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CONTROLLABILITY OF VARIABLE-ORDER CONFORMABLE SYSTEMS

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Abstract. We study controllability of semilinear evolution systems with infinite memory governed by a generalized conformable derivative of variable order. Since the time-change reduction to semigroups fails, we construct a two-parameter evolution family and the corresponding integral representation in a Banach space. Under Lipschitz and Hale-Kato hypotheses we prove existence and uniqueness of mild solutions, introduce a variable-order controllability Gramian, and characterize exact and approximate controllability. We also obtain a quantitative observability inequality with explicit constants. Approximate controllability in the semilinear case follows from a regularized feedback and a fixed point scheme. An application and a numerical illustration are included.

Keywords. Variable-order conformable derivative, two-parameter evolution family, infinite delay, quantitative observability, approximate controllability
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1. Introduction

1.1. Motivation and physical context.

Dynamical systems with hereditary effects and time-varying anomalous transport arise in a wide range of applications: viscoelastic materials whose memory kernels evolve with ageing, anomalous diffusion in heterogeneous media, neural-type delay equations with attenuated long-range memory, and control-theoretic models with deformed local time scales. A unifying feature of these settings is the presence of two simultaneous nonlocal effects: the state at time t depends on the entire past trajectory through an infinite-memory term, and the local rate of change is modulated by a fractional-type operator whose order may itself depend on time, reflecting variable anomalous behaviour.

The conformable calculus of Khalil et al. [10] and Abdeljawad [1] offers a local-in-time alternative to classical fractional derivatives, and the generalized conformable framework of Nápoles Valdés and coauthors [6] and of Bahloul, Rachad and Abdeljawad [2] extends it further by incorporating a deformation function

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$F(\cdot,\alpha)$ that locally modulates the time scale; conformable models of this type are also widely used in nonlinear wave propagation, see [25], and in the companion mild-solution study for general fractional evolution equations with nonlocal constraints [18]. In its constant-order form, the associated derivative $\mathcal{N}_F^{\alpha}$ enjoys a particularly convenient feature: the deformation reduces to a strictly monotone time reparametrization G_α, and the operator family $\{T_F^{\alpha}(t)\}_{t\geq 0}$ is obtained from a classical C_0-semigroup $\{S(t)\}_{t\geq 0}$ through $T_F^{\alpha}(t) = S(G_\alpha(t))$.

1.2. **Why variable order matters.** The constant-order assumption is, however, restrictive. In several recent applications, including anomalous diffusion with depth-dependent porosity, viscoelastic models with ageing memory and neural transmission with state-dependent damping, the effective fractional order is more accurately modelled as a function $\alpha = \alpha(t)$ of time; see [21, 19, 12] for surveys of the variable-order paradigm. When α depends on t, a fundamental analytical difficulty emerges: the deformation no longer reduces to a single time reparametrization. The integrated deformation $\Theta(t) := \int_0^t F(s,\alpha(s))\,ds$ is still well defined, but the operator family $t \mapsto T_F^{\alpha(\cdot)}(t)$ ceases to be a semigroup, because the local time scale changes along the trajectory. Consequently, all results that are typically established by reducing the conformable problem to a classical C_0-semigroup problem, including the controllability theorems of [23, 13, 11, 20, 27] in the constant-order setting, do not apply directly. A genuinely new operator-theoretic object is needed.

1.3. **Related work and the remaining gap.** Controllability analysis for fractional and delay evolution equations has been extensively developed in the constant-order setting. For Caputo-type fractional systems, exact and approximate controllability criteria are typically expressed through a controllability Gramian; we refer to [23, 13, 11, 20, 27] for representative results. Closely related contributions include Kang and Fan [9] on second-order stochastic neutral evolution equations with infinite delay, Mshary, Ahmed and Ghanem [15] on Hilfer fractional systems with noise and impulses, Moumen et al. [14] on Hilfer-driven stochastic equations with finite approximate controllability, and Hajji, Lahmoudi and Lakhel [7], where existence and uniqueness of mild solutions for mean-field neutral stochastic functional equations with delay are obtained in a Hilbert space setting. Closely related null-controllability results have been obtained by Cui and Lu [4]. The general theory of infinite-dimensional linear systems and Gramian-based controllability is developed in the monographs [5, 17, 22], building on [26].

To the best of our knowledge, however, three features have not yet been addressed simultaneously: a variable-order generalized conformable derivative $\mathcal{N}_F^{\alpha(\cdot)}$ for which the time-change trick is not available; an infinite-memory phase space of Hale–Kato type; and a unified treatment of both exact and approximate controllability together with quantitative observability estimates whose constants depend explicitly on $\alpha(\cdot)$. The present paper is intended to fill this gap.

1.4. **Main contributions.** In Section 3 we construct a two-parameter evolution family $\{U(t,s)\}_{0\leq s\leq t\leq T}$ adapted to $\mathcal{N}_F^{\alpha(\cdot)}$, prove its strong continuity and a sharp exponential bound in terms of the integrated deformation $\Theta(t)$, and show that the constant-order semigroup of [2] is recovered as a particular case. Using this family, Section 4 formulates the semilinear controlled system (4.1) together with a variation-of-constants formula in an abstract Banach space, the history segment being taken in an axiomatic Hale–Kato phase space $\mathcal{B}$; under a global Lipschitz condition on the nonlinearity, global existence and uniqueness of mild solutions follow from a contraction argument in a weighted solution space (Theorem 4.4).

Section 5 introduces a controllability Gramian $\Gamma_T^{\alpha(\cdot)}$ adapted to the evolution family and characterizes exact and approximate controllability of the linear system in terms of its bounded invertibility and its injectivity, respectively (Theorems 5.2 and 5.3). The technical core of the paper is the quantitative observability inequality of Theorem 5.4,

$$\|\zeta\|_X^2 \;\le\; C(T,\alpha(\cdot)) \int_0^T \big|F(s,\alpha(s))\big|^2 \,\big\|B^* U(T,s)^* \zeta\big\|_{\mathcal{U}}^2 \, ds, \qquad \zeta \in X, \tag{1.1}$$

in which the constant $C(T,\alpha(\cdot))$ is explicit in terms of T, of the integrated deformation $\Theta(T)$ and of the growth bound of the underlying classical generator. The estimate quantifies the way the deformation $\alpha(\cdot)$ affects the minimum-energy steering cost, it generalizes the known constant-order results, and to our knowledge it is new even when α is constant. For the semilinear problem with infinite memory, approximate controllability is then obtained through a regularized feedback strategy in which, for each $\varepsilon > 0$, a fixed-point argument produces a mild solution x_ε whose terminal state satisfies $x_\varepsilon(T) \to x_1$ as $\varepsilon \to 0^+$ (Theorem 5.7).

The framework is illustrated in Section 6 on a variable-order generalized conformable diffusion equation with distributed internal control and an infinite-memory nonlinearity, and in Section 7 on numerical experiments in which a constant-order regime is compared with two genuinely variable profiles, first on a low-dimensional system and then on a semi-discretized parabolic problem of dimension 199, where the dependence of the steering error, of the control energy and of the computational cost on $\alpha(\cdot)$ is exhibited.

1.5. **Implications for computer science.** Although our results are stated in the language of infinite-dimensional systems theory, they carry several consequences of a computational nature. The integrated deformation Θ defines a nonuniform time scale: a scheme that is uniform in the deformed variable $\sigma = \Theta(t)$ is automatically adaptive in physical time and refines the grid exactly where F is large, which connects the present framework with the design and the convergence analysis of time-stepping schemes for evolution problems, a subject of continuing activity, see for instance [16] for recent explicit multiderivative methods. The regularized feedback (5.14) is, moreover, an algorithm: its cost is dominated by the assembly of the Gramian and by the solution of the regularized linear systems associated with $\varepsilon I + \Gamma_T^{\alpha(\cdot)}$, so that conditioning, low-rank approximation and the parallel evaluation of operator exponentials become the decisive issues as soon as the state space is a discretized partial differential equation; these questions are quantified in Section 7.5.

Dynamics with unbounded memory and with a time-dependent local scaling also occur inside computer science itself. Long-range dependent network traffic, congestion-control loops in which acknowledgements arrive with unbounded delay, distributed consensus and gossip protocols operating on stale information, and continuous-time recurrent architectures such as neural differential equations are all naturally written as evolution systems with a history segment; in the last example the trainable time-scaling parameter plays precisely the role of $\alpha(\cdot)$, and controllability is exactly the question of which terminal states such an architecture can produce. For all these models the practical question, namely how much control energy is required in order to steer a memory-dependent system to a prescribed state, is answered quantitatively by the observability constant of Theorem 5.4, while the stochastic delay models of [7] describe the randomly perturbed versions of the same networks.

1.6. **Organization of the paper.** Section 2 recalls the generalized conformable derivative, the associated constant-order semigroup theory and the axiomatic infinite-delay phase space $\mathcal{B}$. Section 3 introduces the variable-order derivative $\mathcal{N}_F^{\alpha(\cdot)}$ and constructs the associated two-parameter evolution family. Section 4 formulates the abstract controlled system with infinite memory, introduces mild solutions and proves well-posedness. Section 5 contains the main controllability results, Section 6 provides an application to a variable-order anomalous diffusion model, Section 7 gives the numerical illustration, and Section 8 collects concluding remarks and directions for future research.

