section 5 presents the partition of the soft space into clusters, develops an algorithm for constructing the induced partition, establishes its worst-case time and space complexities, and demonstrates the practical utility of the proposed framework through a course-outcomes assessment example.

2. Preliminaries

In this section some foundational definitions related to soft sets and distance measures are recalled. Throughout, U denotes a non-empty universal set and E a collection of parameters (or attributes) associated with U.

Definition 2.1. A *soft set* over U is a mapping $F\colon E \to 2^U$ that assigns to each parameter $e \in E$ a subset $F(e)$ of U. It is denoted by (F, E). Equivalently, a soft set (F, E) can be explicitly represented as

$$(F, E) = \big\{(e,\ F(e)) : e \in E\big\}.$$

The following definitions are adapted from Maji et al. [2].

Definition 2.2. Let (F_1, E) and (F_2, E) be two soft sets over a universal set U. Then (F_1, E) is said to be a *soft subset* of (F_2, E) if

$$F_1(e) \subseteq F_2(e), \qquad \forall\, e \in E.$$

The relation is denoted by

$$(F_1, E) \mathrel{\widetilde{\sqsubseteq}} (F_2, E).$$

Definition 2.3. Let (F_1, E) and (F_2, E) be two soft sets.

(1) The *intersection* of (F_1, E) and (F_2, E) is the soft set (F, E), where $F : E \to 2^U$ is defined by

$$F(e) = F_1(e) \cap F_2(e), \qquad \forall\, e \in E.$$

It is denoted by

$$(F_1, E) \mathbin{\widetilde{\sqcap}} (F_2, E) = (F, E).$$

(2) The *union* of (F_1, E) and (F_2, E) is the soft set (F, E), where $F : E \to 2^U$ is defined by

$$F(e) = F_1(e) \cup F_2(e), \qquad \forall\, e \in E.$$

It is denoted by

$$(F_1, E) \mathbin{\widetilde{\sqcup}} (F_2, E) = (F, E).$$

Definition 2.4. Let (F, E) be a soft set. The *complement* of (F, E) is the soft set (F^c, E), where

$$F^c(e) = U \setminus F(e), \qquad \forall\, e \in E.$$

Definition 2.5. A soft set (F, E) is said to be a *null soft set* if

$$F(e) = \emptyset, \qquad \forall\, e \in E.$$

It is denoted by $(F_\emptyset, E)$.

Definition 2.6. A soft set (F, E) is called an *absolute soft set* if

$$F(e) = U, \qquad \forall\, e \in E.$$

It is denoted by (F_U, E).

Definition 2.7. Two soft sets (F_1, E) and (F_2, E) are said to be *equal* if each is a soft subset of the other. In this case, we write

$$(F_1, E) = (F_2, E).$$

Definition 2.8. Two soft sets (F_1, E) and (F_2, E) are said to be *disjoint* if

$$(F_1, E) \,\widetilde{\sqcap}\, (F_2, E) = (F_\emptyset, E).$$

Definition 2.9. A function $A^n : [0, 1]^n \to [0, 1]$ is called an *aggregation function* [22] if it satisfies the following conditions:

(1) A^n is non-decreasing in each argument.
(2) $A^n(0, 0, \ldots, 0) = 0$ and $A^n(1, 1, \ldots, 1) = 1$.

3. Measure of Soft Sets and Partition of Soft Spaces

3.1. **Measure of soft sets.** Let U be a finite non-empty universal set. Throughout this section, let $m : 2^U \to [0, 1]$ be a measure satisfying the following properties:

(1) $m(\emptyset) = 0$.
(2) If $X \subseteq Y$, then $m(X) \le m(Y)$.

Let $(F, E) = \{(e, F(e)) : e \in E\}$ be a soft set, and let I_E denote the index set of E. For each $e_i \in E$, where $i \in I_E$, the set $F(e_i)$ is an element of 2^U. Hence, the measure m assigns to each $F(e_i)$ a value $m(F(e_i)) \in [0, 1]$.

The measure of the soft set (F, E) is obtained by aggregating the measures of its parameterized subsets. Let

$$\theta : [0, 1]^n \to [0, 1]$$

be an aggregation operator. Then the vector

$$\big(m(F(e_1)), m(F(e_2)), \ldots, m(F(e_n))\big)$$

is mapped by θ to a real number in $[0, 1]$, which is taken as the measure of the soft set (F, E).

Throughout this section, the term *aggregation operator* refers to an operator that is monotone in each argument and satisfies

$$\theta(0, 0, \ldots, 0) = 0.$$

Definition 3.1. A function

$$m_\theta : S(U) \to [0,1]$$

defined by

$$m_\theta(F,E) = \theta\big(m(F(e_1)), m(F(e_2)), \ldots, m(F(e_n))\big), \tag{3.1}$$

for every soft set $(F,E) \in S(U)$, is called the *measure* of the soft set (F,E).

The aggregation operator θ may be chosen as the arithmetic mean, weighted arithmetic mean, maximum, minimum, or any other suitable aggregation operator. Consequently, the properties of the measure m_θ depend on the choice of θ. Therefore, the aggregation operator should be selected according to the nature and requirements of the application under consideration.

Remark 3.2. The choice of the aggregation operator has a significant influence on the measure m_θ and, consequently, on the measure-based partition of $S(U)$. Different aggregation operators emphasize different characteristics of a soft set and may therefore induce different partitions of $S(U)$.

Example 3.3. The arithmetic mean aggregation operator is defined by

$$\theta\big(m(F(e_1)), \ldots, m(F(e_n))\big) = \frac{1}{n}\sum_{i=1}^{n} m(F(e_i)).$$

The arithmetic mean combines the parameter-wise measures of a soft set by taking their average. It provides an overall measure of the soft set under the assumption that all parameters contribute equally to the evaluation. Consequently, it is appropriate when no parameter is considered more important than the others. However, this operator may not be suitable in situations where different parameters have different levels of importance or when the overall assessment is expected to depend primarily on the most or the least significant parameter. Furthermore, the arithmetic mean may not be appropriate when the values of $m(F(e_i))$ differ considerably, since the resulting measure may not adequately reflect the variation among the parameter-wise measures.

Example 3.4. The weighted arithmetic mean aggregation operator is defined by

$$\theta\big(m(F(e_1)), m(F(e_2)), \ldots, m(F(e_n))\big) = \sum_{i=1}^{n} w_i\, m(F(e_i)),$$

where

$$\sum_{i=1}^{n} w_i = 1, \qquad w_i \geq 0, \quad i = 1, 2, \ldots, n.$$

Unlike the arithmetic mean, this aggregation operator allows different parameters to be assigned different weights according to their relative importance. Consequently, it is suitable when the parameters of a soft set do not contribute equally to the overall evaluation or when certain parameters are considered more significant than others.

