Annals of Mathematics and Computer Science
ISSN: 2789-7206
Vol 8 (2022) 21-25

ON THE RELATIONSHIP BETWEEN FUZZY TOPOLOGICAL ORDERED SPACES AND FUZZY BITOPOLOGICAL SPACES

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ABSTRACT. The parallel between the theory of bitopological spaces and the theory of topological ordered spaces was given by M.J. Canfell in his thesis "Semialgebras and rings of continuous functions", but he left the question open when a bitopological space may be treated as a topological ordered space. More precisely, if $(X, \mathcal{T}_1, \mathcal{T}_2)$ is a bitopological space and $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ is the coarest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$,to find conditions under which there exists a partial order $\leq$ on X such that $\mathcal{T}_1$ coincides with the upper topology and $\mathcal{T}_2$ with the lower topology of $(X, \mathcal{T}, \leq)$. In this paper we give a solution to this problem in the fuzzy setting.

1. Introduction

J.C.Kelley [1] introduced bitopological spaces in 1963,since then the bitopological spaces have been subject of intensive investigation for many topologists.There has been a lot of research on the correct generalization of basic topological properties to the bitopological setting.To study of the interdependence between fuzzy topology and order, Katsaras [7] inroduced fuzzy topological ordered spaces in 1981.Here fuzzy topological ordered space is a triplet $(X, \mathcal{T}, \leq)$ where $\mathcal{T}$ is a fuzzy topology on X and $\leq$ is a crisp order on X.

If $(X, \mathcal{T}, \leq)$ is a fuzzy topological ordered space,then we can show that

$$\mathcal{T}_u = \{\mu \in \mathcal{T} \mid \mu \text{ is increasing}\}$$

and

$$\mathcal{T}_l = \{\nu \in \mathcal{T} \mid \nu \text{ is decreasing}\}$$

are fuzzy topologies for X ,called respectively the upper and lower fuzzy topologies.

Then, $(X, \mathcal{T}_u, \mathcal{T}_l)$ is a fuzzy bitopological space.

We are interested in the interplay between fuzzy topological ordered space and fuzzy bitopological space.

Date: Received: Jul 28, 2022; Accepted: Aug 25, 2022.

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2020 *Mathematics Subject Classification.* 54A40,03E72,54F05.

Key words and phrases. fuzzy topology,fuzzy topological ordered space, fuzzy bitopology .

2. Preliminaries

Definition 2.1. [7] Let X be a nonempty set . A fuzzy topological ordered space is a triple $(X, \mathcal{T}, \leq)$ where $\mathcal{T}$ is a fuzzy topology on X and $\leq$ is a partial order on X.

Definition 2.2. A fuzzy set μ in a fuzzy topological space $(X, \mathcal{T})$ is called a neighborhood of a point $x \in X$ if there exists an fuzzy open set μ_1,with $\mu_1 \leq \mu$ and $\mu_1(x) = \mu(x) > 0$.

A fuzzy set μ is open if and only if μ is a neighborhood of each $x \in X$ for which $\mu(x) > 0$.

Definition 2.3. A family $\mathcal{F}$ of neighborhoods of a point x is called a base for the system of neighborhoods of x if the following condition is satisfied:
For each neighborhood μ of x and each θ , with $0 < \theta < \mu(x)$,there exists $\mu_1 \in \mathcal{F}$ with $\mu_1 \leq \mu$ and $\mu_1(x) > \theta$.

Definition 2.4. A function f from a fuzzy topological space $(X, \mathcal{T})$ to a fuzzy topological space $(Y, \mathcal{S})$ is called fuzzy continuous if $f^{-1}(\mu)$ is open in X for each open set μ in Y.

The function f is continuous at some $x \in X$ if $f^{-1}(\mu)$ is a neighborhood of x for each neighborhood μ of $f(x)$.

f is continuous on X iff f is continuous at each $x \in X$.

Definition 2.5. A fuzzy set μ in a ordered set X is called

i) increasing if $x \leq y \Rightarrow \mu(x) \leq \mu(y)$
ii) decreasing if $x \leq y \Rightarrow \mu(x) \geq \mu(y)$
iii) orderconvex if $x \leq z \leq y \Rightarrow \mu(z) \geq (\mu(x) \wedge \mu(y))$

Note that : Constant functions are increasing,decreasing and orderconvex.
If μ is increasing then $1 - \mu$ is decreasing.
$\{\mu_i \mid i \in I\}$ is increasing(resp.decreasing) then $\mu = \wedge\{\mu_i \mid i \in I\}$is also increasing(resp. decreasing).

Definition 2.6. Let μ be a fuzzy set in a ordered set X then the smallest increasing set containing μ, smallest decreasing set containing μ and smallest convex set containing μ will be denoted by $i(\mu), d(\mu)$ and $c(\mu)$ respectively. Katsaras shown that
i) $i(\mu)(x) = \bigvee\{\mu(y) \mid y \leq x\}$.
ii) $d(\mu)(x) = \bigvee\{\mu(y) \mid y \geq x\}$.
iii) $c(\mu)(x) = \bigvee\{\mu(x_1) \wedge \mu(x_2) \mid x_1 \leq x \leq x_2\}$

Note that: $c(\mu) = i(\mu) \wedge d(\mu)$.
The smallest increasing closed set containing μ, the smallest decreasing closed set containing μ and the smallest convex closed set containing μ will be denoted by $I(\mu), D(\mu)$ and $C(\mu)$ respectively.