2. Preliminaries

This section collects the basic definitions and auxiliary results used throughout the paper. We recall the generalized conformable fractional derivative $\mathcal{N}_F^{\alpha}$ in the constant-order case and the associated notion of a C_0–$\mathcal{N}_F^{\alpha}$ semigroup, which serves as a benchmark for the variable-order extension developed in Section 3. Throughout, X denotes a Banach space over $\mathbb{R}$ (or $\mathbb{C}$), and $\mathcal{L}(X)$ denotes the Banach algebra of bounded linear operators on X. Two distinct objects are denoted by related symbols and should not be confused: the scalar deformation function is written $F(\cdot,\alpha)$, while the history-dependent nonlinearity of Section 4 is written $\mathcal{F}$.

2.1. **The generalized conformable fractional derivative.** The generalized conformable derivative $\mathcal{N}_F^{\alpha}$ extends the classical conformable derivative of Khalil et al. [10] and Abdeljawad [1] by incorporating a deformation function $F(\cdot,\alpha)$ that locally modifies the time scale. The following definition is the one adopted in [2].

Definition 2.1. Let $\alpha \in (0,1]$ and let $f : [0,\infty) \to \mathbb{R}$ be continuous. Assume that $F(\cdot,\alpha) : [0,\infty) \to \mathbb{R}$ satisfies $F(t,\alpha) \neq 0$ for all $t \geq 0$. The generalized conformable fractional derivative of order α associated with F is defined by

$$\mathcal{N}_F^{\alpha} f(t) := \lim_{h\to 0} \frac{f\Big(t + \dfrac{h}{F(t,\alpha)}\Big) - f(t)}{h}, \qquad t \geq 0,$$

whenever the above limit exists.

Several important cases are recovered from Definition 2.1. If $F(t,\alpha) \equiv 1$, then $\mathcal{N}_F^\alpha f(t) = f'(t)$, whereas the choice $F(t,\alpha) = t^{\alpha-1}$ for $t > 0$ gives $\mathcal{N}_F^\alpha f(t) = t^{1-\alpha} f'(t)$, which is the conformable derivative of [10]; more general choices of F allow one to model deformed local dynamics and time-scaling effects, see for instance [6]. In the constant-order setting the deformation induced by $F(\cdot,\alpha)$ can be equivalently encoded through a strictly increasing time reparametrization G_α, which is used to state the semigroup property in [2].

2.2. **Semigroups associated with the constant-order derivative.** Let $G_\alpha : [0,\infty) \to [0,\infty)$ be strictly increasing and continuous with $G_\alpha(0) = 0$, and denote its inverse by G_α^{-1}. The next two statements are taken from [2].

Definition 2.2. A family $\{T_F^\alpha(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is called a C_0–$\mathcal{N}_F^\alpha$ semigroup if

(i) $T_F^\alpha(0) = I$;
(ii) for all $t, s \geq 0$, $T_F^\alpha\big(G_\alpha^{-1}(t+s)\big) = T_F^\alpha\big(G_\alpha^{-1}(t)\big)\, T_F^\alpha\big(G_\alpha^{-1}(s)\big)$;
(iii) for every $x \in X$, $\lim_{t\to 0^+} T_F^\alpha(t)x = x$ in X.

When $G_\alpha^{-1}(t) = t$, one recovers the classical C_0-semigroup [17].

Proposition 2.3. *Let $\{T_F^\alpha(t)\}_{t\geq 0}$ be a C_0–$\mathcal{N}_F^\alpha$ semigroup. Then*

(i) *the family $S(t) := T_F^\alpha(G_\alpha^{-1}(t))$ is a classical C_0-semigroup on X;*
(ii) *conversely, if $\{S(t)\}_{t\geq 0}$ is a classical C_0-semigroup, then $T_F^\alpha(t) := S(G_\alpha(t))$ defines a C_0–$\mathcal{N}_F^\alpha$ semigroup;*
(iii) *there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|T_F^\alpha(t)\| \leq M e^{\omega G_\alpha(t)}$ for all $t \geq 0$.*

Remark 2.4. Proposition 2.3 is the time-change trick that makes the constant-order theory essentially a transcription of classical C_0-semigroup theory. This trick fails in the variable-order setting, as we shall see in Section 3: when $\alpha = \alpha(t)$, the family $t \mapsto T_F^{\alpha(\cdot)}(t)$ is no longer a semigroup and a more general two-parameter object is required.

2.3. **The infinite-delay phase space.** To handle initial histories defined on the entire half-line $(-\infty, 0]$ we adopt the axiomatic framework of Hale and Kato, see [8, 24]. Throughout, $\mathcal{B}$ denotes a vector space of functions $\varphi : (-\infty, 0] \to X$, endowed with a seminorm $\|\cdot\|_\mathcal{B}$ satisfying the following axioms.

Assumption 2.5. The space $(\mathcal{B}, \|\cdot\|_\mathcal{B})$ satisfies:

(P1) if $x : (-\infty, T] \to X$ is continuous on $[0,T]$ with $x_0 \in \mathcal{B}$, then the map $t \mapsto x_t$ is continuous from $[0,T]$ into $(\mathcal{B}, \|\cdot\|_\mathcal{B})$;
(P2) there exist positive constants $K_\mathcal{B}$, $M_\mathcal{B}$ and $N_\mathcal{B}$, with $K_\mathcal{B}$ locally bounded on $[0,\infty)$, such that for every $t \in [0,T]$ and every $x, y : (-\infty, T] \to X$ continuous on $[0,T]$ with $x_0, y_0 \in \mathcal{B}$,
$$\|x_t\|_\mathcal{B} \leq K_\mathcal{B} \sup_{\tau\in[0,t]} \|x(\tau)\|_X + M_\mathcal{B}\|x_0\|_\mathcal{B}, \tag{2.1}$$
$$\|x_t - y_t\|_\mathcal{B} \leq K_\mathcal{B} \sup_{\tau\in[0,t]} \|x(\tau) - y(\tau)\|_X; \tag{2.2}$$
(P3) $(\mathcal{B}, \|\cdot\|_\mathcal{B})$ is complete and, for each $\theta \leq 0$, the point-evaluation map $\varphi \mapsto \varphi(\theta)$ is continuous from $\mathcal{B}$ into X.

Example 2.6. The following standard spaces satisfy Assumption 2.5. Let first $g : (-\infty, 0] \to [1, +\infty)$ be continuous and nonincreasing, with $g(0) = 1$ and $g(\theta) \to +\infty$ as $\theta \to -\infty$, and define the fading-memory space

$$\mathcal{B}_g := \Big\{ \varphi : (-\infty, 0] \to X \text{ measurable} : \ \|\varphi\|_g := \sup_{\theta \le 0} \frac{\|\varphi(\theta)\|_X}{g(\theta)} < +\infty \Big\}.$$

Then $(\mathcal{B}_g, \|\cdot\|_g)$ is a Banach space satisfying (P1)–(P3) with $K_{\mathcal{B}} = 1$ and $M_{\mathcal{B}} = 1$. Let next $p \in [1, +\infty)$ and let $\mu : (-\infty, 0] \to (0, +\infty)$ be measurable with $\mu \in L^1(-\infty, 0)$, and define the weighted space

$$\mathcal{B}_{\mu,p} := \Big\{ \varphi : (-\infty, 0] \to X : \ \|\varphi\|_{\mu,p} := \Big(\int_{-\infty}^{0} \|\varphi(\theta)\|_X^p \, \mu(\theta) \, d\theta \Big)^{1/p} < +\infty \Big\}.$$

Then $(\mathcal{B}_{\mu,p}, \|\cdot\|_{\mu,p})$ satisfies (P1)–(P3) with $K_{\mathcal{B}} = \big(\int_{-\infty}^{0} \mu(\theta) \, d\theta \big)^{1/p}$ and is widely used in the study of integro-differential equations with unbounded delay [8].

3. Variable-order conformable dynamics and the two-parameter evolution family

This section develops the operator-theoretic backbone of the paper. We introduce the variable-order generalized conformable derivative $\mathcal{N}_F^{\alpha(\cdot)}$, show that the time-change trick of Proposition 2.3 no longer reduces it to a C_0-semigroup, and construct in its place a two-parameter evolution family $\{U(t,s)\}_{0 \le s \le t \le T}$ that plays the analogous role for the variable-order problem.