Furthermore, if the decision-maker wishes to evaluate the contribution of a single parameter, a weight of one may be assigned to that parameter and weights of zero to all the remaining parameters. In this case, the measure of the soft set reduces to the measure of the selected parameter.

Example 3.5. The maximum aggregation operator is defined by

$$\theta\big(m(F(e_1)), m(F(e_2)), \ldots, m(F(e_n))\big) = \max_{e_i \in E} m(F(e_i)).$$

The maximum aggregation operator evaluates a soft set by selecting the largest of its parameter-wise measures. It is appropriate when the assessment is intended to depend solely on the most significant parameter, that is, the parameter $e_i \in E$ for which $m(F(e_i))$ attains its maximum value. However, this operator may not be suitable when all parameters are expected to contribute to the overall evaluation, since it completely ignores the contributions of the remaining parameters.

Example 3.6. The minimum aggregation operator is defined by

$$\theta\big(m(F(e_1)), m(F(e_2)), \ldots, m(F(e_n))\big) = \min_{e_i \in E} m(F(e_i)).$$

The minimum aggregation operator evaluates a soft set by selecting the smallest of its parameter-wise measures. It is appropriate when the assessment is intended to depend on the least significant parameter, that is, the parameter $e_i \in E$ for which $m(F(e_i))$ attains its minimum value. This operator is particularly useful in applications where the overall evaluation is determined by the weakest or the most critical parameter. However, it may not be suitable when all parameters are expected to contribute to the overall evaluation, since it completely ignores the contributions of the remaining parameters.

Several other aggregation operators may also be employed to construct measures of soft sets. The choice of the aggregation operator depends on the nature of the problem and the objectives of the decision-making.

Example 3.7. Let $U = \{u_1, u_2, u_3\}$ and $E = \{e_1, e_2\}$. Consider the soft set

$$(F, E) = \{(e_1, \{u_2, u_3\}), (e_2, \{u_3\})\}.$$

Let the measure $m : 2^U \to [0, 1]$ be defined by

$$m(A) = \frac{|A|}{|U|}, \qquad A \subseteq U.$$

Then

$$m(F(e_1)) = \frac{2}{3}, \qquad m(F(e_2)) = \frac{1}{3}.$$

If θ is the arithmetic mean aggregation operator, then

$$m_\theta(F, E) = \frac{1}{2}\left(\frac{2}{3} + \frac{1}{3}\right) = \frac{1}{2}.$$

Hence, the measure of the soft set (F, E) is $\frac{1}{2}$.

Similar computations may be carried out using the other aggregation operators discussed above, yielding different measures depending on the choice of the aggregation operator.

Proposition 3.8. *Let $\theta : [0,1]^n \to [0,1]$ be an aggregation operator, and let $m_\theta : S(U) \to [0,1]$ be the measure of a soft set defined by (3.1). Then the following statements hold.*

(1) $m_\theta(F_\emptyset, E) = 0$.

(2) *If* $(F,E) \,\widetilde{\sqsubseteq}\, (G,E)$, *then* $m_\theta(F,E) \le m_\theta(G,E)$.

Proof. (1) We have

$$(F_\emptyset, E) = \{(e, F(e)) : F(e) = \emptyset,\ \forall\, e \in E\}.$$

Since $m(\emptyset) = 0$, we have $m(F(e)) = 0$ for every $e \in E$. Hence,

$$m_\theta(F_\emptyset, E) = \theta(0, 0, \ldots, 0) = 0.$$

(2) Suppose $(F,E) \,\widetilde{\sqsubseteq}\, (G,E)$. Then $F(e) \subseteq G(e)$ for every $e \in E$. Since m is monotone, $m(F(e)) \le m(G(e))$ for every $e \in E$. As θ is non-decreasing,

$$\theta\big(m(F(e_1)), \ldots, m(F(e_n))\big) \le \theta\big(m(G(e_1)), \ldots, m(G(e_n))\big),$$

which implies

$$m_\theta(F,E) \le m_\theta(G,E).$$

□

3.2. **Partition of the Soft Space Based on the Measure of Soft Sets.** We now partition the soft space into disjoint classes based on the measure of soft sets. To this end, we define the following relation on the soft space $S(U)$.

Definition 3.9. Let $S(U)$ be the soft space. Define a relation $\underset{m_\theta}{\simeq}$ on $S(U)$ by

$$(F,E) \underset{m_\theta}{\simeq} (G,E)$$

if and only if

$$m_\theta(F,E) = m_\theta(G,E).$$

Proposition 3.10. *The relation $\underset{m_\theta}{\simeq}$ on $S(U)$ defined by*

$$(F,E) \underset{m_\theta}{\simeq} (G,E)$$

if and only if

$$m_\theta(F,E) = m_\theta(G,E),$$

is an equivalence relation.

Proof. (i) *Reflexive.* Since $m_\theta(F,E) = m_\theta(F,E)$, we have

$$(F,E) \underset{m_\theta}{\simeq} (F,E).$$

(ii) *Symmetric.* Suppose

$$(F,E) \underset{m_\theta}{\simeq} (G,E).$$

Then

$$m_\theta(F,E) = m_\theta(G,E).$$

Hence,

$$m_\theta(G,E) = m_\theta(F,E),$$

which implies

$$(G, E) \underset{m_\theta}{\doteq} (F, E).$$

(iii) *Transitive.* Suppose

$$(F, E) \underset{m_\theta}{\doteq} (G, E) \quad \text{and} \quad (G, E) \underset{m_\theta}{\doteq} (H, E).$$

Then

$$m_\theta(F, E) = m_\theta(G, E)$$

and

$$m_\theta(G, E) = m_\theta(H, E).$$

Therefore,

$$m_\theta(F, E) = m_\theta(H, E),$$

which implies

$$(F, E) \underset{m_\theta}{\doteq} (H, E).$$

□

Corollary 3.11. *The equivalence relation* $\underset{m_\theta}{\doteq}$ *partitions the soft space* $S(U)$ *into disjoint equivalence classes, called the* blocks of the partition*. The equivalence class containing* (F, E) *is given by*

$$[(F, E)]_{\underset{m_\theta}{\doteq}} = \{(G, E) \in S(U) : m_\theta(F, E) = m_\theta(G, E)\}.$$

Thus, each block of the partition consists of all soft sets having the same measure m_θ*.*

4. Reference-Based Radial Distance Functions

In this section, we introduce the concept of reference-based radial distance functions on a soft space. A fixed soft set is chosen as the reference soft set, and the distances of all other soft sets are computed relative to it. The resulting distance of a soft set from the reference soft set is referred to as its radial distance.

