

CERTAIN GEOMETRIC PROPERTIES OF FUNCTIONS INVOLVING GALUË TYPE STRUVE FUNCTION

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ABSTRACT. In this work, we make use of the Galuë type Struve function (GTSF) $\alpha w_{p,b,c,\xi}^{\lambda,\mu}(z)$ to define a normalized analytic function

$$g_{p,b,c,\xi}(z) = z + \sum_{n=1}^{\infty} \frac{\left(\frac{-c}{4}\right)^n}{(\mu)_{\lambda n}(\gamma)_{\alpha n}} z^{n+1}.$$

Strong star-likeness, strong convexity and uniform convexity properties results for the normalized analytic function are obtained. Relevant connections of these results with certain results for the Struve functions are pointed out as corollaries.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of a function $f(z)$ normalized by $f(0) = f'(0) - 1 = 0$ and is of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Denote S to be the subclass of $\mathcal{A}$ consisting of univalent functions. Also, let S^* , C , $S^*(\alpha)$, $\tilde{S}^*(\alpha)$ and $\tilde{C}(\alpha)$ denote the subclasses of S which are star-like, convex, star-like of order α , strongly star-like of order α , and strongly convex of order α respectively, and defined as:

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$$\begin{aligned}
S^* &= \{f : f \in S \text{ and } \Re \left(\frac{zf'(z)}{f(z)} \right) > 0, z \in U\}, \\
C &= \{f : f \in S \text{ and } \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, z \in U\}, \\
S^*(\alpha) &= \{f : f \in S \text{ and } \Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha, z \in U, 0 \leq \alpha < 1\}, \\
\tilde{S}^*(\alpha) &= \{f : f \in S \text{ and } \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| > \frac{\alpha\pi}{2}, z \in U, 0 \leq \alpha < 1\}, \\
&\text{and} \\
\tilde{C}^*(\alpha) &= \{f : f \in S \text{ and } \left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| > \frac{\alpha\pi}{2}, z \in U, 0 \leq \alpha < 1\},
\end{aligned}$$

It is evident that

$$\tilde{S}^*(1) = S^*(0) = S^*, \quad \tilde{C}^*(1) = C(0) = C.$$

For more insight on these classes, see ([2], [3], [4] and [5]).

Among the special functions whose interest has attracted the attention of many researchers is the Struve function and its other variant, the generalized Struve function.

The function H_p is called the Struve function of order p and is defined as a particular solution of the second-order inhomogeneous differential equation

$$z^2 w''(z) + z w'(z) + (z^2 - p^2) w(z) = \frac{4 \left(\frac{z}{2}\right)^{p+1}}{\sqrt{\pi} \Gamma(p + \frac{1}{2})} \quad (1.2)$$

such that

$$H_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n + \frac{3}{2}) \Gamma(p + n + \frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \quad (z \in U). \quad (1.3)$$

Also, let us consider the following differential equation which only differs in the coefficient of w :

$$z^2 w''(z) + z w'(z) - (z^2 + p^2) w(z) = \frac{4 \left(\frac{3}{2}\right)^{p+1}}{\sqrt{\pi} \Gamma(p + \frac{1}{2})} \quad (1.4)$$

and whose particular solution $L_p(z)$ is called the modified Struve function of order p and is defined as:

$$L_p(z) = -i e^{\frac{-ip\pi}{2}} H_p(iz) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n + \frac{3}{2}) \Gamma(p + n + \frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \quad (z \in U) \quad (1.5)$$

For more in-depth discussion on Struve function and other works related to it, interested reader may refer to ([6] and [7]).