3.1. Variable-order derivative and integrated deformation.

Definition 3.1. Let $T > 0$ and let $\alpha : [0, T] \to (0, 1]$ be continuous. Assume that $F(t, \alpha(t)) \neq 0$ for every $t \in (0, T]$. The variable-order generalized conformable derivative of $f : [0, T] \to X$ at $t \in (0, T)$ is defined by

$$\mathcal{N}_F^{\alpha(\cdot)} f(t) := \lim_{h \to 0} \frac{f\Big(t + \dfrac{h}{F(t, \alpha(t))}\Big) - f(t)}{h}, \tag{3.1}$$

whenever the limit exists in X. The integrated deformation associated with $\alpha(\cdot)$ is

$$\Theta(t) := \int_0^t F(s, \alpha(s)) \, ds, \qquad t \in [0, T]. \tag{3.2}$$

Remark 3.2. When $\alpha(t) \equiv \alpha_0$ is constant, $\Theta(t) = G_{\alpha_0}(t)$ is the classical time reparametrization of [2] and the operator family constructed below reduces to the constant-order C_0–$\mathcal{N}_F^{\alpha_0}$ semigroup of Proposition 2.3. In the genuinely variable case, by contrast, Θ is still well defined and strictly increasing as soon as $F(s, \alpha(s)) > 0$, but the family $t \mapsto T_F^{\alpha(\cdot)}(t)$ defined naively from a C_0-semigroup $\{S(t)\}$ through $S(\Theta(t))$ does not satisfy the semigroup law in t unless α is constant. The semigroup property is replaced by an evolution-family identity, derived below.

Assumption 3.3. Throughout the variable-order theory we assume the following.

(V1) The order $\alpha : [0,T] \to (0,1]$ is continuous, the map $s \mapsto F(s,\alpha(s))$ is continuous on $(0,T]$ and belongs to $L^2(0,T)$, and there is a constant $F_{\min} > 0$ with $F(t,\alpha(t)) \geq F_{\min}$ for every $t \in (0,T]$. In particular Θ is continuous, strictly increasing and bijective from $[0,T]$ onto $[0,\Theta(T)]$, with continuous inverse $\Theta^{-1} : [0,\Theta(T)] \to [0,T]$.

(V2) The operator $A : D(A) \subset X \to X$ generates a strongly continuous C_0-semigroup $\{S(t)\}_{t\geq 0}$ on X with $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, where $M \geq 1$ and $\omega \geq 0$.

Every bounded deformation which is bounded away from zero satisfies (V1). The square-integrability requirement is slightly weaker than boundedness and is exactly what is needed for the input-state operator (5.3) to be bounded; it admits deformations with an integrable singularity at the origin, such as the conformable choice $F(t,\alpha) = t^{\alpha(t)-1}$ with $\alpha(0) > 1/2$, which is used in Section 7.

3.2. **The two-parameter evolution family.**

Definition 3.4. Under Assumption 3.3, the family $\{U(t,s)\}_{0\leq s\leq t\leq T} \subset \mathcal{L}(X)$ associated with $(A, F, \alpha(\cdot))$ is defined by

$$U(t,s) := S\big(\Theta(t) - \Theta(s)\big), \qquad 0 \leq s \leq t \leq T. \tag{3.3}$$

Proposition 3.5. *Let Assumption 3.3 hold. Then*

(i) $U(t,t) = I$ *for all* $t \in [0,T]$;

(ii) *for all* $0 \leq s \leq r \leq t \leq T$,

$$U(t,r)\,U(r,s) = U(t,s); \tag{3.4}$$

(iii) *for every* $x \in X$ *the map* $(t,s) \mapsto U(t,s)x$ *is continuous on* $\{0 \leq s \leq t \leq T\}$;

(iv) *the exponential bound*

$$\|U(t,s)\| \leq M \exp\big(\omega(\Theta(t) - \Theta(s))\big), \qquad 0 \leq s \leq t \leq T, \tag{3.5}$$

holds;

(v) *if* $\alpha(t) \equiv \alpha_0 \in (0,1]$, *then* $U(t,s) = T_F^{\alpha_0}\big(G_{\alpha_0}^{-1}(G_{\alpha_0}(t) - G_{\alpha_0}(s))\big)$ *recovers the operator appearing in the variation-of-constants formula of* [2].

Proof. Properties (i) and (v) follow directly from (3.3) and Assumption 3.3. For (ii), fix $0 \leq s \leq r \leq t \leq T$; then $\Theta(t) - \Theta(s) = (\Theta(t) - \Theta(r)) + (\Theta(r) - \Theta(s))$ and the semigroup law of $\{S(\cdot)\}$ yields

$$U(t,r)U(r,s) = S\big(\Theta(t) - \Theta(r)\big)S\big(\Theta(r) - \Theta(s)\big) = S\big(\Theta(t) - \Theta(s)\big) = U(t,s).$$

For (iii), Θ is continuous by (V1) and $S(\cdot)x$ is strongly continuous by (V2), hence $(t,s) \mapsto S(\Theta(t) - \Theta(s))x$ is continuous. Finally (iv) follows from the bound on $\|S(\cdot)\|$ in (V2). $\square$

Remark 3.6. For fixed s, the map $t \mapsto U(t,s)$ is not a C_0-semigroup in t when α is genuinely variable: in general $U(t_1 + t_2, s) \neq U(t_1,s)U(t_2,s)$. The correct algebraic structure is the two-parameter identity (3.4), which is the natural analogue for variable-order conformable dynamics.

Lemma 3.7. *Let Assumption 3.3 hold and let $f \in L^1(0,T;X)$. Then, for sufficiently regular initial data $x_0 \in X$, the unique mild solution of the abstract Cauchy problem*

$$\mathcal{N}_F^{\alpha(\cdot)} x(t) = A\,x(t) + f(t), \quad t \in (0,T], \qquad x(0) = x_0, \tag{3.6}$$

is given by

$$x(t) = U(t,0)\,x_0 + \int_0^t F(s,\alpha(s))\,U(t,s)\,f(s)\,ds, \qquad t \in [0,T]. \tag{3.7}$$

Sketch of proof. For sufficiently smooth x the variable-order derivative (3.1) reduces to $\mathcal{N}_F^{\alpha(\cdot)} x(t) = F(t,\alpha(t))^{-1} x'(t)$, so that (3.6) is equivalent to $x'(t) = F(t,\alpha(t))\big(Ax(t) + f(t)\big)$. Setting $y(\tau) := x(\Theta^{-1}(\tau))$ with $\tau = \Theta(t)$ and applying the chain rule gives $y'(\tau) = Ay(\tau) + \tilde{f}(\tau)$, where $\tilde{f}(\tau) = f(\Theta^{-1}(\tau))$. The classical variation-of-constants formula for A then yields $y(\tau) = S(\tau)x_0 + \int_0^\tau S(\tau - \sigma)\tilde{f}(\sigma)\,d\sigma$, and undoing the change of variable $\sigma = \Theta(s)$, $d\sigma = F(s,\alpha(s))\,ds$, produces (3.7). The argument is standard once justified in the regular case, and it extends to general $f \in L^1(0,T;X)$ by density, using the strong continuity of U. □

4. Problem formulation and well-posedness

We now formulate the abstract semilinear control system driven by $\mathcal{N}_F^{\alpha(\cdot)}$ with infinite memory, define the associated notion of mild solution and prove well-posedness.

4.1. **The controlled system.** Let $(X, \|\cdot\|)$ be a Banach space, let $\mathcal{U}$ be a separable Hilbert space, the control space, and let $\mathcal{L}(\mathcal{U}, X)$ denote the bounded linear operators from $\mathcal{U}$ into X. For a trajectory $x : (-\infty, T] \to X$ and $t \in [0,T]$, the history segment is $x_t(\theta) := x(t+\theta)$ for $\theta \le 0$, and the initial datum is $\varphi \in \mathcal{B}$. We consider the variable-order controlled system

$$\mathcal{N}_F^{\alpha(\cdot)} x(t) = A\,x(t) + \mathcal{F}(t, x_t) + B\,u(t), \qquad t \in [0,T], \qquad x_0 = \varphi \in \mathcal{B}, \tag{4.1}$$

where $A : D(A) \subset X \to X$ generates the C_0-semigroup $\{S(t)\}_{t \ge 0}$ of (V2) and $\{U(t,s)\}$ is the associated evolution family of Section 3, $B \in \mathcal{L}(\mathcal{U}, X)$ is the control operator, $u \in L^2(0,T;\mathcal{U})$ is the control, and $\mathcal{F} : [0,T] \times \mathcal{B} \to X$ is the history-dependent nonlinearity.

Assumption 4.1. In addition to Assumptions 2.5 and 3.3, we suppose the following.

- (H1) There exist $M \ge 1$ and $\omega \ge 0$ such that $\|S(t)\| \le Me^{\omega t}$ for all $t \ge 0$; equivalently, by Proposition 3.5(iv), $\|U(t,s)\| \le Me^{\omega(\Theta(t) - \Theta(s))}$.
- (H2) The map $\mathcal{F} : [0,T] \times \mathcal{B} \to X$ is continuous and globally Lipschitz in its second argument, that is, there exists $L_{\mathcal{F}} > 0$ with

  $$\|\mathcal{F}(t,\phi_1) - \mathcal{F}(t,\phi_2)\|_X \le L_{\mathcal{F}} \|\phi_1 - \phi_2\|_{\mathcal{B}}, \qquad t \in [0,T],\ \phi_1, \phi_2 \in \mathcal{B},$$

  and $\sup_{t \in [0,T]} \|\mathcal{F}(t,0)\|_X \le m_{\mathcal{F}} < +\infty$.
- (H3) There exists $C_B > 0$ with $\|Bu\|_X \le C_B \|u\|_{\mathcal{U}}$ for all $u \in \mathcal{U}$.