Throughout this section, a soft set (F, E) is denoted by $\mathcal{S}$. In particular, the fixed reference soft set is denoted by $\mathcal{S}_r$, while arbitrary soft sets are denoted by $\mathcal{S}_i$, $\mathcal{S}_j$, and so on.

Definition 4.1. Let $\mathcal{S}_r \in S(U)$ be a fixed reference soft set. A radial distance on $S(U)$ is a function $r_{\mathcal{S}_r} : S(U) \to [0, \infty)$ that assigns to each soft set $\mathcal{S}_i \in S(U)$ its distance from the reference soft set $\mathcal{S}_r$.

We now introduce the radial Hamming distance and the radial Euclidean distance, together with their normalized forms, and establish their basic properties.

Definition 4.2. Let $\mathcal{S} = (F, E)$ be a soft set in $S(U)$, and let $\mathcal{S}_r = (F_r, E)$ be a fixed non-empty reference soft set in $S(U)$. The *radial Hamming distance* is the function

$$H_{\mathcal{S}_r} : S(U) \to [0, \infty)$$

defined by

$$H_{\mathcal{S}_r}(\mathcal{S}) = \sum_{e \in E} |F(e) \triangle F_r(e)|, \tag{4.1}$$

where $|F(e) \triangle F_r(e)|$ denotes the cardinality of the symmetric difference $F(e) \triangle F_r(e)$.

Proposition 4.3. *Let $\mathcal{S} = (F, E)$ be a soft set in $S(U)$, and let $\mathcal{S}_r = (F_r, E)$ be a fixed reference soft set. Then the following statements hold.*

(1) $H_{\mathcal{S}_r}(\mathcal{S}) \geq 0$. *(Non-negativity)*
(2) $H_{\mathcal{S}_r}(\mathcal{S}) = H_{\mathcal{S}}(\mathcal{S}_r)$. *(Symmetry)*
(3) $H_{\mathcal{S}_r}(\mathcal{S}) = 0$ *if and only if* $\mathcal{S} = \mathcal{S}_r$. *(Identity of indiscernibles)*

Proof. (1) Since $|F(e) \triangle F_r(e)| \geq 0$ for every $e \in E$, we have $H_{\mathcal{S}_r}(\mathcal{S}) \geq 0$.

(2) Since $|F(e) \triangle F_r(e)| = |F_r(e) \triangle F(e)|$ for every $e \in E$, it follows that $H_{\mathcal{S}_r}(\mathcal{S}) = H_{\mathcal{S}}(\mathcal{S}_r)$.

(3) Suppose $H_{\mathcal{S}_r}(\mathcal{S}) = 0$. Then $|F(e) \triangle F_r(e)| = 0$ for every $e \in E$. Hence, $F(e) = F_r(e)$ for every $e \in E$, and therefore $\mathcal{S} = \mathcal{S}_r$.
Conversely, if $\mathcal{S} = \mathcal{S}_r$, then $F(e) = F_r(e)$ for every $e \in E$. Hence, $H_{\mathcal{S}_r}(\mathcal{S}) = 0$.

□

Remark 4.4. For any two soft sets $\mathcal{S}_i, \mathcal{S}_j \in S(U), H_{\mathcal{S}_r}(\mathcal{S}_i) = H_{\mathcal{S}_r}(\mathcal{S}_j)$ does not necessarily imply that $\mathcal{S}_i = \mathcal{S}_j$.

Example 4.5. Let $U = \{u_1, u_2, u_3\}$ and $E = \{e_1, e_2\}$. Let

$$\mathcal{S}_r = (F_r, E) = \{(e_1, \{u_1, u_2\}), (e_2, \{u_2, u_3\})\}$$

be the reference soft set. Consider the soft sets

$$\mathcal{S}_i = (F_i, E) = \{(e_1, \{u_1\}), (e_2, \{u_2\})\}$$

and

$$\mathcal{S}_j = (F_j, E) = \{(e_1, \{u_2\}), (e_2, \{u_3\})\}.$$

Then

$$H_{\mathcal{S}_r}(\mathcal{S}_i) = 2 \quad \text{and} \quad H_{\mathcal{S}_r}(\mathcal{S}_j) = 2,$$

but

$$\mathcal{S}_i \neq \mathcal{S}_j.$$

Remark 4.6. The radial Hamming distance is not necessarily monotone with respect to the soft subset relation.

Example 4.7. Let $U = \{u_1, u_2\}$ and $E = \{e\}$. Let

$$\mathcal{S}_r = (F_r, E) = \{(e, \{u_2\})\}$$

be the reference soft set. Consider the soft sets

$$\mathcal{S}_1 = (F_1, E) = \{(e, \{u_1\})\}$$

and

$$\mathcal{S}_2 = (F_2, E) = \{(e, \{u_1, u_2\})\}.$$

Clearly,

$$(F_1, E) \mathrel{\widetilde{\sqsubseteq}} (F_2, E).$$

Moreover,

$$|F_1(e) \triangle F_r(e)| = 2 \quad \text{and} \quad |F_2(e) \triangle F_r(e)| = 1.$$

Hence,

$$H_{\mathcal{S}_r}(\mathcal{S}_1) = 2 \quad \text{and} \quad H_{\mathcal{S}_r}(\mathcal{S}_2) = 1.$$

Therefore,

$$H_{\mathcal{S}_r}(\mathcal{S}_1) > H_{\mathcal{S}_r}(\mathcal{S}_2),$$

even though

$$(F_1, E) \,\widetilde{\sqsubseteq}\, (F_2, E).$$

Thus, the radial Hamming distance is not monotone with respect to the soft subset relation.

Theorem 4.8. *Let $\mathcal{S}_i = (F_i, E)$ and $\mathcal{S}_j = (F_j, E)$ be two soft sets in $S(U)$, and let $\mathcal{S}_r = (F_r, E)$ be the reference soft set. If*

$$\mathcal{S}_r \,\widetilde{\sqsubseteq}\, \mathcal{S}_i \,\widetilde{\sqsubseteq}\, \mathcal{S}_j,$$

then

$$H_{\mathcal{S}_r}(\mathcal{S}_i) \le H_{\mathcal{S}_r}(\mathcal{S}_j).$$

That is, the radial Hamming distance is monotone whenever the reference soft set is a soft subset of both soft sets being compared.