Next, consider the following differential equation [8]:

$$z^2 w''(z) + bz w'(z) + [cz^2 - p^2 + (1-b)p] w(z) = \frac{4 \left(\frac{z}{2}\right)^{p+1}}{\sqrt{\pi} \Gamma(p + \frac{1}{2})} \quad (1.6)$$

when $b, c, p \in \mathbb{C}$. If $b = 1$ and $c = 1$, then equation (1.6) becomes (1.2), and if $b = 1$ and $c = -1$, then we get (1.4). Which means that (1.6) generalizes (1.2) and (1.4). The generalization given by (1.6) permits the study of (1.2) and (1.4) together. A particular solution of (1.6) which is denoted by $w_{p,b,c}(z)$ and whose series representation is defined as

$$w_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{\Gamma(n + \frac{3}{2}) \Gamma(p + n + \frac{b+2}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \quad (z \in U) \quad (1.7)$$

is referred to as the generalized Struve function of order p .

Although, the series defined by (1.7) is convergent everywhere in the complex plane $\mathbb{C}$, the function is generally not univalent in U (see [9]).

Now, consider the function $u_{p,b,c}(z)$ defined as follows:

$$u_{p,b,c}(z) = 2^p \sqrt{\pi} \Gamma\left(p + \frac{b+2}{2}\right) z^{\frac{-p-1}{2}} w_{p,b,c}(\sqrt{z}). \quad (1.8)$$

Using the Pochhammer (or Appell) symbol, defined in terms of Euler's gamma function, by $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \gamma(\gamma+1)\cdots(\gamma+n-1)$, the following form of the function $u_{p,b,c}(z)$ can be written:

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{-c}{4}\right)^n}{\left(\frac{3}{2}\right)_n (k)_n} z^n = b_0 + b_1 z + b_2 z^2 + \cdots, \quad (1.9)$$

when $k = p + \frac{b+2}{2} \neq 0, -1, -2, \dots$ and

$$b_n = \frac{(-1)^n c^n \Gamma(\frac{3}{2}) \Gamma(k)}{4^n \Gamma(n + \frac{3}{2}) \Gamma(n+k)}, \quad \text{for } n \geq 0.$$

This function defined in (1.9) is not analytic in $\mathbb{C}$ but satisfies the second-order inhomogeneous differential equation

$$4z^2 u''(z) + 2(2p+b+3)z u'(z) + (cz + 2p+b)u(z) = 2p+b \quad (1.10)$$

as well as the condition $u_{p,b,c}(0) = 1$.

We note that the normalized form of the Struve function defined by (1.9) which is analytic in $\mathbb{C}$ is defined as

$$\begin{aligned} \nu_{p,b,c}(z) &= zu_{p,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{\left(\frac{-c}{4}\right)^n}{\left(\frac{3}{2}\right)_n (k)_n} z^{n+1} \\ &= z + \sum_{n=2}^{\infty} \frac{\left(\frac{-c}{4}\right)^{n-1}}{\left(\frac{3}{2}\right)_{n-1} (k)_{n-1}} z^n. \end{aligned} \quad (1.11)$$

The functions defined by (1.9) and (1.11) were first introduced and studied in [6] and very recently, the function defined by (1.11) was further investigated in ([10], [11]).

Mahmood et al. [12], studied the geometric properties of the generalized Struve function defined by (1.11) and obtained the following results:

Theorem 1.1. *If $q \geq \frac{7|c|}{12}$, then $V_{L,b,c} \in \tilde{S}^*(\alpha)$,*

$$\text{where: } \alpha = \frac{2}{\pi} \arcsin \left(\psi \sqrt{1 - \frac{\psi^2}{4}} + \frac{\psi}{2} \sqrt{1 - \psi^2} \right)$$

$$\text{and } \psi = \frac{4|c|}{3(4q - |c|)} \text{ is such that } \arcsin \frac{\psi}{2} + \arcsin \psi \in \left[\frac{-\pi}{2}, \frac{\pi}{2} \right].$$

Theorem 1.2. *If $q \geq \frac{4|c|}{3}$, then $V_{L,b,c} \in \tilde{C}(\alpha)$,*

$$\text{where: } \alpha = \frac{2}{\pi} \arcsin \left(\psi \sqrt{1 - \frac{\psi^2}{4}} + \frac{\psi}{2} \sqrt{1 - \psi^2} \right)$$

$$\text{and } \psi = \frac{2|c|}{3q - 2|c|} \text{ is such that } \arcsin \frac{\psi}{2} + \arcsin \psi \in \left[\frac{-\pi}{2}, \frac{\pi}{2} \right].$$