4.2. **Mild solutions.**

Definition 4.2. Assume that Assumption 4.1 holds and fix $u \in L^2(0,T;\mathcal{U})$ and $\varphi \in \mathcal{B}$. A function $x : (-\infty, T] \to X$ is a mild solution of (4.1) on $[0,T]$ if $x|_{[0,T]} \in C([0,T];X)$, $x_0 = \varphi$ in $\mathcal{B}$, and for every $t \in [0,T]$

$$x(t) = U(t,0)\varphi(0) + \int_0^t F(s,\alpha(s))U(t,s)\mathcal{F}(s,x_s)\,ds + \int_0^t F(s,\alpha(s))U(t,s)Bu(s)\,ds. \tag{4.2}$$

The mild-solution formulation (4.2) is consistent with the variation-of-constants approach developed for general fractional evolution equations with nonlocal constraints in [18].

Remark 4.3. When $\alpha(t) \equiv \alpha_0$, formula (4.2) reduces to the constant-order representation through the identification $U(t,s) = T_F^{\alpha_0}\big(G_{\alpha_0}^{-1}(G_{\alpha_0}(t) - G_{\alpha_0}(s))\big)$.

4.3. **Existence and uniqueness.** Define the space of admissible trajectories

$$\mathcal{X}_T := \big\{x : (-\infty, T] \to X : \ x_0 = \varphi,\ x|_{[0,T]} \in C([0,T];X)\big\},$$

endowed with $\|x\|_{\infty,T} := \sup_{t\in[0,T]} \|x(t)\|_X$, and let $\Lambda : \mathcal{X}_T \to \mathcal{X}_T$ be given by

$$(\Lambda x)(t) := \begin{cases} \varphi(t), & t \le 0, \\ U(t,0)\varphi(0) + \displaystyle\int_0^t F(s,\alpha(s))U(t,s)\mathcal{F}(s,x_s)\,ds & \\ \qquad + \displaystyle\int_0^t F(s,\alpha(s))U(t,s)Bu(s)\,ds, & t \in [0,T]. \end{cases} \tag{4.3}$$

Theorem 4.4. *Under Assumption 4.1, set*

$$M_\Theta := Me^{\omega\Theta(T)}, \qquad q := M_\Theta L_{\mathcal{F}} K_{\mathcal{B}} \Theta(T). \tag{4.4}$$

If $q < 1$, then system (4.1) admits a unique mild solution on $[0,T]$.

Proof. The control terms in (4.3) cancel in a difference, so for $x, y \in \mathcal{X}_T$ and $t \in [0,T]$ the bounds (H1)–(H2) together with (2.2) give

$$\|(\Lambda x)(t) - (\Lambda y)(t)\|_X \le M_\Theta L_{\mathcal{F}} K_{\mathcal{B}} \Big(\int_0^t F(s,\alpha(s))\,ds\Big) \|x-y\|_{\infty,T} \le q\,\|x-y\|_{\infty,T},$$

because $\int_0^t F(s,\alpha(s))\,ds = \Theta(t) \le \Theta(T)$. Since $q < 1$, the Banach fixed-point theorem applies to Λ on the complete metric space $(\mathcal{X}_T, \|\cdot\|_{\infty,T})$ and yields a unique mild solution. □

Remark 4.5. If $q \ge 1$, condition (4.4) still holds on a sufficiently small interval $[0,T_0]$ on which $\Theta(T_0)$ is small enough, which gives local existence and uniqueness; a continuation argument then provides a maximal interval of existence.

5. Exact and approximate controllability

In this section X and $\mathcal{U}$ are Hilbert spaces and adjoints are denoted by $(\cdot)^*$. We establish the variable-order analogues of the classical Gramian-based criteria, together with our central quantitative observability inequality.

5.1. **The linear system and the variable-order Gramian.** The linear control system associated with (4.1) is

$$\mathcal{N}_F^{\alpha(\cdot)} x(t) = Ax(t) + Bu(t), \qquad t \in [0,T], \qquad x(0) = x_0 \in X, \tag{5.1}$$

whose mild solution is

$$x(t) = U(t,0)x_0 + \int_0^t F(s,\alpha(s))\, U(t,s)\, Bu(s)\, ds. \tag{5.2}$$

The input-state operator $W_T : L^2(0,T;\mathcal{U}) \to X$ is

$$W_T u := \int_0^T F(s,\alpha(s))\, U(T,s)\, Bu(s)\, ds, \tag{5.3}$$

which is bounded because $F(\cdot,\alpha(\cdot)) \in L^2(0,T)$ by (V1), and the variable-order controllability Gramian $\Gamma_T^{\alpha(\cdot)} : X \to X$ is

$$\Gamma_T^{\alpha(\cdot)} := W_T W_T^* = \int_0^T F(s,\alpha(s))^2\, U(T,s)\, BB^*\, U(T,s)^*\, ds. \tag{5.4}$$

Definition 5.1. System (5.1) is exactly controllable on $[0,T]$ if $\mathrm{Ran}(W_T) = X$, and approximately controllable on $[0,T]$ if $\overline{\mathrm{Ran}(W_T)} = X$.

5.2. **Exact and approximate criteria.**

Theorem 5.2. *The linear system (5.1) is exactly controllable on $[0,T]$ if and only if $\Gamma_T^{\alpha(\cdot)}$ is boundedly invertible on X. In this case, for any target $x_1 \in X$, the control*

$$u(t) = F(t,\alpha(t))\, B^*\, U(T,t)^* \big(\Gamma_T^{\alpha(\cdot)}\big)^{-1} \big(x_1 - U(T,0)x_0\big) \tag{5.5}$$

steers (5.1) from $x(0) = x_0$ to $x(T) = x_1$.

Proof. At $t = T$, (5.2) reads $x(T) = U(T,0)x_0 + W_T u$, so exact controllability amounts to $\mathrm{Ran}(W_T) = X$, which in a Hilbert space is equivalent to the bounded invertibility of $\Gamma_T^{\alpha(\cdot)} = W_T W_T^*$. Setting $u = W_T^*(\Gamma_T^{\alpha(\cdot)})^{-1}(x_1 - U(T,0)x_0)$ gives $W_T u = x_1 - U(T,0)x_0$, hence $x(T) = x_1$. □

Theorem 5.3. *The linear system (5.1) is approximately controllable on $[0,T]$ if and only if*

$$\ker\big(\Gamma_T^{\alpha(\cdot)}\big) = \{0\}. \tag{5.6}$$

Proof. Since $\Gamma_T^{\alpha(\cdot)} = W_T W_T^*$ is self-adjoint and nonnegative, $\langle \Gamma_T^{\alpha(\cdot)}\zeta, \zeta\rangle = \|W_T^*\zeta\|^2$, so that $\Gamma_T^{\alpha(\cdot)}\zeta = 0$ if and only if $W_T^*\zeta = 0$. In a Hilbert space $\overline{\mathrm{Ran}(W_T)} = X$ is equivalent to $\ker(W_T^*) = \{0\}$, which is (5.6). □

5.3. **A quantitative observability inequality.** The Gramian-based criteria above are qualitative. We now derive a quantitative variant which makes explicit the way the deformation $\alpha(\cdot)$ enters the steering cost; this is the technical core of the paper.

Theorem 5.4. *Let Assumptions 2.5, 3.3 and 4.1 hold and assume that the underlying C_0-semigroup $\{S(t)\}$ satisfies the constant-order observability inequality: there exists $c_S > 0$ such that*

$$\|\zeta\|_X^2 \leq c_S \int_0^{\Theta(T)} \left\|B^* S(\Theta(T)-\sigma)^*\zeta\right\|_{\mathcal{U}}^2 \, d\sigma, \qquad \zeta \in X. \tag{5.7}$$

Then the variable-order observability inequality

$$\|\zeta\|_X^2 \leq C(T,\alpha(\cdot)) \int_0^T \left|F(s,\alpha(s))\right|^2 \left\|B^* U(T,s)^*\zeta\right\|_{\mathcal{U}}^2 \, ds, \qquad \zeta \in X, \tag{5.8}$$

holds with the explicit constant

$$C(T,\alpha(\cdot)) = \frac{c_S}{\inf\limits_{s\in(0,T]} F(s,\alpha(s))} \;\leq\; \frac{c_S}{F_{\min}}. \tag{5.9}$$

In particular $C(T,\alpha(\cdot))$ collapses to c_S in the homogeneous case $F(s,\alpha(s)) \equiv 1$, and it is finite whenever the deformation is bounded away from zero on $(0,T]$.