Proof. Suppose

$$\mathcal{S}_r \,\widetilde{\sqsubseteq}\, \mathcal{S}_i \,\widetilde{\sqsubseteq}\, \mathcal{S}_j.$$

Then

$$F_r(e) \subseteq F_i(e) \subseteq F_j(e), \qquad \forall\, e \in E.$$

Hence,

$$F_i(e) \triangle F_r(e) \subseteq F_j(e) \triangle F_r(e),$$

and therefore

$$|F_i(e) \triangle F_r(e)| \le |F_j(e) \triangle F_r(e)|, \qquad \forall\, e \in E.$$

Summing over all $e \in E$, we obtain

$$\sum_{e \in E} |F_i(e) \triangle F_r(e)| \le \sum_{e \in E} |F_j(e) \triangle F_r(e)|,$$

which implies

$$H_{\mathcal{S}_r}(\mathcal{S}_i) \le H_{\mathcal{S}_r}(\mathcal{S}_j).$$

Thus, the radial Hamming distance is monotone under the given condition. □

Definition 4.9. Let $\mathcal{S}_r = (F_r, E)$ be a fixed reference soft set. The *normalized radial Hamming distance* is the function

$$\overline{H}_{\mathcal{S}_r} : S(U) \to [0, 1]$$

defined by

$$\overline{H}_{\mathcal{S}_r}(\mathcal{S}) = \frac{\displaystyle\sum_{e \in E} |F(e) \triangle F_r(e)|}{|U|\,|E|}, \tag{4.2}$$

where $|\cdot|$ denotes the cardinality of a set.

Proposition 4.10. *Let $\mathcal{S} = (F, E)$ be a soft set in $S(U)$, and let $\mathcal{S}_r = (F_r, E)$ be a fixed reference soft set. Then the following statements hold.*

(1) $\overline{H}_{\mathcal{S}_r}(\mathcal{S}) \geq 0.$ *(Non-negativity)*
(2) $\overline{H}_{\mathcal{S}_r}(\mathcal{S}) = \overline{H}_{\mathcal{S}}(\mathcal{S}_r).$ *(Symmetry)*
(3) $\overline{H}_{\mathcal{S}_r}(\mathcal{S}) = 0$ *if and only if* $\mathcal{S} = \mathcal{S}_r$. *(Identity of indiscernibles)*

Proof. The proof follows directly from Proposition 4.3 by dividing both sides of the defining equation by the positive constant $|U||E|$. Hence, the details are omitted. □

Theorem 4.11. *Let $\mathcal{S}_i = (F_i, E)$ and $\mathcal{S}_j = (F_j, E)$ be two soft sets in $S(U)$, and let $\mathcal{S}_r = (F_r, E)$ be the reference soft set. If*

$$\mathcal{S}_r \,\widetilde{\sqsubseteq}\, \mathcal{S}_i \,\widetilde{\sqsubseteq}\, \mathcal{S}_j,$$

then

$$\overline{H}_{\mathcal{S}_r}(\mathcal{S}_i) \leq \overline{H}_{\mathcal{S}_r}(\mathcal{S}_j).$$

.

Proof. By theorem 4.8,

$$H_{\mathcal{S}_r}(\mathcal{S}_i) \leq H_{\mathcal{S}_r}(\mathcal{S}_j).$$

Dividing both sides by the positive constant $|U||E|$, we obtain

$$\overline{H}_{\mathcal{S}_r}(\mathcal{S}_i) \leq \overline{H}_{\mathcal{S}_r}(\mathcal{S}_j).$$

Hence, the normalized radial Hamming distance is monotone under the given condition. □

Theorem 4.12. *Let $\mathcal{S} = (F, E)$ be a soft set and let $\mathcal{S}_r = (F_r, E)$ be a reference soft set in $S(U)$. Let*

$$\mathcal{S}^c = (F^c, E)$$

be the complement of $\mathcal{S}$, where

$$F^c(e) = U \setminus F(e), \qquad \forall\, e \in E.$$

Then

$$\overline{H}_{\mathcal{S}_r}(\mathcal{S}^c) = 1 - \overline{H}_{\mathcal{S}_r}(\mathcal{S}).$$

Proof. For any subsets $A, B \subseteq U$, we have

$$A \triangle (U \setminus B) = U \setminus (A \triangle B). \tag{4.3}$$

Using (4.3), for every $e \in E$,

$$\begin{aligned} |F_r(e) \triangle F^c(e)| &= |F_r(e) \triangle (U \setminus F(e))| \\ &= |U \setminus (F_r(e) \triangle F(e))| \\ &= |U| - |F_r(e) \triangle F(e)|. \end{aligned}$$

Summing over all $e \in E$, we obtain

$$\sum_{e \in E} |F_r(e) \triangle F^c(e)| = \sum_{e \in E} \left(|U| - |F_r(e) \triangle F(e)| \right).$$

Hence,

$$H_{\mathcal{S}_r}(\mathcal{S}^c) = |U||E| - H_{\mathcal{S}_r}(\mathcal{S}).$$

Dividing both sides by $|U||E|$ yields

$$\overline{H}_{\mathcal{S}_r}(\mathcal{S}^c) = 1 - \overline{H}_{\mathcal{S}_r}(\mathcal{S}).$$

□

Having established the properties of the radial Hamming distance and its normalized form, we now introduce the radial Euclidean distance and its normalized version.

Definition 4.13. Let $\mathcal{S} = (F, E)$ be a soft set in $S(U)$, and let $\mathcal{S}_r = (F_r, E)$ be a fixed reference soft set. The *radial Euclidean distance* is the function

$$E_{\mathcal{S}_r} : S(U) \to [0, \infty)$$

defined by

$$E_{\mathcal{S}_r}(\mathcal{S}) = \sqrt{\sum_{e \in E} |F(e) \triangle F_r(e)|^2}, \tag{4.4}$$

where $|F(e) \triangle F_r(e)|$ denotes the cardinality of the symmetric difference $F(e) \triangle F_r(e)$.

Proposition 4.14. *Let $\mathcal{S} = (F, E)$ be a soft set in $S(U)$, and let $\mathcal{S}_r = (F_r, E)$ be a fixed reference soft set. Then the following statements hold.*

(1) $E_{\mathcal{S}_r}(\mathcal{S}) \geq 0$. *(Non-negativity)*
(2) $E_{\mathcal{S}_r}(\mathcal{S}) = E_{\mathcal{S}}(\mathcal{S}_r)$. *(Symmetry)*
(3) $E_{\mathcal{S}_r}(\mathcal{S}) = 0$ *if and only if* $\mathcal{S} = \mathcal{S}_r$. *(Identity of indiscernibles)*

Proof. (1) By definition,

$$E_{\mathcal{S}_r}(\mathcal{S}) = \sqrt{\sum_{e \in E} |F(e) \triangle F_r(e)|^2}.$$

Since $|F(e) \triangle F_r(e)|^2 \geq 0$ for every $e \in E$, we have

$$E_{\mathcal{S}_r}(\mathcal{S}) \geq 0.$$

(2) Since

$$|F(e) \triangle F_r(e)| = |F_r(e) \triangle F(e)|$$

for every $e \in E$, it follows that

$$E_{\mathcal{S}_r}(\mathcal{S}) = E_{\mathcal{S}}(\mathcal{S}_r).$$

(3) Suppose $E_{\mathcal{S}_r}(\mathcal{S}) = 0$. Then

$$\sum_{e \in E} |F(e) \triangle F_r(e)|^2 = 0.$$

Since each term in the summation is non-negative, it follows that

$$|F(e) \triangle F_r(e)| = 0, \qquad \forall\, e \in E.$$

Hence, $F(e) = F_r(e)$ for every $e \in E$, and therefore $\mathcal{S} = \mathcal{S}_r$.
Conversely, if $\mathcal{S} = \mathcal{S}_r$, then $F(e) = F_r(e)$ for every $e \in E$.
Hence, $E_{\mathcal{S}_r}(\mathcal{S}) = 0$.