Theorem 1.3. *Let $q > \frac{19|c|}{12}$, then $V_{L,b,c} \in U \subset V$.*

Furthermore, arising from the function $u_{p,b,c}(z)$ is the function $T_{p,b,c}(z)$ which is obtained from the convolution of $f \in \mathcal{A}$ and $g_{p,b,c}(z) = z u_{p,b,c}(z)$, i.e.,

$$T_{p,b,c}(z) = (f * g_{p,b,c})(z) = z + \sum_{n=2}^{\infty} \frac{\left(\frac{-c}{4}\right)^n}{\left(\frac{3}{2}\right)_{n-1} (k)_{n-1}} a_n z^n, \quad (z \in U) \quad (1.12)$$

where $p, b, c \in \mathbb{C}$ and $\alpha = p + \frac{b+2}{2}$. $T_{p,b,c}(z)$ was first introduced by Raza and Yagmur [11] while for other works on the geometric properties for the function $T_{p,b,c}(z)$, one may refer to ([13] and [14]).

In [1], the Galuê type Struve function (GTSF), another generalized Struve function was introduced by Nisar et al. and is defined as:

$$\alpha w_{p,b,c,\xi}^{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(\lambda n + \mu) \Gamma(an + \frac{p}{\xi} + \frac{p+2}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \quad \alpha \in \mathbb{N}, p, b, c \in \mathbb{C} \quad (1.13)$$

where $\lambda > 0$, $\xi > 0$ and μ is an arbitrary parameter.

It is evident that when $\lambda = \alpha = 1$, $\mu = \frac{3}{2}$ and $\xi = 1$ in (1.13), it reduces to generalization of Struve function which is defined by Orhan and Yagmur ([9, 10]).

Let us now consider the function $u_{p,b,c,\xi}(z)$ which is defined as follows:

$$\begin{aligned} u_{p,b,c,\xi}(z) &= 2^p \sqrt{\pi} \Gamma\left(\frac{p}{\xi} + \frac{p+2}{2}\right) z^{\frac{-(p+1)}{2}} \alpha w_{p,b,c,\xi}^{\lambda,\mu}(\sqrt{z}) \\ &= \frac{\sqrt{\pi}}{2} \Gamma\left(\frac{p}{\xi} + \frac{p+2}{2}\right) \sum_{n=0}^{\infty} \frac{(-c)^n}{4^n \Gamma(\lambda n + \mu) \Gamma(\alpha n + \frac{p}{\xi} + \frac{b+2}{2})} z^n \end{aligned} \quad (1.14)$$

Let $\gamma = \left(\frac{p}{\xi} + \frac{p+2}{2}\right) \neq 0, -1, -2, \dots$ in (1.14), we get

$$u_{p,b,c,\xi}(z) = \Gamma\left(\frac{3}{2}\right) \Gamma(\gamma) \sum_{n=0}^{\infty} \frac{(-c)^n}{4^n \Gamma(\lambda n + \mu) \Gamma(\alpha n + \gamma)} z^n. \quad (1.15)$$

Using Pochhammer (or Appell) defined in terms of the Euler's gamma function by $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \gamma(\gamma+1)\cdots(\gamma+n-1)$. By working on (1.15) it is possible to have the following form of $u_{p,b,c,\xi}(z)$:

$$u_{p,b,c,\xi}(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{-c}{4}\right)^n}{(\mu)_{\lambda n} (\gamma)_{\alpha n}} z^n = b_0 + b_1 z + b_2 z^2 + \dots \quad (1.16)$$

where $(\gamma)_{\alpha n} = \frac{\Gamma(\alpha n + \gamma)}{\Gamma(\gamma)}$, $(\mu)_{\lambda n} = \frac{\Gamma(\lambda n + \mu)}{\Gamma(\mu)}$, $\gamma = \frac{p}{\xi} + \frac{b+2}{2}$ and μ is an arbitrary parameter.