Proof. Fix $\zeta \in X$ and perform the change of variable $\sigma = \Theta(s)$ in (5.7), so that $d\sigma = F(s,\alpha(s))\,ds$ and $\sigma \in [0,\Theta(T)]$ corresponds to $s \in [0,T]$. By (3.3) we have $S(\Theta(T)-\sigma) = S(\Theta(T)-\Theta(s)) = U(T,s)$ and, taking adjoints, $S(\Theta(T)-\sigma)^* = U(T,s)^*$. Substitution into (5.7) gives

$$\|\zeta\|_X^2 \leq c_S \int_0^T \left\|B^* U(T,s)^*\zeta\right\|_{\mathcal{U}}^2 F(s,\alpha(s))\,ds.$$

Writing the weight as $F = |F|^2/F$ and using the lower bound of (V1) in the denominator, we get

$$\|\zeta\|_X^2 \leq \frac{c_S}{\inf_{r\in(0,T]} F(r,\alpha(r))} \int_0^T \left|F(s,\alpha(s))\right|^2 \left\|B^* U(T,s)^*\zeta\right\|_{\mathcal{U}}^2 \, ds,$$

which is (5.8) with the constant (5.9). Its finiteness follows from (V1), which provides the uniform positive lower bound $F_{\min}$. □

Remark 5.5. Inequality (5.8) is the variable-order analogue of the classical observability estimate that underlies the controllability of finite-energy linear systems, and it quantifies a basic principle: the slower the deformed local time scale, that is, the smaller $\inf F$, the larger the energy required in order to steer the system. When F is in addition bounded above one may replace (5.9) by the cruder bound

$$C(T,\alpha(\cdot)) \leq c_S \, \frac{\sup_{s\in(0,T]} F(s,\alpha(s))}{\left(\inf_{s\in(0,T]} F(s,\alpha(s))\right)^2},$$

in which the oscillation of the deformation is apparent; the form (5.9), however, remains finite for deformations that are unbounded near the origin, such as the conformable choice $F(t,\alpha) = t^{\alpha(t)-1}$ used in Section 7, and it is therefore the one we retain. This makes the variable-order Gramian framework not only qualitative but also quantitatively useful.

Corollary 5.6. *Under the assumptions of Theorem 5.4, the linear system (5.1) is approximately controllable on $[0,T]$ and the minimum-energy control satisfies*

$$\inf\big\{\|u\|^2_{L^2(0,T;\mathcal{U})} :\ x(T)=x_1\big\} \le C(T,\alpha(\cdot))\,\|x_1-U(T,0)x_0\|_X^2 \tag{5.10}$$

for every reachable target $x_1 \in \operatorname{Ran}(W_T)$.

Proof. Approximate controllability follows from Theorem 5.3, since (5.8) implies that $W_T^*\zeta = 0$ forces $\zeta = 0$, hence $\ker(\Gamma_T^{\alpha(\cdot)}) = \{0\}$. For (5.10), inequality (5.8) reads $\|\zeta\|^2 \le C(T,\alpha(\cdot))\|W_T^*\zeta\|_{L^2}^2$, that is, $\Gamma_T^{\alpha(\cdot)} \ge C(T,\alpha(\cdot))^{-1}I$ on $\operatorname{Ran}(W_T)$; the minimum-energy control associated with a reachable target x_1 is $u = W_T^*(\Gamma_T^{\alpha(\cdot)})^{-1}(x_1 - U(T,0)x_0)$, whose squared norm is $\langle(\Gamma_T^{\alpha(\cdot)})^{-1}z, z\rangle$ with $z = x_1 - U(T,0)x_0$, and this is bounded by $C(T,\alpha(\cdot))\|z\|^2$. See [22, Ch. 6] for the corresponding constant-order argument. □

5.4. **Regularized control and convergence.** For $\varepsilon > 0$ define the regularized linear control

$$u_\varepsilon(t) := F(t,\alpha(t))\,B^*\,U(T,t)^*\big(\varepsilon I + \Gamma_T^{\alpha(\cdot)}\big)^{-1}\big(x_1 - U(T,0)x_0\big). \tag{5.11}$$

The associated terminal state is

$$x_\varepsilon(T) = U(T,0)x_0 + \Gamma_T^{\alpha(\cdot)}\big(\varepsilon I + \Gamma_T^{\alpha(\cdot)}\big)^{-1}\big(x_1 - U(T,0)x_0\big), \tag{5.12}$$

and $x_\varepsilon(T) \to x_1$ as $\varepsilon \to 0^+$ whenever $\ker(\Gamma_T^{\alpha(\cdot)}) = \{0\}$.

5.5. **Approximate controllability for the semilinear system.** For $x \in \mathcal{X}_T$ define the accumulated nonlinear term

$$\Phi(x) := \int_0^T F(s,\alpha(s))\,U(T,s)\,\mathcal{F}(s,x_s)\,ds, \tag{5.13}$$

and, for $\varepsilon > 0$ and a target $x_1 \in X$, the feedback control

$$u_\varepsilon^x(t) := F(t,\alpha(t))\,B^*\,U(T,t)^*\big(\varepsilon I + \Gamma_T^{\alpha(\cdot)}\big)^{-1}\big(x_1 - U(T,0)\varphi(0) - \Phi(x)\big). \tag{5.14}$$

Theorem 5.7. *Assume Assumption 4.1 and $\ker(\Gamma_T^{\alpha(\cdot)}) = \{0\}$. For each $\varepsilon > 0$ define $\Psi_\varepsilon : \mathcal{X}_T \to \mathcal{X}_T$ by*

$$(\Psi_\varepsilon x)(t) := \begin{cases} \varphi(t), & t \le 0,\\ U(t,0)\varphi(0) + \displaystyle\int_0^t F(s,\alpha(s))U(t,s)\mathcal{F}(s,x_s)\,ds & \\ \qquad\qquad + \displaystyle\int_0^t F(s,\alpha(s))U(t,s)Bu_\varepsilon^x(s)\,ds, & t \in [0,T]. \end{cases}$$

If Ψ_ε is a contraction on $(\mathcal{X}_T, \|\cdot\|_{\infty,T})$, a sufficient condition being given in Lemma 5.8 below, then Ψ_ε admits a unique fixed point x_ε, which is a mild solution of (4.1) driven by $u_\varepsilon^{x_\varepsilon}$. If, moreover, the family $\{\Phi(x_\varepsilon)\}_{\varepsilon>0}$ is bounded in X, then

$$x_\varepsilon(T) \longrightarrow x_1 \qquad \text{as } \varepsilon \to 0^+, \tag{5.15}$$

and (4.1) is approximately controllable on $[0,T]$.

Proof. A fixed point x_ε of Ψ_ε is a mild solution by construction, and at $t = T$ it satisfies $x_\varepsilon(T) = U(T,0)\varphi(0) + \Phi(x_\varepsilon) + W_T u_\varepsilon^{x_\varepsilon}$. By (5.14),

$$W_T u_\varepsilon^{x_\varepsilon} = \Gamma_T^{\alpha(\cdot)}\big(\varepsilon I + \Gamma_T^{\alpha(\cdot)}\big)^{-1}\big(x_1 - U(T,0)\varphi(0) - \Phi(x_\varepsilon)\big),$$

hence $x_\varepsilon(T) - x_1 = -\varepsilon(\varepsilon I + \Gamma_T^{\alpha(\cdot)})^{-1}\big(x_1 - U(T,0)\varphi(0) - \Phi(x_\varepsilon)\big)$. Since $\ker(\Gamma_T^{\alpha(\cdot)}) = \{0\}$, the spectral theorem gives $\varepsilon(\varepsilon I + \Gamma_T^{\alpha(\cdot)})^{-1} z \to 0$ in X for every $z \in X$, which together with the boundedness of $\{\Phi(x_\varepsilon)\}$ yields (5.15). □

Lemma 5.8. *Let $C_W := \sup_{t\in[0,T]} \|W_t\|_{\mathcal{L}(L^2(0,t;\mathcal{U}),X)}$. If*

$$q + \frac{q\, C_W \|W_T\|}{\varepsilon} < 1, \tag{5.16}$$

with q as in (4.4), then Ψ_ε is a contraction on $(\mathcal{X}_T, \|\cdot\|_{\infty,T})$.

Proof. For $x, y \in \mathcal{X}_T$ write $(\Psi_\varepsilon x - \Psi_\varepsilon y)(t) = I_1(t) + I_2(t)$, where I_1 collects the nonlinear terms and I_2 the control terms. Arguing as in Theorem 4.4 gives $\|I_1\|_{\infty,T} \le q\|x - y\|_{\infty,T}$. For I_2, one has $u_\varepsilon^x - u_\varepsilon^y = W_T^*(\varepsilon I + \Gamma_T^{\alpha(\cdot)})^{-1}(\Phi(y) - \Phi(x))$ together with $\|(\varepsilon I + \Gamma_T^{\alpha(\cdot)})^{-1}\| \le 1/\varepsilon$ and $\|\Phi(x) - \Phi(y)\| \le q\|x - y\|_{\infty,T}$, whence $\|u_\varepsilon^x - u_\varepsilon^y\|_{L^2} \le (q\|W_T\|/\varepsilon)\|x - y\|_{\infty,T}$ and $\|I_2\|_{\infty,T} \le (C_W q\|W_T\|/\varepsilon)\|x - y\|_{\infty,T}$. Condition (5.16) then implies contractivity. □

6. Application to variable-order anomalous diffusion

We illustrate the abstract theory on a variable-order generalized conformable diffusion equation with infinite memory and distributed internal control. This model is motivated by anomalous transport in heterogeneous media with a time-varying anomaly index [21, 19, 12].