□

Remark 4.15. The radial Euclidean distance is not necessarily monotone with respect to the soft subset relation.

Example 4.16. Let $U = \{u_1, u_2\}$ and $E = \{e\}$. Let
$$\mathcal{S}_r = (F_r, E) = \{(e, \{u_2\})\}$$
be the reference soft set. Consider the soft sets
$$\mathcal{S}_1 = (F_1, E) = \{(e, \{u_1\})\}$$
and
$$\mathcal{S}_2 = (F_2, E) = \{(e, \{u_1, u_2\})\}.$$
Clearly,
$$(F_1, E) \,\widetilde{\sqsubseteq}\, (F_2, E).$$
Moreover,
$$|F_1(e) \triangle F_r(e)| = 2 \quad \text{and} \quad |F_2(e) \triangle F_r(e)| = 1.$$
Hence,
$$E_{\mathcal{S}_r}(\mathcal{S}_1) = \sqrt{2^2} = 2,$$
and
$$E_{\mathcal{S}_r}(\mathcal{S}_2) = \sqrt{1^2} = 1.$$
Therefore,
$$E_{\mathcal{S}_r}(\mathcal{S}_1) > E_{\mathcal{S}_r}(\mathcal{S}_2),$$
even though
$$(F_1, E) \,\widetilde{\sqsubseteq}\, (F_2, E).$$
Thus, the radial Euclidean distance is not monotone with respect to the soft subset relation.

Theorem 4.17. *Let $\mathcal{S}_i = (F_i, E)$ and $\mathcal{S}_j = (F_j, E)$ be two soft sets in $S(U)$, and let $\mathcal{S}_r = (F_r, E)$ be the reference soft set. If*
$$\mathcal{S}_r \,\widetilde{\sqsubseteq}\, \mathcal{S}_i \,\widetilde{\sqsubseteq}\, \mathcal{S}_j,$$
then
$$E_{\mathcal{S}_r}(\mathcal{S}_i) \leq E_{\mathcal{S}_r}(\mathcal{S}_j).$$
That is, the radial Euclidean distance is monotone.

Proof. Suppose
$$\mathcal{S}_r \,\widetilde{\sqsubseteq}\, \mathcal{S}_i \,\widetilde{\sqsubseteq}\, \mathcal{S}_j.$$
Then
$$F_r(e) \subseteq F_i(e) \subseteq F_j(e), \qquad \forall\, e \in E.$$
Hence,
$$F_i(e) \triangle F_r(e) \subseteq F_j(e) \triangle F_r(e),$$
which implies
$$|F_i(e) \triangle F_r(e)|^2 \leq |F_j(e) \triangle F_r(e)|^2, \qquad \forall\, e \in E.$$
Summing over all $e \in E$, we obtain
$$\sum_{e \in E} |F_i(e) \triangle F_r(e)|^2 \leq \sum_{e \in E} |F_j(e) \triangle F_r(e)|^2.$$

Since the square root function is increasing on $[0, \infty)$,

$$E_{\mathcal{S}_r}(\mathcal{S}_i) \leq E_{\mathcal{S}_r}(\mathcal{S}_j).$$

Thus, the radial Euclidean distance is monotone under the given condition. $\square$

We now introduce the normalized radial Euclidean distance.

Definition 4.18. Let $\mathcal{S}_r = (F_r, E)$ be a fixed reference soft set. The *normalized radial Euclidean distance* is the function

$$\overline{E}_{\mathcal{S}_r} : S(U) \to [0, 1]$$

defined by

$$\overline{E}_{\mathcal{S}_r}(\mathcal{S}) = \frac{\sqrt{\sum_{e \in E} |F(e) \triangle F_r(e)|^2}}{|U|\sqrt{|E|}}, \tag{4.5}$$

where $|\cdot|$ denotes the cardinality of a set.

Remark 4.19. The normalization constants $|U||E|$ and $|U|\sqrt{|E|}$ represent the maximum possible values of the radial Hamming distance and the radial Euclidean distance, respectively. Consequently, the normalized radial Hamming distance and the normalized radial Euclidean distance take values in the interval $[0, 1]$.

Proposition 4.20. *Let $\mathcal{S} = (F, E)$ be a soft set in $S(U)$, and let $\mathcal{S}_r = (F_r, E)$ be a fixed reference soft set. Then the following statements hold.*

(1) $\overline{E}_{\mathcal{S}_r}(\mathcal{S}) \geq 0.$ *(Non-negativity)*
(2) $\overline{E}_{\mathcal{S}_r}(\mathcal{S}) = \overline{E}_{\mathcal{S}}(\mathcal{S}_r).$ *(Symmetry)*
(3) $\overline{E}_{\mathcal{S}_r}(\mathcal{S}) = 0$ *if and only if* $\mathcal{S} = \mathcal{S}_r.$ *(Identity of indiscernibles)*

Proof. The proof follows directly from Proposition 4.14 by dividing both sides of the defining equation by the positive constant $|U|\sqrt{|E|}$. Hence, the details are omitted. $\square$

Theorem 4.21. *Let $\mathcal{S}_i = (F_i, E)$ and $\mathcal{S}_j = (F_j, E)$ be two soft sets in $S(U)$, and let $\mathcal{S}_r = (F_r, E)$ be the reference soft set. If*

$$\mathcal{S}_r \,\widetilde{\sqsubseteq}\, \mathcal{S}_i \,\widetilde{\sqsubseteq}\, \mathcal{S}_j,$$

then

$$\overline{E}_{\mathcal{S}_r}(\mathcal{S}_i) \leq \overline{E}_{\mathcal{S}_r}(\mathcal{S}_j).$$

That is, the normalized radial Euclidean distance is monotone.

Proof. The proof follows directly from Theorem 4.17 by dividing both sides of the defining equation by the positive constant $|U|\sqrt{|E|}$. Hence, the details are omitted. $\square$

The radial distances introduced in this section are reference-based distance functions rather than metrics in the classical sense. They satisfy the properties of non-negativity, symmetry, and identity of indiscernibles, but they do not, in general, satisfy the triangle inequality. This is because they are defined with respect to a fixed reference soft set $\mathcal{S}_r$, rather than between arbitrary pairs of soft sets.