Next, we let

$$\begin{aligned} g_{p,b,c,\xi}(z) &= z u_{p,b,c,\xi}(z) = z + \sum_{n=1}^{\infty} \left(\frac{\left(\frac{-c}{4}\right)^n}{(\mu)_{\lambda n} (\gamma)_{\alpha n}} \right) z^{n+1} \\ &= z + \sum_{n=2}^{\infty} \left(\frac{\left(\frac{-c}{4}\right)^{n-1}}{(\mu)_{\lambda(n-1)} (\gamma)_{\alpha(n-1)}} \right) z^n, \quad z \in U \end{aligned} \quad (1.17)$$

The equation defined in (1.17) is the normalized form of the Galuê type Struve function and is analytic on $\mathbb{C}$.

Consequently, by making use of the concept of convolution on (1.1) and (1.17), we define another function

$$\begin{aligned} \alpha L_{p,b,c,\xi}(z) &= (f * g_{p,b,c,\xi})(z) \\ &= z + \sum_{n=2}^{\infty} \left(\frac{\left(\frac{-c}{4}\right)^{n-1}}{(\mu)_{\lambda(n-1)} (\gamma)_{\alpha(n-1)}} \right) a_n z^n, \quad z \in U \end{aligned} \quad (1.18)$$

where $p, b, c \in \mathbb{C}$, $\gamma = \frac{p}{\xi} + \frac{b+2}{2} \neq 0, -1, -2, \dots$, $\alpha \in \mathbb{N}$, $\lambda, \xi > 0$ (or $\lambda > 0, \xi > 0$) and μ is an arbitrary parameter.

Motivated by the above geometric properties of generalized Struve function given in section 3 of [1] and as given by the Theorem 1.1 – Theorem 1.3 above in section 2, we now derive the geometric properties of the function $g_{p,b,c,\xi}(z)$ involving the

Galuê type Struve function as a generalization of the one in section 3 of [1]. In order to achieve this objective, the following results are indispensable:

Lemma 1.4 ([15]). *Let $G(z)$ be convex and univalent in the open unit disk with condition $G(0) = 1$. Let $f(z)$ be analytic in the open unit disk with condition $F(0) = 1$ and $F \prec G$ in the open unit disk. Then, $\forall n \in \mathbb{N} \cup \{0\}$, we obtain*

$$(n+1) z^{-1-n} \int_0^z t^n F(t) dt \prec (n+1) z^{-1-n} \int_0^z t^n G(t) dt.$$

Lemma 1.5 ([16]). *If $g \in \mathcal{A}$ satisfies: $\left| \frac{z g''(z)}{g'(z)} \right| < \frac{1}{2}$, then $g \in U \in V$.*

2. MAIN RESULTS

The Geometric properties of Galuê type Struve function are shown in the theorems below. Unless, it is otherwise stated, every parameter used henceforth is as earlier defined.

Theorem 2.1. *If $\gamma^\alpha \geq \frac{7(2[\frac{\mu}{2}])_\lambda |c|}{12}$, then $\delta_{p,b,c,\xi} \in \tilde{S}^*(\alpha)$, where*

$$\beta = \frac{2}{\pi} \arcsin \left(\phi \sqrt{1 - \frac{\phi^2}{4}} + \frac{\phi}{2} \sqrt{1 - \phi^2} \right), \quad (2.1)$$

and $\phi = \frac{4|c|}{2(2[\frac{\mu}{2}])_\lambda (4\gamma - |c|)}$ is such that $\arcsin \frac{\phi}{2} + \arcsin \phi \in \left[\frac{-\pi}{2}, \frac{\pi}{2} \right]$.