6.1. Model description.

Let $\Omega \subset \mathbb{R}^d$, $d \ge 1$, be a bounded domain with smooth boundary, let $T > 0$ and let $\alpha : [0,T] \to (0,1]$ be continuous. Fix a nonempty open subset $\omega \subset \Omega$, the control region, with characteristic function χ_ω. Let $\mu : (-\infty,0] \to (0,+\infty)$ satisfy $\mu \in L^1(-\infty,0)$ and let $k : (-\infty,0] \to \mathbb{R}$ satisfy

$$\int_{-\infty}^{0} |k(\theta)|^2 \mu(\theta)\, d\theta < +\infty. \tag{6.1}$$

We consider the variable-order controlled system

$$\begin{cases} \mathcal{N}_F^{\alpha(\cdot)} y(t,\xi) = \Delta y(t,\xi) + \displaystyle\int_{-\infty}^{0} k(\theta)\, \sigma\big(y(t+\theta,\xi)\big)\, d\theta \\ \qquad\qquad\qquad + \chi_\omega(\xi) v(t,\xi), \qquad (t,\xi) \in (0,T] \times \Omega, \\ y(t,\xi) = 0, \qquad (t,\xi) \in (0,T] \times \partial\Omega, \\ y(\theta,\xi) = y_0(\theta,\xi), \qquad (\theta,\xi) \in (-\infty,0] \times \Omega. \end{cases} \tag{6.2}$$

6.2. **Abstract reformulation and verification of the hypotheses.** Set $X := L^2(\Omega), \mathcal{U} := L^2(\omega)$ and $Az := \Delta z$ with $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$; then A generates an analytic C_0-semigroup $\{S(t)\}_{t\geq 0}$ on X [17]. Define $(Bu)(\xi) := \chi_\omega(\xi)u(\xi)$, choose the phase space $\mathcal{B} := \mathcal{B}_{\mu,2}$ of Example 2.6 and the nonlinearity

$$\big(\mathcal{F}(t,\phi)\big)(\xi) := \int_{-\infty}^{0} k(\theta)\, \sigma\big(\phi(\theta)(\xi)\big)\, d\theta. \tag{6.3}$$

With $x(t) := y(t,\cdot)$ and $u(t) := v(t,\cdot)|_\omega$, system (6.2) takes the abstract form (4.1), and the two-parameter family $U(t,s) = S(\Theta(t) - \Theta(s))$ of Section 3 governs its evolution.

The hypotheses of Assumption 4.1 are readily verified. The bound (H1) holds by Proposition 3.5(iv). If σ is globally Lipschitz with constant L_σ and $\sigma(0) = 0$, then by the Cauchy–Schwarz inequality and Fubini's theorem

$$\|\mathcal{F}(t,\phi) - \mathcal{F}(t,\psi)\|_X \leq L_{\mathcal{F}} \|\phi - \psi\|_{\mathcal{B}}, \qquad L_{\mathcal{F}} := L_\sigma \Big(\int_{-\infty}^{0} |k(\theta)|^2 \mu(\theta)\, d\theta \Big)^{1/2} < +\infty, \tag{6.4}$$

which gives (H2), while $\|Bu\|_{L^2(\Omega)} \leq \|u\|_{L^2(\omega)}$ gives (H3) with $C_B = 1$.

6.3. **Well-posedness and approximate controllability.**

Proposition 6.1. *Let σ be globally Lipschitz with $\sigma(0) = 0$, let k satisfy (6.1), let $u \in L^2(0,T;\mathcal{U})$ and assume that*

$$q_{\text{app}} := M e^{\omega \Theta(T)} L_{\mathcal{F}} K_{\mathcal{B}} \Theta(T) < 1. \tag{6.5}$$

Then system (6.2) admits a unique mild solution $y \in C([0,T]; L^2(\Omega))$.

Proof. This is a direct consequence of Theorem 4.4. □

Proposition 6.2. *Under the hypotheses of Proposition 6.1, assume in addition that $\ker(\Gamma_T^{\alpha(\cdot)}) = \{0\}$. Then, for every target $x_1 \in X$ and every $\varepsilon > 0$, there is a control u_ε in $L^2(0,T;\mathcal{U})$ for which the terminal states converge to the target, that is,*

$$\|y_\varepsilon(T,\cdot) - x_1\|_{L^2(\Omega)} \longrightarrow 0 \qquad \text{as } \varepsilon \to 0^+.$$

Proof. This follows from Theorem 5.7. The injectivity $\ker(\Gamma_T^{\alpha(\cdot)}) = \{0\}$ reduces, through the change of variable $\sigma = \Theta(s)$ used in the proof of Theorem 5.4, to the standard approximate-observability property of the heat semigroup with internal control on ω, which holds for every nonempty open set $\omega \subset \Omega$ [22, 5]. □

Remark 6.3. Three points deserve emphasis. First, the Lipschitz constant $L_{\mathcal{F}}$ in (6.4) can be made arbitrarily small by choosing a rapidly decaying memory kernel, which ensures (6.5). Second, exact controllability typically fails for parabolic equations in infinite dimensions, so that approximate controllability is the natural achievable property in this setting; this is confirmed numerically in Section 7.4, where the Gramian of a fine discretization is seen to be injective but severely ill-conditioned. Third, and most importantly, the variable order $\alpha(\cdot)$ does not destroy this property, because the change of variable $\sigma = \Theta(s)$ used to prove Theorem 5.4 transports the unique-continuation property of the heat semigroup from physical time t to deformed time σ and back, with explicit constants tracking $\alpha(\cdot)$.

7. Numerical illustration: constant versus variable order

We present numerical experiments that make the convergence $x_\varepsilon(T) \to x_1$ explicit and, above all, exhibit the impact of the variable order $\alpha(\cdot)$ on the steering cost. A low-dimensional system is treated first, and a semi-discretized parabolic problem is treated in Section 7.4.

7.1. **Setup.** Let $X = \mathcal{U} = \mathbb{R}^2$, $T = 1$ and consider the linear system (5.1) with

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \qquad B = I_2, \qquad x_0 = (0,0)^\top, \qquad x_1 = (1,1)^\top.$$

The deformation is $F(t,\alpha(t)) = t^{\alpha(t)-1}$ for $t > 0$, so that $F \equiv 1$ when $\alpha \equiv 1$ and the conformable family is recovered when α is constant. We compare three regimes: a constant order $\alpha(t) \equiv 3/4$, denoted (R1), which reproduces the constant-order case of the original framework and for which $\Theta(t) = (4/3)t^{3/4}$; a monotone increasing order $\alpha(t) = 3/5 + 2t/5$, denoted (R2), which ranges from $3/5$ at $t = 0$ to 1 at $t = 1$ and models a memory that fades as time progresses; and an oscillating order $\alpha(t) = 3/4 + (1/4)\sin(2\pi t)$, denoted (R3), which models a time-periodic anomaly. The classical semigroup is $S(t) = \operatorname{diag}(e^{-t}, e^{-2t})$, so that

$$U(t,s) = S(\Theta(t) - \Theta(s)) = \operatorname{diag}\big(e^{-(\Theta(t)-\Theta(s))},\, e^{-2(\Theta(t)-\Theta(s))}\big).$$

The three profiles are admissible in the sense of (V1). Indeed, for the conformable deformation one has $F(s,\alpha(s)) \geq 1$ on $(0,1]$, so that $F_{\min} = 1$ and, by Theorem 5.4, $C(T,\alpha(\cdot)) = c_S$ in each regime, whereas $|F(s,\alpha(s))|^2$ behaves like $s^{2\alpha(0)-2}$ near the origin and is integrable precisely when

$$\alpha(0) > \tfrac{1}{2}. \tag{7.1}$$

Condition (7.1) is the sharp admissibility threshold for the conformable deformation: it is what makes the Gramian (5.4) a convergent integral, and it is the reason why the increasing profile (R2) is started at $\alpha(0) = 3/5$ rather than at the borderline value $1/2$.

7.2. **The variable-order Gramian.** With $B = I_2$ and $x_0 = 0$ the Gramian $\Gamma_T^{\alpha(\cdot)}$ is diagonal,

$$\Gamma_T^{\alpha(\cdot)} = \operatorname{diag}\big(\gamma_1^{\alpha(\cdot)}, \gamma_2^{\alpha(\cdot)}\big), \qquad \gamma_i^{\alpha(\cdot)} := \int_0^T F(s,\alpha(s))^2 e^{-2\lambda_i(\Theta(T)-\Theta(s))}\, ds,$$

with $\lambda_1 = 1$ and $\lambda_2 = 2$. The integrals are evaluated numerically in each regime by adaptive quadrature graded towards the endpoint singularity at $s = 0$; for regime (R1) one recovers $\gamma_1^{(R1)} \approx 0.5609$ and $\gamma_2^{(R1)} \approx 0.2715$, in agreement with the constant-order computation of the original framework. Representative values are summarized in Table 1.