The measure-based partition developed in Section 3 and the reference-based radial distance functions introduced in Section 4 provide a natural framework for partitioning the soft space according to the degree of similarity or dissimilarity with respect to the reference soft set. Motivated by these concepts, we now develop a framework for partitioning the soft space into disjoint clusters.

5. Partition of the soft space using reference based radial distance functions

In this section, we develop a framework for partitioning the soft space $S(U)$ using the reference based normalized radial distance functions introduced in the previous section.

Throughout this section, let $\rho_{\mathcal{S}_r} : S(U) \to [0,1]$ denote any normalized radial distance function with respect to the fixed reference soft set $\mathcal{S}_r$.

Definition 5.1. Let $S(U)$ be the soft space, and let

$$\rho_{\mathcal{S}_r} : S(U) \to [0,1]$$

be a normalized radial distance function with respect to the fixed reference soft set $\mathcal{S}_r$. Let

$$P = \{I_1, I_2, \ldots, I_m\}$$

be a partition of the interval $[0,1]$ such that

$$\rho_{\mathcal{S}_r}^{-1}(I_j) \neq \emptyset, \qquad \forall\, I_j \in P.$$

Then $\rho_{\mathcal{S}_r}$ induces the partition

$$\Pi = \{\rho_{\mathcal{S}_r}^{-1}(I_j) : I_j \in P\}$$

of the soft space $S(U)$.

Remark 5.2. Let $x_0 < x_1 < \cdots < x_m$ be a sequence of thresholds in $[0,1]$. These thresholds induce the partition

$$\{[x_0, x_1], (x_1, x_2], \ldots, (x_{m-1}, x_m]\}$$

of the interval $[0,1]$. Consequently, the normalized radial distance function $\rho_{\mathcal{S}_r} : S(U) \to [0,1]$ induces the partition

$$\Pi = \{\rho_{\mathcal{S}_r}^{-1}(I_1), \rho_{\mathcal{S}_r}^{-1}(I_2), \ldots, \rho_{\mathcal{S}_r}^{-1}(I_m)\}$$

of the soft space $S(U)$, where

$$\rho_{\mathcal{S}_r}^{-1}(I_1) = \{\mathcal{S}_i \in S(U) : x_0 \leq \rho_{\mathcal{S}_r}(\mathcal{S}_i) \leq x_1\},$$

and

$$\rho_{\mathcal{S}_r}^{-1}(I_j) = \{\mathcal{S}_i \in S(U) : x_{j-1} < \rho_{\mathcal{S}_r}(\mathcal{S}_i) \leq x_j\}, \qquad j = 2, 3, \ldots, m.$$

Thus, $\rho_{\mathcal{S}_r}^{-1}(I_j)$ contains all soft sets whose normalized radial distance from the reference soft set $\mathcal{S}_r$ lies in the interval I_j.

Proposition 5.3. *Let $P = \{I_1, I_2, \dots, I_m\}$ be a partition of the interval $[0,1]$, and let*

$$\Pi = \{\rho_{\mathcal{S}_r}^{-1}(I_j) : I_j \in P\}$$

be the induced family of non-empty subsets of the soft space $S(U)$. Then Π is a partition of $S(U)$.

Proof. Since $\rho_{\mathcal{S}_r} : S(U) \to [0,1]$ is a function, every soft set $\mathcal{S} \in S(U)$ is assigned a unique value $\rho_{\mathcal{S}_r}(\mathcal{S}) \in [0,1]$.. Consequently, every $I_j \in P$ contains soft sets whose $\rho_{\mathcal{S}_r}(\mathcal{S})$ values fall in the sub interval I_j

As P is a partition of $[0,1]$, there exists a unique interval $I_j \in P$ such that

$$\rho_{\mathcal{S}_r}(\mathcal{S}) \in I_j.$$

Hence,

$$\mathcal{S} \in \rho_{\mathcal{S}_r}^{-1}(I_j).$$

Since $\mathcal{S}$ was arbitrary, it follows that

$$\bigcup_{I_j \in P} \rho_{\mathcal{S}_r}^{-1}(I_j) = S(U).$$

Further, since the intervals in P are pairwise disjoint, their inverse images under $\rho_{\mathcal{S}_r}$ are also pairwise disjoint. Therefore,

$$\rho_{\mathcal{S}_r}^{-1}(I_i) \cap \rho_{\mathcal{S}_r}^{-1}(I_j) = \emptyset, \qquad i \neq j.$$

Thus, Π is a partition of the soft space $S(U)$. □

Remark 5.4. The choice of the thresholds $\{x_k\}_{k=0}^{m}$ that determine the partition $\Pi = \{\rho_{\mathcal{S}_r}^{-1}(I_j) : I_j \in P\}$ is application dependent. The thresholds may be selected using any established binning method, depending on the underlying data distribution and the objectives of the analysis.

Now we formulate the problem of reference-based partitioning of a soft space as a computational problem and present an algorithm for constructing the induced partition using the normalized radial distance functions introduced in the previous sections.

Instance.

A finite universe$U$ with $|U| = n$.
A finite parameter set E be with $|E| = k$.
A finite collection of soft sets,

$$S(U) = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_N\},$$

where each

$$\mathcal{S}_i = (F_i, E)$$

is a soft set over U defined

$$F_i : E \to 2^U.$$

A fixed reference soft set

$$\mathcal{S}_r = (F_r, E) \in S(U)$$

A normalized radial distance function

$$\rho_{\mathcal{S}_r} : S(U) \to [0,1]$$

, where $\rho_{\mathcal{S}_r}$ denotes either the normalized radial Hamming distance or the normalized radial Euclidean distance.
A positive integer m together with a threshold sequence

$$0 = x_0 < x_1 < \cdots < x_m = 1,$$

which induces the partition

$$P = \{I_1, I_2, \ldots, I_m\}$$

of the interval $[0,1]$, where

$$I_1 = [x_0, x_1], \qquad I_j = (x_{j-1}, x_j], \quad j = 2, 3, \ldots, m.$$

Computational Objective. Construct the induced partition

$$\Pi = \{\Pi_1, \Pi_2, \ldots, \Pi_t\}, \qquad t \le m,$$

of the soft space $S(U)$, where each non-empty cluster is given by

$$\Pi_j = \{\mathcal{S}_i \in S(U) : \rho_{\mathcal{S}_r}(\mathcal{S}_i) \in I_j\},$$

for $j = 1, 2, \ldots, m$, and every soft set belongs to exactly one cluster.

Algorithm 5.5. Reference-Based Partition of a Soft Space

Input.