Proof. From equation (1.17) we have that

$$g'_{p,b,c,\xi}(z) - 1 = \sum_{n=1}^{\infty} \frac{(-c)^n (n+1)}{(\mu)_{\lambda n} 4^n (\gamma)_{\alpha n}} z^n,$$

so that by triangle inequality,

$$|g'_{p,b,c,\xi}(z) - 1| \leq \sum_{n=1}^{\infty} \frac{|c|^n (n+1)}{(\mu)_{\lambda n} 4^n (\gamma)_{\alpha n}}.$$

Now, if $(\mu)_{\lambda n} \geq (2[\frac{\mu}{2}])_\lambda (n+1)$, $(\gamma)_{\alpha n} \geq \gamma^{\alpha n}$, $\forall n \geq 1$. Then

$$\begin{aligned} \left| g'_{p,b,c,\xi}(z) - 1 \right| &\leq \frac{|c|}{4(2[\frac{\mu}{2}])_\lambda \gamma^\alpha} \sum_{n=1}^{\infty} \left(\frac{|c|}{4\gamma^\alpha} \right)^{n-1} \\ &= \frac{4|c|}{2(2[\frac{\mu}{2}])_\lambda (4\gamma^\alpha - |c|)} = \phi, \quad \gamma^\alpha > \frac{|c|}{4} \end{aligned} \quad (2.2)$$

For $\gamma^\alpha \geq \frac{7|c|}{12}$, it is clear that $\phi \in (0, 1]$.

Again, from (2.2), we can conclude that

$$g'_{p,b,c,\xi}(z) \prec 1 + \phi(z) \quad (2.3)$$

$$\Rightarrow |\arg(g'_{p,b,c,\xi}(z))| < \arcsin \phi$$

By making use of Lemma 1.4, taking $n = 0$, with $F(z) = g'_{p,b,c,\xi}(z)$ and $G(z) = 1 + \phi z$, we obtain

$$\frac{g'_{p,b,c,\xi}(z)}{z} \prec 1 + \frac{\phi}{2}z \quad (2.4)$$

So that

$$\left| \arg \left(\frac{g'_{p,b,c,\xi}(z)}{z} \right) \right| < \arcsin \frac{\phi}{2} \quad (2.5)$$

Using the relations (2.3) and (2.4), we get

$$\begin{aligned} \left| \arg \left(\frac{z g'_{p,b,c,\xi}(z)}{g_{p,b,c,\xi}(z)} \right) \right| &= \left| \arg \left(\frac{z}{g_{p,b,c,\xi}(z)} \right) - \arg(g'_{p,b,c,\xi}(z)) \right| \\ &\leq \left| \arg \left(\frac{z}{g_{p,b,c,\xi}(z)} \right) \right| + \left| \arg \left(\frac{z}{g'_{p,b,c,\xi}(z)} \right) \right| \\ &< \arcsin \frac{\phi}{2} + \arcsin \phi \end{aligned}$$

Since $\phi \in (0, 1]$, one can write the above last expression as:

$$\left| \arg \left(\frac{z g'_{p,b,c,\xi}(z)}{g_{p,b,c,\xi}(z)} \right) \right| < \arcsin \left(\phi \sqrt{1 - \frac{\phi^2}{4}} + \frac{\phi}{2} \sqrt{1 - \phi^2} \right), \quad (2.6)$$

which shows that $\delta_{p,b,c,\xi}(z) \in \tilde{S}^*(\alpha)$ for

$$\beta = \frac{2}{\pi} \arcsin \left(\phi \sqrt{1 - \frac{\phi^2}{4}} + \frac{\phi}{2} \sqrt{1 - \phi^2} \right).$$

□

Remark 2.2. Since $\phi = \frac{4|c|}{2(2[\frac{p}{2}](4\gamma - |c|))}$ when $\gamma = \frac{p}{\xi} + \frac{b+2}{2}$ and μ is an arbitrary parameter.

By suitably varying the parameters involved, such that for $\mu = \frac{3}{2}, \xi = \lambda = \alpha = 1$; we have the following corollary

Corollary 2.3. If $q \geq \frac{7|c|}{12}$, then $V_{L,b,c} \in \tilde{S}^*(\alpha)$, where:

$$\alpha = \frac{2}{\pi} \arcsin \left(\psi \sqrt{1 - \frac{\psi^2}{4}} + \frac{\psi}{2} \sqrt{1 - \psi^2} \right)$$

and $\phi = \frac{4|c|}{3(4q - |c|)} = \psi$ when $\mu = \frac{3}{2}$, $\xi = 1$, $\lambda = \alpha = 1$ is such that $\arcsin \frac{\psi}{2} + \arcsin \psi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ which is Theorem 1 as established in [1].