In all three regimes $\gamma_1^{\alpha(\cdot)}, \gamma_2^{\alpha(\cdot)} > 0$, so that the system is both exactly and approximately controllable in each case. The monotone regime (R2), in which $\alpha(\cdot)$ increases from $3/5$ to 1, produces the largest integrated deformation and the

TABLE 1. Integrated deformation and variable-order Gramian eigenvalues in the three regimes. The constant-order case (R1) reproduces the values of the underlying constant-order theory.

Regime	Profile $\alpha(t)$	$\Theta(T)$	$\gamma_1^{\alpha(\cdot)}$	$\gamma_2^{\alpha(\cdot)}$
(R1)	$3/4$	1.3333	0.5609	0.2715
(R2)	$3/5 + 2t/5$	1.5192	0.6936	0.2743
(R3)	$3/4 + (1/4)\sin(2\pi t)$	1.1966	0.5305	0.2738

largest Gramian eigenvalues, whereas the oscillating regime (R3), in which $\alpha(\cdot)$ varies around the constant baseline, yields a slightly smaller $\Theta(T)$ and eigenvalues close to those of (R1). The qualitative behaviour of $\alpha(\cdot)$, monotone against oscillating, therefore has a clearly visible impact on the controllability map even before passing to the steering error.

7.3. **Regularized control and convergence to the target.** With $x_0 = 0$ the regularized control (5.11) gives the terminal state

$$x_\varepsilon(T) = \begin{pmatrix} \gamma_1^{\alpha(\cdot)}/(\varepsilon + \gamma_1^{\alpha(\cdot)}) \\ \gamma_2^{\alpha(\cdot)}/(\varepsilon + \gamma_2^{\alpha(\cdot)}) \end{pmatrix}, \tag{7.2}$$

and therefore the steering error

$$\|x_\varepsilon(T) - x_1\| = \left(\frac{\varepsilon^2}{(\varepsilon + \gamma_1^{\alpha(\cdot)})^2} + \frac{\varepsilon^2}{(\varepsilon + \gamma_2^{\alpha(\cdot)})^2} \right)^{1/2}. \tag{7.3}$$

TABLE 2. Steering error as the regularization parameter tends to zero in regimes (R1)–(R3). All three regimes exhibit linear decay in the parameter, with a prefactor controlled by the Gramian eigenvalues of Table 1.

ε	(R1) error	(R2) error	(R3) error
1	1.0144	0.9821	1.0214
10^{-1}	0.3088	0.2954	0.3110
10^{-2}	0.0396	0.0379	0.0398
10^{-3}	0.00408	0.00391	0.00410
10^{-4}	0.000409	0.000392	0.000411
10^{-5}	0.0000409	0.0000392	0.0000411

Table 2 confirms that $x_\varepsilon(T) \to x_1 = (1,1)^\top$ as $\varepsilon \to 0^+$ in all three regimes, in full agreement with Theorems 5.3 and 5.4. The comparison is illuminating: the monotone variable-order regime (R2), which produces the largest Gramian eigenvalues, yields the smallest steering error at any fixed ε, while (R1) and (R3) remain very close to each other. Since $F_{\min} = 1$ in all three cases, the observability constant (5.9) is the same for the three profiles, and the differences seen in Table 2 are entirely due to the size of the Gramian, that is, to the way the deformation redistributes the control effort along $[0, T]$: an order that increases towards 1 keeps the deformed clock running fast over most of the interval and enlarges $\Theta(T)$, whereas an oscillating order essentially compensates its own excursions.

Remark 7.1. Three comments are in order. For the conformable choice $\alpha(t) \equiv \alpha_0$ one has $G_{\alpha_0}(t) = t^{\alpha_0}/\alpha_0$, and as $\alpha_0 \to 1$ the Gramian coefficients converge to the classical values $\gamma_i^* = (1 - e^{-2\lambda_i})/(2\lambda_i)$, $i = 1, 2$. Next, (7.3) gives

$$\|x_\varepsilon(T) - x_1\| \leq \varepsilon \Big(1/(\gamma_1^{\alpha(\cdot)})^2 + 1/(\gamma_2^{\alpha(\cdot)})^2 \Big)^{1/2} + O(\varepsilon^2),$$

so the decay is linear in ε with a prefactor depending explicitly on $\alpha(\cdot)$ through the Gramian eigenvalues. Finally, when the nonlinearity $\mathcal{F}(t, x_t)$ is present, the feedback law (5.14) replaces x_1 by $x_1 - U(T, 0)\varphi(0) - \Phi(x)$ at each fixed-point step; under the contractivity condition of Lemma 5.8 the unique fixed point still satisfies $x_\varepsilon(T) \to x_1$ as $\varepsilon \to 0^+$, with the same rate up to a memory-dependent prefactor.

7.4. **A semi-discretized diffusion problem.** The two-dimensional example above isolates the role of $\alpha(\cdot)$. It does not, however, reflect the infinite-dimensional setting of Sections 5 and 6, and we therefore repeat the experiment on a semi-discretization of the model (6.2). Let $\Omega = (0, 1)$ and let A be the operator $\nu\,\partial_\xi^2$ with $\nu = 0.05$ and homogeneous Dirichlet boundary conditions, discretized by centred finite differences on $n = 199$ interior nodes. The control acts on $\omega = (0.3, 0.7)$ through $Bu = \chi_\omega u$, the initial state is zero, and the memory term is switched off, so that the linear theory of Section 5 applies verbatim. The target is the smooth profile

$$x_1(\xi) = \sin(\pi\xi) + \tfrac{1}{2}\sin(3\pi\xi),$$

and the three order profiles of Section 7 are used again.

The Gramian (5.4) is assembled in the orthonormal eigenbasis $\{q_j\}$ of the discrete Laplacian, in which $U(T, s)$ is diagonal with entries $e^{\lambda_j(\Theta(T) - \Theta(s))}$: writing $P := Q^\top BB^*Q$ one has

$$\big(Q^\top \Gamma_T^{\alpha(\cdot)} Q\big)_{jl} = P_{jl} \int_0^T F(s, \alpha(s))^2 e^{(\lambda_j + \lambda_l)(\Theta(T) - \Theta(s))}\, ds, \tag{7.4}$$

and the scalar integrals are evaluated with a quadrature graded towards $s = 0$, which is required by (7.1). Only the first $m = 60$ modes are retained, the remaining ones contributing below machine precision. Table 3 reports the relative steering error $\|x_\varepsilon(T) - x_1\|/\|x_1\|$ and the relative control energy $\|u_\varepsilon\|_{L^2(0,T;\mathcal{U})}/\|x_1\|$.

The picture is consistent with the theory and complements the two-dimensional experiment in three ways. First, the ordering of the regimes is unchanged: the monotone profile (R2) is the cheapest and the oscillating profile (R3) the most expensive, at every value of ε, exactly as predicted by the Gramian eigenvalues of Table 1. Second, the convergence $x_\varepsilon(T) \to x_1$ is accompanied by a steady growth

Table 3. Semi-discretized variable-order diffusion problem with $n = 199$: relative steering error (left) and relative control energy (right) for the three order profiles.

	relative steering error			relative control energy		
ε	(R1)	(R2)	(R3)	(R1)	(R2)	(R3)
1	0.7316	0.6456	0.7558	0.42	0.44	0.41
10^{-2}	0.2007	0.1679	0.2148	2.63	2.34	2.73
10^{-4}	0.0201	0.0168	0.0217	5.07	4.29	5.41
10^{-6}	0.00172	0.00145	0.00184	6.57	5.54	7.02
10^{-8}	0.000140	0.000118	0.000149	7.44	6.28	7.95

of the control energy, which increases by more than one order of magnitude between $\varepsilon = 1$ and $\varepsilon = 10^{-8}$; this is the numerical signature of Remark 6.3, namely that only approximate, and not exact, controllability can be expected for parabolic dynamics. Third, the spectrum of the discretized Gramian illustrates the same phenomenon quantitatively: in regime (R1) its eigenvalues decay from $6.97 \cdot 10^{-1}$ to $1.46 \cdot 10^{-2}$, $3.07 \cdot 10^{-3}$, $7.28 \cdot 10^{-4}$ and $1.00 \cdot 10^{-7}$ for the first, fifth, tenth, twentieth and fortieth modes, so that the Gramian is injective, as required by Theorem 5.3, but is not boundedly invertible uniformly in the discretization parameter.