Soft space

$$S(U) = \{S_1, S_2, \ldots, S_N\},$$

where each $S_i = (F_i, E)$; reference soft set $S_r = (F_r, E)$; distance type

$$\texttt{Distance Type} \in \{\texttt{NRHD}, \texttt{NRED}\},$$

where NRHD and NRED denote the Normalized Radial Hamming Distance and the Normalized Radial Euclidean Distance, respectively; number of clusters m; threshold selection method

$$M \in \{\texttt{Equal Width}, \texttt{Equal Frequency}, \texttt{User Defined}\}.$$

Output.

The induced partition

$$\Pi = \{\Pi_1, \Pi_2, \ldots, \Pi_t\}, \qquad t \le m,$$

together with the cluster label assigned to each soft set.

Phase 1. Distance Computation.

(1) For each $i = 1, 2, \ldots, N$, initialize $d[i] \leftarrow 0$.
(2) For each $j = 1, 2, \ldots, k$, compute

$$h = |F_i(e_j) \triangle F_r(e_j)|\,.$$

(3) If `Distance Type` = `NRHD`, set

$$d[i] \leftarrow d[i] + h;$$

otherwise set

$$d[i] \leftarrow d[i] + h^2.$$

(4) After processing all parameters,
if `Distance Type` = `NRHD`, set

$$d[i] \leftarrow \frac{d[i]}{nk};$$

otherwise set

$$d[i] \leftarrow \frac{\sqrt{d[i]}}{n\sqrt{k}}.$$

(5) Repeat Steps 1–4 for every soft set in $S(U)$.

Phase 2. Threshold Construction.

(1) If

$$M = \texttt{Equal Width},$$

construct the threshold sequence

$$x_j = \frac{j}{m}, \qquad j = 0, 1, \ldots, m.$$

(2) Else if

$$M = \texttt{Equal Frequency},$$

sort the values

$$d[1], d[2], \ldots, d[N]$$

and determine the threshold sequence

$$0 = x_0 < x_1 < \cdots < x_m = 1$$

using the equal-frequency binning procedure.

(3) Otherwise, read the user-defined threshold sequence

$$0 = x_0 < x_1 < \cdots < x_m = 1.$$

(4) Construct the interval partition

$$I_1 = [x_0, x_1],$$

and

$$I_j = (x_{j-1}, x_j], \qquad j = 2, 3, \ldots, m.$$

Phase 3. Cluster Assignment.

(1) Initialize

$$\Pi_j = \emptyset, \qquad j = 1, 2, \ldots, m.$$

(2) For each soft set S_i, determine the unique interval I_j satisfying

$$d[i] \in I_j.$$

(3) Assign

$$S_i$$

to the corresponding cluster

$$\Pi_j \leftarrow \Pi_j \cup \{S_i\},$$

and record its cluster label.

(4) Remove all empty clusters.

(5) Return

$$\Pi = \{\Pi_j : \Pi_j \neq \emptyset,\ 1 \leq j \leq m\},$$

together with the cluster labels assigned to the soft sets.

Algorithm 5.5 computes the induced partition defined in Definition 5.1 by assigning each soft set to the unique interval containing its normalized radial distance from the reference soft set.

Theorem 5.6. *Let U be a finite universe with $|U| = n$, let E be a finite parameter set with $|E| = k$, and let $S(U)$ be a finite collection of N soft sets. Then the worst-case running time of Algorithm 1 is $O(Nnk)$.*

Proof. We analyze each phase of Algorithm 5.5 separately.

Phase 1. Distance Computation. For each soft set $S_i \in S(U)$, the algorithm computes its normalized radial distance from the reference soft set S_r. For every parameter $e \in E$, it computes the cardinality of the symmetric difference

$$|F_i(e) \triangle F_r(e)|.$$

Since each subset is contained in the universe U, the computation of one symmetric difference requires $O(n)$ time. As there are k parameters, the computation of the normalized radial distance for one soft set requires $O(nk)$ time. Hence, processing all N soft sets requires

$$O(Nnk)$$

time.

Phase 2. Threshold Construction. The threshold sequence consists of m intervals determined according to the prescribed threshold selection method. Since m is chosen independently of the input sizes N, n, and k, the computational cost of this phase does not affect the asymptotic running time.

Phase 3. Cluster Assignment. Each of the N computed distance values is compared with the threshold sequence and assigned to its corresponding cluster. This requires one assignment for each soft set and therefore requires

$$O(N)$$

time.

Hence, the total running time of Algorithm 1 is

$$O(Nnk) + O(N).$$

Since $n \geq 1$ and $k \geq 1$, we have

$$N \leq Nnk.$$

Therefore, the term $O(N)$ is dominated by $O(Nnk)$, and consequently,

$$O(Nnk) + O(N) = O(Nnk).$$

Hence, the worst-case running time of Algorithm 1 is $O(Nnk)$. □

Theorem 5.7 (Worst-Case Space Complexity)**.** *Let U be a finite universe with $|U| = n$, let E be a finite parameter set with $|E| = k$, and let $S(U)$ be a finite collection of N soft sets. Then the worst-case space complexity of Algorithm 1 is $O(Nnk)$.*

Proof. The algorithm stores the collection of N soft sets. Each soft set consists of k parameterized subsets, and each subset contains at most n elements of the universe. Hence, the storage required for the input soft sets is

$$O(Nnk).$$

In addition, the algorithm stores one normalized radial distance and one cluster label for each soft set, requiring

$$O(N)$$

additional space. The storage required for the threshold sequence and the partition cells is independent of the input sizes N, n, and k, and therefore contributes only constant additional space.

Hence, the total space required by Algorithm 1 is

$$O(Nnk) + O(N).$$

Since $n \geq 1$ and $k \geq 1$, we have

$$N \leq Nnk.$$

Therefore, the term $O(N)$ is dominated by $O(Nnk)$, and consequently,

$$O(Nnk) + O(N) = O(Nnk).$$

Hence, the worst-case space complexity of Algorithm 1 is $O(Nnk)$. □

Example 5.8. We now illustrate the applicability of the proposed radial distance functions through a practical decision-making problem arising in course outcomes assessment.

Suppose a university prescribes a standard framework of assessment tools for evaluating the course outcomes of a course across all its affiliated colleges. This framework serves as the reference model. Individual teachers may either adopt the prescribed assessment tools or employ alternative combinations of assessment tools. The objective is to measure the degree to which the assessment strategy adopted by each teacher deviates from the university's prescribed framework and to group teachers into clusters according to their level of deviation. For this purpose, the university collects the relevant assessment information from all the affiliated colleges.

Consider a course having five course outcomes, namely, CO1, CO2, CO3, CO4, and CO5. Let U denote the set of all possible assessment tools that may be used to evaluate the attainment of these course outcomes.