Definition 2.7. Let λ be a fuzzy set of X and μ be a fuzzy set of Y then $\lambda \times \mu$ is fuzzy set of $X \times Y$,defined as

$$(\lambda \times \mu)(x, y) = \lambda(x) \wedge \mu(y) \text{ for each } (x, y) \in X \times Y$$

.

Definition 2.8. Let X, Y be ordered sets and f be a function from X to Y.Then f is called increasing (resp.decreasing) if $x \leq y$ in X implies $f(x) \leq f(y)$(respectively $f(y) \leq f(x)$).

Proposition 2.9. *Let X, Y be ordered fuzzy topological spaces.A function $f : X \to Y$ is increasing if and only if $f^{-1}(\mu)$ is increasing in X for every increasing set μ in Y.*

Proof:First, suppose that,$f : X \to Y$ is increasing and μ is increasing set in Y. Let $x \leq y$ in X .Then, $f^{-1}(\mu(x)) = \mu(f(x)), f^{-1}(\mu(y)) = \mu(f(y))$. Since f is increasing,$\mu(f(x)) \leq \mu(f(y))$. As μ is increasing, we have $\mu(f(x)) \leq \mu(f(y))$.
$\therefore f^{-1}(\mu)(x) \leq f^{-1}(\mu)(y)$.So,$f^{-1}(\mu)$ is an increasing function.
Similarly,we can prove the converse.

3. Fuzzy Bitopology

Definition 3.1. [5] Let X be a nonempty set,$\mathcal{T}_1, \mathcal{T}_2$ be fuzzy topologies on X .Then,$(X, \mathcal{T}_1, \mathcal{T}_2)$ is called a fuzzy bitopological space.

A topological property can be generalized to the bitopological setting in several ways:
Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a bitopological space and $\mathcal{T} = \mathcal{T}_1 \vee \mathcal{T}_2$.
For a topological property P , we say $(X, \mathcal{T}_1, \mathcal{T}_2)$ is
- bi-P if both $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are P.
- join-P if $(X, \mathcal{T})$ is P.
-$(1, 2)P$ if P for $\mathcal{T}_1$ w.r.to $\mathcal{T}_2$.
-$(2, 1)P$ if P for $\mathcal{T}_2$ w.r.to $\mathcal{T}_1$.
-pP (pairwise) if $(1, 2)P \wedge (2, 1)P$

Definition 3.2. $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a fuzzy bitopological space. Then

i) $(X, \mathcal{T}_1, \mathcal{T}_2)$ is $p - T_1$ iff X is $bi - T_1$.
ii) $(X, \mathcal{T}_1, \mathcal{T}_2)$ is $p - T_2$ if for each pair of distinct points $x, y \in X$ there exists a $\mathcal{T}_1$ open set λ and a $\mathcal{T}_2$ open set μ such that $\lambda \wedge \mu = 0$.
iii) $(X, \mathcal{T}_1, \mathcal{T}_2)$ is (i, j) regular if for each point $x \in X$ and each i-closed set ν such that $\nu(x) = 0$ there exists an i-open set λ and j-open set μ such that $x \in \lambda, \nu \leq \mu$ and $\lambda \wedge \mu = 0$.
iv) $(X, \mathcal{T}_1, \mathcal{T}_2)$ is p-normal if for every pair of fuzzy sets λ, μ in X such that $\lambda \wedge \mu = 0$,where λ is $\mathcal{T}_1$closed and μ is $\mathcal{T}_2$ closed there exists a $\mathcal{T}_2$ open set ν such that $\lambda \leq \nu$ and a $\mathcal{T}_1$ open set δ such that $\mu \leq \delta$ and $\nu \wedge \delta = 0$.
v) $(X, \mathcal{T}_1, \mathcal{T}_2)$ is hereditary p-normal if its every bitopological subspace is p-normal.

4. Relationship between fuzzy topological ordered spaces and fuzzy bitopological spaces

M.J. Canfell [3] draw the parallel between the theories of order topological spaces and bitopological spaces in the following manner:
To each order topological $(X, \mathcal{T}, \leq)$ there corresponds a bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$,where $\mathcal{T}_1$ and $\mathcal{T}_2$ are respectively the upper and lower topologies w,r,to the partial order $\leq$.
However he left the question open in what cases a bitopological space can be treated as a ordered topological space i.e. if $(X, \mathcal{T}_1, \mathcal{T}_2)$ is a bitopogical space and $\mathcal{T} = \mathcal{T}_1 \vee \mathcal{T}_2$, what conditions must be satisfied for the existence of a partial order $\leq$ on X such that $\mathcal{T}_1$ will coincide with the upper topology and $\mathcal{T}_2$ with the lower topology of $(X, \mathcal{T}, \leq)$?