7.5. **Computational cost of the regularized feedback.**

Remark 7.2. The scheme used above is directly applicable to large discretized systems, and its cost is dominated by two operations. Assembling (7.4) requires, at each of the N_q quadrature nodes, the action of $U(T, s) = S(\Theta(T) - \Theta(s))$; for a symmetric discretization one may diagonalize A once, at cost $O(n^3)$, after which each node costs $O(m^2)$ in the $m \ll n$ retained modes, which gives $O(n^3 + N_q m^2)$ in total, whereas for large sparse nonsymmetric problems one replaces the diagonalization by Krylov approximations of the exponential action, at cost $O(N_q \,\mathrm{nnz}(A)\, \kappa)$ with κ Krylov steps. The assembly is embarrassingly parallel in the quadrature nodes. Evaluating the control (5.14) then amounts to solving a system with the symmetric positive definite matrix $\varepsilon I + \Gamma_T^{\alpha(\cdot)}$, at cost $O(m^3)$ by a direct factorization, or $O(\sqrt{\kappa_\varepsilon}\, m^2)$ by conjugate gradients, where $\kappa_\varepsilon \le 1 + \lambda_{\max}(\Gamma_T^{\alpha(\cdot)})/\varepsilon$; the rapid spectral decay reported above means that a low-rank truncation at a tolerance matched to ε reduces this to $O(rm)$ per application with r of the order of ten in the experiment of Section 7.4. Two further points are specific to the variable-order setting. The quadrature must be graded near $s = 0$, since by (7.1) the integrand behaves like $s^{2\alpha(0)-2}$ there, and a uniform grid degrades the accuracy of Θ and of the Gramian; and, in the semilinear case, each iteration of the fixed-point map of Theorem 5.7 requires one further evaluation of Φ, so that the total cost is that of the linear scheme multiplied by the number of contraction steps, which is $O(\log(1/\mathrm{tol}))$ under (5.16).

8. Conclusion and outlook

This paper has developed a unified controllability theory for semilinear evolution systems with infinite memory governed by the variable-order generalized conformable derivative $\mathcal{N}_F^{\alpha(\cdot)}$. The central technical device is the two-parameter evolution family $\{U(t,s)\}_{0\leq s\leq t\leq T}$, defined by $U(t,s) = S(\Theta(t) - \Theta(s))$ with $\Theta(t) = \int_0^t F(s,\alpha(s))\,ds$, which replaces the constant-order semigroup and remains well defined precisely where the time-change trick breaks down.

Well-posedness was established under a Lipschitz condition on the nonlinearity and a smallness condition on $\Theta(T)$, by a contraction argument in $\mathcal{X}_T$. For the linear system, necessary and sufficient conditions for exact and approximate controllability were derived in terms of the invertibility and the injectivity of the variable-order Gramian $\Gamma_T^{\alpha(\cdot)}$. A central new result is the quantitative observability inequality with the explicit constant (5.9), which collapses to the known constant-order estimate when α is constant and remains meaningful for deformations that are unbounded at the origin. For the semilinear case, approximate controllability was obtained by a regularized feedback strategy combined with a fixed-point argument, and the numerical experiments, both in low dimension and on a semi-discretized parabolic problem, illustrate the dependence of the steering error, of the control energy and of the computational cost on $\alpha(\cdot)$.

Several directions remain open: nonlocal initial conditions [3, 18], impulsive effects, state-dependent delay, stochastic perturbations in the spirit of [7], sharp observability constants in specific geometries, structure-preserving and adaptive discretizations of the deformed time scale Θ in the spirit of [16], and the scaling of the feedback scheme of Section 7.5 to genuinely large-scale discretizations through model reduction. We hope that the two-parameter framework introduced here will provide a useful starting point for these developments.

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References

1. Abdeljawad, T. (2015). On conformable fractional calculus. Journal of Computational and Applied Mathematics, 279, 57–66. https://doi.org/10.1016/j.cam.2014.10.016
2. Bahloul, R., Rachad, H., & Abdeljawad, T. (2025). N_F^{α}-fractional semi-groups of operators. Methods of Functional Analysis and Topology, 31(1), 39–46. https://doi.org/10.31392/MFAT-npu26_1.2025.04
3. Byszewski, L. (1991). Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. Journal of Mathematical Analysis and Applications, 162, 494–505. https://doi.org/10.1016/0022-247X(91)90255-W

4. Cui, L., & Lu, J. (2024). Exact null controllability of string equations with Neumann boundaries. Journal of Mathematics, 2024, Article 8890544. https://doi.org/10.1155/2024/8890544
5. Curtain, R. F., & Zwart, H. (1995). An introduction to infinite-dimensional linear systems theory. Springer. https://doi.org/10.1007/978-1-4612-4224-6
6. Fleitas, A., Nápoles Valdés, J. E., Rodríguez, J. M., & Sigarreta, J. M. (2021). Note on the generalized conformable derivative. Revista de la Unión Matemática Argentina, 62(2), 443–457. https://doi.org/10.33044/revuma.1930
7. Hajji, S., Lahmoudi, A., & Lakhel, E. H. (2023). Transportation inequalities for mean-field neutral stochastic functional differential equation driven by a fractional Brownian motion. Annals of Mathematics and Computer Science, 18. https://doi.org/10.56947/amcs.v18.203
8. Hale, J. K., & Verduyn Lunel, S. M. (1993). Introduction to functional differential equations. Springer. https://doi.org/10.1007/978-1-4612-4342-7
9. Kang, X., & Fan, H. (2024). Approximate controllability for second-order stochastic neutral evolution equations with infinite delay. Evolution Equations and Control Theory, 13, 140–159. https://doi.org/10.3934/eect.2023040
10. Khalil, R., Al Horani, M., Yousef, A., & Sababheh, M. (2014). A new definition of fractional derivative. Journal of Computational and Applied Mathematics, 264, 65–70. https://doi.org/10.1016/j.cam.2014.01.002
11. Liu, J., Wei, W., & Xu, W. (2022). Approximate controllability of non-instantaneous impulsive stochastic evolution systems driven by fractional Brownian motion with Hurst parameter in the interval from zero to one half. Fractal and Fractional, 6, Article 440. https://doi.org/10.3390/fractalfract6080440
12. Lorenzo, C. F., & Hartley, T. T. (2002). Variable order and distributed order fractional operators. Nonlinear Dynamics, 29, 57–98. https://doi.org/10.1023/A:1016586905654
13. Mahmudov, N. I. (2013). Approximate controllability of fractional Sobolev-type evolution equations in Banach spaces. Abstract and Applied Analysis, 2013, Article 502839. https://doi.org/10.1155/2013/502839
14. Moumen, A., Shafqat, R., Alsinai, A., Boulares, H., Cancan, M., & Jeelani, M. B. (2023). Analysis of fractional stochastic evolution equations by using Hilfer derivative of finite approximate controllability. AIMS Mathematics, 8, 16094–16114. https://doi.org/10.3934/math.2023821
15. Mshary, N., Ahmed, H. M., & Ghanem, A. S. (2024). Existence and controllability of nonlinear evolution equation involving Hilfer fractional derivative with noise and impulsive effect via Rosenblatt process and Poisson jumps. AIMS Mathematics, 9, 9746–9769. https://doi.org/10.3934/math.2024477
16. Olaniyan, A., Akanbi, M., Wusu, A., & Shonibare, K. (2024). A four-stage multiderivative explicit Runge-Kutta method for the solution of first order ordinary differential equations. Annals of Mathematics and Computer Science, 20. https://doi.org/10.56947/amcs.v20.224
17. Pazy, A. (1983). Semigroups of linear operators and applications to partial differential equations. Springer. https://doi.org/10.1007/978-1-4612-5561-1
18. Rachad, H., & Bahloul, R. (2026). A mild solution approach to general fractional evolution equations with nonlocal constraints. Gulf Journal of Mathematics, 22, 1–11. https://doi.org/10.56947/gjom.v22i1.3834
19. Samko, S. G., & Ross, B. (1993). Integration and differentiation to a variable fractional order. Integral Transforms and Special Functions, 1, 277–300. https://doi.org/10.1080/10652469308819027
20. Selvam, A. P., & Govindaraj, V. (2024). Investigation of controllability and stability of fractional dynamical systems with delay in control. Mathematics and Computers in Simulation, 220, 89–104. https://doi.org/10.1016/j.matcom.2024.01.011

21. Sun, H., Chang, A., Zhang, Y., & Chen, W. (2019). A review on variable-order fractional differential equations: Mathematical foundations, physical models, numerical methods and applications. Applied Mathematical Modelling, 74, 462–482. https://doi.org/10.1016/j.apm.2019.05.027
22. Tucsnak, M., & Weiss, G. (2009). Observation and control for operator semigroups. Birkhäuser. https://doi.org/10.1007/978-3-7643-8994-9
23. Wang, J., & Zhou, Y. (2012). Complete controllability of fractional evolution systems. Communications in Nonlinear Science and Numerical Simulation, 17, 4346–4355. https://doi.org/10.1016/j.cnsns.2012.02.029
24. Wu, J. (1996). Theory and applications of partial functional differential equations. Springer. https://doi.org/10.1007/978-1-4612-4050-1
25. Yaşar, E., & Kopçasız, B. (2023). Novel multi-wave solutions for the fractional order dual-mode nonlinear Schrödinger equation. Annals of Mathematics and Computer Science, 16, 100–111. https://doi.org/10.56947/amcs.v16.186
26. Yosida, K. (1948). On the differentiability and the representation of one-parameter semi-groups of linear operators. Journal of the Mathematical Society of Japan, 1, 15–21. https://doi.org/10.2969/jmsj/00110015
27. Zang, Y., & Li, J. (2013). Approximate controllability of fractional impulsive neutral stochastic differential equations with nonlocal conditions. Boundary Value Problems, 2013, Article 193. https://doi.org/10.1186/1687-2770-2013-193

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