Let

$$E = \{e_1, e_2, e_3, e_4, e_5\},$$

where

$$e_1 = \mathrm{CO}_1,\ e_2 = \mathrm{CO}_2,\ e_3 = \mathrm{CO}_3,\ e_4 = \mathrm{CO}_4,\ e_5 = \mathrm{CO}_5.$$

Let

$$U = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9\}$$

be the set of all possible assessment tools.

For each course outcome $e \in E$, the university specifies the assessment tools that are to be used for evaluating the corresponding course outcome. These recommendations constitute the reference soft set $\mathcal{S}_r = (F_r, E)$, where

$$(F_r, E) = \{(e_1, \{t_3, t_7\}), (e_2, \{t_1, t_5, t_6\}), (e_3, \{t_1\}), (e_4, \{t_3, t_4\}), (e_5, \{t_1, t_2\})\}.$$

The assessment tools employed by each teacher are collected from the affiliated colleges. This information gives rise to a soft set $\mathcal{S}_i = (F_i, E)$, for each teacher $i = 1, 2, \ldots, 8$, as given below.

$$(F_1, E) = \{(e_1, \{t_3, t_7\}), (e_2, \{t_1, t_5\}), (e_3, \{t_1\}), (e_4, \{t_3, t_4\}), (e_5, \{t_1, t_2\})\}.$$
$$(F_2, E) = \{(e_1, \{t_3, t_7\}), (e_2, \{t_1, t_5, t_6\}), (e_3, \{t_1\}), (e_4, \{t_3, t_4\}), (e_5, \{t_1, t_2\})\}.$$
$$(F_3, E) = \{(e_1, \{t_3, t_4, t_7\}), (e_2, \{t_1, t_5, t_6\}), (e_3, \{t_1, t_8\}), (e_4, \{t_3, t_4\}), (e_5, \{t_1, t_2\})\}.$$
$$(F_4, E) = \{(e_1, \{t_7\}), (e_2, \{t_1, t_4, t_5, t_6\}), (e_3, \{t_1\}), (e_4, \{t_4\}), (e_5, \{t_1, t_2, t_8\})\}.$$
$$(F_5, E) = \{(e_1, \{t_3, t_4, t_7\}), (e_2, \{t_1, t_5, t_6, t_8\}), (e_3, \{t_1, t_8\}), (e_4, \{t_3, t_4, t_8\}), (e_5, \{t_1, t_2\})\}.$$
$$(F_6, E) = \{(e_1, \{t_3, t_7\}), (e_2, \{t_1, t_5, t_6, t_8\}), (e_3, \{t_1\}), (e_4, \{t_3, t_4, t_8\}), (e_5, \{t_1, t_2\})\}.$$
$$(F_7, E) = \{(e_1, \{t_3, t_7\}), (e_2, \{t_1, t_5\}), (e_3, \{t_1\}), (e_4, \{t_4, t_8\}), (e_5, \{t_2\})\}.$$
$$(F_8, E) = \{(e_1, \{t_3, t_4, t_7, t_8\}), (e_2, \{t_1, t_5, t_6, t_8\}), (e_3, \{t_1, t_8\}), (e_4, \{t_3, t_4, t_8\}), (e_5, \{t_1, t_2, t_8\})\}.$$

The normalized radial Hamming distances between each soft set $\mathcal{S}_i$, $1 \le i \le 8$, and the reference soft set $\mathcal{S}_r$ are given by

$$\overline{H}_{\mathcal{S}_r}(\mathcal{S}_1) = \frac{1}{45}, \quad \overline{H}_{\mathcal{S}_r}(\mathcal{S}_2) = 0, \quad \overline{H}_{\mathcal{S}_r}(\mathcal{S}_3) = \frac{2}{45}, \quad \overline{H}_{\mathcal{S}_r}(\mathcal{S}_4) = \frac{4}{45},$$
$$\overline{H}_{\mathcal{S}_r}(\mathcal{S}_5) = \frac{4}{45}, \quad \overline{H}_{\mathcal{S}_r}(\mathcal{S}_6) = \frac{2}{45}, \quad \overline{H}_{\mathcal{S}_r}(\mathcal{S}_7) = \frac{4}{45}, \quad \overline{H}_{\mathcal{S}_r}(\mathcal{S}_8) = \frac{6}{45}.$$

For illustrative purposes, the thresholds are chosen using the user-defined thresholding strategy. Let

$$P = \{I_1, I_2, I_3\},$$

where

$$I_1 = \left[0, \frac{2}{45}\right], \qquad I_2 = \left(\frac{2}{45}, \frac{4}{45}\right], \qquad I_3 = \left(\frac{4}{45}, 1\right].$$

The induced partition of the soft space is

$$\Pi = \{\Pi_1, \Pi_2, \Pi_3\},$$

where

$$\Pi_1 = \overline{H}_{S_r}^{-1}(I_1) = \{S_1, S_2, S_3, S_6, S_r\},$$
$$\Pi_2 = \overline{H}_{S_r}^{-1}(I_2) = \{S_4, S_5, S_7\},$$

and

$$\Pi_3 = \overline{H}_{S_r}^{-1}(I_3) = \{S_8\}.$$

Cluster Π_1 contains teachers with low deviation from the reference soft set, Cluster Π_2 contains teachers with moderate deviation, and Cluster Π_3 contains teachers with high deviation.

This example demonstrates the practical applicability of the proposed normalized radial Hamming distance for clustering soft sets. Likewise, any normalized radial distance function

$$\rho_{\mathcal{S}_r} : S(U) \to [0, 1],$$

including the normalized radial Euclidean distance, can be used to partition the soft space into clusters according to the degree of deviation from the reference soft set.

6. Conclusion

This paper introduced a framework for partitioning soft sets. We first defined a scalar measure for soft sets, by combining a classical monotone measure on the parameterized subsets with an appropriate aggregation operator. The proposed measure induces a partition of the soft space into clusters consisting of soft sets having identical measure values.

Furthermore, we introduced reference-based radial Hamming and radial Euclidean distances, together with their normal forms, between a reference soft set and any soft set in a soft space. We established their fundamental properties and showed that, although these distances are not monotonic in general, they become monotonic whenever the reference soft set $\mathcal{S}_r$ is a subset of the soft sets being compared. These normalized radial distances induce a partition of the soft space into clusters, where each cluster consists of soft sets whose distances from the reference soft set lie within a prescribed interval.

We also formulated the partitioning task as a computational problem, developed an algorithm for constructing the induced partition, and established its worst-case time and space complexities.

Finally, the proposed method was illustrated through an example based on the assessment of course outcomes in accordance with university guidelines. The normalized radial distances were used to cluster teachers into groups representing low, moderate, and high levels of deviation from the prescribed assessment framework, demonstrating the practical applicability of the proposed methodology.

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