Theorem 2.4. If $\gamma^\alpha \geq \frac{8(2[\frac{\mu}{2}])_\lambda |c|}{9}$, then $g_{p,b,c,\xi} \in \tilde{C}(\alpha)$, where:

$$\beta = \frac{2}{\pi} \arcsin \left(\theta \sqrt{1 - \frac{\theta^2}{4}} + \frac{\theta}{2} \sqrt{1 - \theta^2} \right), \quad (2.7)$$

and $\theta = \frac{4(2[\frac{\mu}{2}])_\lambda |c|}{9\gamma^\alpha - 4(2[\frac{\mu}{2}])_\lambda |c|}$ is such that $\arcsin \frac{\theta}{2} + \arcsin \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Proof. By using the triangular inequality : $|z_1 + z_2| \leq |z_1| + |z_2|$,

with the inequalities: $(\mu)_{\lambda n} \leq \frac{4}{3}(2[\frac{\mu}{2}])_\lambda$, $(n+1)^2 \leq 4^n$, $(\gamma)_n \geq \gamma^n \forall n \in \mathbb{N}$

$$\begin{aligned} \left(z g'_{p,b,c,\xi}(z) \right)' - 1 &= \frac{4(2[\frac{\mu}{2}])_\lambda (-c)}{3\gamma^\alpha} \sum_{n=1}^{\infty} \frac{4^{n-1} (-c)^{n-1} [(2[\frac{\mu}{2}])_\lambda]^{n-1}}{3^{n-1} (\gamma^\alpha)^{n-1}} \\ \left| g'_{p,b,c,\xi}(z) - 1 \right| &\leq \frac{4|c|(2[\frac{\mu}{2}])_\lambda}{3\gamma^\alpha} \sum_{n=1}^{\infty} \frac{4^{n-1} |c|^{n-1} [(2[\frac{\mu}{2}])_\lambda]^{n-1}}{3^{n-1} (\gamma^\alpha)^{n-1}} \\ &\leq \frac{4|c|(2[\frac{\mu}{2}])_\lambda}{3\gamma^\alpha} \sum_{n=1}^{\infty} \left(\frac{4(2[\frac{\mu}{2}])_\lambda |c|}{3\gamma^\alpha} \right)^{n-1} = \frac{4|c|(2[\frac{\mu}{2}])_\lambda}{3\gamma^\alpha - 4(2[\frac{\mu}{2}])_\lambda |c|} = \rho \end{aligned} \quad (2.8)$$

It is clear that $0 < \rho \leq 1$ for $\gamma^\alpha \geq \frac{8(2[\frac{\mu}{2}])_\lambda |c|}{9}$ and from (2.8) we conclude that:

$$\left(z g'_{p,b,c,\xi}(z) \right)' \prec 1 + \rho z \implies \left| \arg \left(z g'_{p,b,c,\xi}(z) \right)' \right| \prec \arcsin \rho \quad (2.9)$$

With the help of Lemma 1.4, take $n = 0$ with $F(z) = \left(z g'_{p,b,c,\xi}(z) \right)'$ and

$$G(z) = 1 + \rho z$$

and we get:

$$\frac{z g'_{p,b,c,\xi}(z)}{z} \prec 1 + \frac{\rho}{2} z \quad (2.10)$$

This implies that

$$g'_{p,b,c,\xi}(z) \prec 1 + \frac{\rho}{2} z.$$

$$\text{and} \quad \left| \arg z \delta'_{p,b,c,\xi}(z) \right| < \arcsin \frac{\rho}{2} \quad (2.11)$$