A partial answer to this problem in crisp case was given by H.A.Priestley.[6] in 1972.
Here we solve the problem in the fuzzy setting.

Theorem 4.1. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a fuzzy bitopological space and $\mathcal{T} = \mathcal{T}_1 \vee \mathcal{T}_2$ be compact.Then there exists a closed partial order $\leq$ on $(X, \mathcal{T})$ such that $\mathcal{T}_1$ and $\mathcal{T}_2$ are the upper and lower topology of $(X, \mathcal{T}, \leq)$ respectively if*
i) $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise T_1.
ii)$(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise regular.

Proof:Suppose $(X, \mathcal{T}_1, \mathcal{T}_2)$ satisfy i).ii) and $\mathcal{T} = \mathcal{T}_1 \vee \mathcal{T}_2$ is compact.
We define a relation on X by $x \leq y$ iff $x \in cl_{\mathcal{T}_1}\{y\}$.
Since X is a pairwise regular space the quasiorder $y \leq x$ may be equivalently defined as $y \in cl_{\mathcal{T}_2}\{x\}$.
i) implies that $\leq$ is a partial order.
If $x \not\leq y$,$i(x), d(y)$ are disjoint then they are respectively $\mathcal{T}_2$-closed and $\mathcal{T}_1$-closed.
Pairwise regularity together with compactness of $\mathcal{T}$,implies that there exist sets $\lambda \in \mathcal{T}_1$ and $\mu \in \mathcal{T}_2$ such that $x \in \lambda, y \in \mu$.
Clearly,λ is $\mathcal{T}$-open.
Suppose $z \in 1 - \lambda, w \leq z$.
Then w belongs to the to the smallest $\mathcal{T}_1$-closed set containing z and hence to $1 - \lambda$.
Therefore λ is increasing.
Similarly we can show μ is $\mathcal{T}$-open and decreasing.
So,$(X, \mathcal{T}, \leq)$ is a compact ordered topological space and $\mathcal{T}_1 \leq \mathcal{U}, \mathcal{T}_2 \leq \mathcal{L}$. To obtain the reverse inclusion,
Let $\lambda \in \mathcal{U}$ by Urysohn lemma ,we can find for each $y \in \lambda$ a continuous function

$$f_y : X \to [0, 1]$$

with

$$f_y|_{1-\lambda} = 0, f_y(y) = 1$$

Therefore,

$$\begin{aligned} 1 - \lambda &= \textstyle\bigwedge_{y \in \lambda}\{x \in X \mid f_y(x) = 0\} \\ &= \textstyle\bigwedge_{y \in \lambda}\{x \in X \mid f_y(x) \leq 0\} \end{aligned}$$

So, $1-\lambda$ is $\mathcal{T}_1$-closed if each function f_y is l.s.c..
Hence to show $\mathcal{U} \leq \mathcal{T}_1, \mathcal{L} \leq \mathcal{T}_2$ it is sufficient to show that every continuous function from $(X, \mathcal{T}, \leq)$ to $[0,1]$ is $\mathcal{T}_1$-l.s.c. and $\mathcal{T}_2$-u.s.c.
Kelley shows that,$\mathcal{F}$ the set of real valued $\mathcal{T}_1$-l.s.c. and $\mathcal{T}_2$-u.s.c. functions is contained in $\mathcal{G}$,the lattice of all real valued $\mathcal{T}$-continuous increasing functions.
Further,$\mathcal{F}$ is a sublattice of $\mathcal{G}$,is closed in the uniform norm.
Hence, by Kakutani-Stone theorem,$\mathcal{F}$ is a dense in $\mathcal{G}$,$\mathcal{G} = \mathcal{F}$.
If given $x, y \in X$ and $g \in \mathcal{G} \exists f \in \mathcal{F}$ such that $f(x) = g(x), f(y) = g(y)$.
If $g(x) = g(y)$, f may be taken to be a constant function.
If $g(y) < g(x)$as g is increasing we have $x \not\leq y$.
So, there exists $h \in \mathcal{F}$ with $h|_{i(x)} = 1, h|_{d(y)} = 0$.
Then,$f = (g(x) - g(y))h + g(y) \in \mathcal{F}$ satisfies the requirement $f(x) = g(x), f(y) = g(y)$.

Proposition 4.2. *Let $(X, \mathcal{T}, \leq)$ be a fuzzy topological ordered space and $(X, \mathcal{T}_1, \mathcal{T}_2)$ be the corresponding bitopological space then the following statements are valid:*
i) If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise T_1 then $(X, \mathcal{T}, \leq)$ is strongly T_1 ordered.
ii) If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise T_2 then $(X, \mathcal{T}, \leq)$ is strongly T_2 ordered.
iii) $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise regular if and only if $(X, \mathcal{T}, \leq)$ is strongly regular ordered.
iv)$(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise normal if and only if $(X, \mathcal{T}, \leq)$ is strongly normal ordered.

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