By making use of relations (2.9) and (2.11), we obtain:

$$\begin{aligned} \left| \arg \left(\frac{(z g'_{p,b,c,\xi}(z))'}{g'_{p,b,c,\xi}(z)} \right) \right| &= \left| \arg (z g'_{p,b,c,\xi}(z))' - \arg (g_{p,b,c,\xi}(z)) \right| \\ &\leq \left| \arg (z g'_{p,b,c,\xi}(z))' \right| + \left| \arg (g_{p,b,c,\xi}(z)) \right| \\ &< \arcsin \frac{\rho}{2} + \arcsin \rho. \end{aligned}$$

As $0 < \rho \leq 1$, it is possible to write the above last expression as:

$$\left| \arg \left(\frac{(z \delta'_{p,b,c,\xi}(z))'}{\delta'_{p,b,c,\xi}(z)} \right) \right| < \arcsin \left(\rho \sqrt{1 - \frac{\rho^2}{4}} + \frac{\rho}{2} \sqrt{1 - \rho^2} \right),$$

which shows that $\delta_{p,b,c,\xi}(z) \in \tilde{C}(\alpha)$ for

$$\alpha = \frac{\pi}{2} \arcsin \left(\rho \sqrt{1 - \frac{\rho^2}{4}} + \frac{\rho}{2} \sqrt{1 - \rho^2} \right).$$

□

Remark 2.5. Since $\rho = \frac{4(2[\frac{\mu}{2}])_\lambda |c|}{3\gamma^\alpha - 4(2[\frac{\mu}{2}])_\lambda |c|}$, when $\gamma = \frac{p}{\xi} + \frac{b+2}{2} \neq 0, -1, -2, \dots$ for $\alpha = 1$ and μ is an arbitrary parameter, by suitably varying the parameters involved such that $\lambda = 1$ in Theorem 2.4, one would have the result in Theorem 2 of [1]. That is, Theorem 2.4 reduces to the Theorem 2 established in [1].

Theorem 2.6. Let $\gamma^\alpha > \frac{(8 + (2[\frac{\mu}{2}])_\lambda)|c|}{4(2[\frac{\mu}{2}])_\lambda}$, then $\delta_{p,b,c,\xi} \in U \subset V$.

Proof. From equation (1.17)

$$g'_{p,b,c,\xi}(z) = 1 + \sum_{n=1}^{\infty} \frac{(n+1)(\frac{-c}{4})^n}{(\mu)_{\lambda n}(\gamma)_{\alpha n}} z^n \quad (2.12)$$

$$z g''_{p,b,c,\xi}(z) = \sum_{n=1}^{\infty} \frac{n(n+1)(\frac{-c}{4})^n}{(\mu)_{\lambda n}(\gamma)_{\alpha n}} z^{n+1} \quad (2.13)$$

using the reverse of the well known triangular inequality and the inequalities $(\mu)_{\lambda n} \geq (2[\frac{\mu}{2}])_\lambda(n+1)$, $(\gamma)_{\alpha n} \geq \gamma^{\alpha n}$ ($n \in \mathbb{N}$) on (2.12), we have the following:

$$\begin{aligned}
\left| g'_{p,b,c,\xi}(z) \right| &\geq \left| 1 - \frac{c}{4(2[\frac{\mu}{2}])_\lambda \gamma^\alpha} \sum_{n=1}^{\infty} \frac{(-c)^{n-1}}{4^{n-1}(\gamma^\alpha)^{n-1}} z^n \right| \\
&\geq 1 - \frac{|c|}{4(2[\frac{\mu}{2}])_\lambda \gamma^\alpha} \sum_{n=1}^{\infty} \left(\frac{|c|}{4\gamma^\alpha} \right)^{n-1} \\
&= \frac{(2[\frac{\mu}{2}])_\lambda (4\gamma^\alpha - |c|) - |c|}{4(2[\frac{\mu}{2}])_\lambda (4\gamma^\alpha - |c|)}, \quad \left(\gamma^\alpha > \frac{|c|}{4} \right).
\end{aligned} \tag{2.14}$$

Also by making use of the well known triangular inequality and the inequalities $(\mu)_{\lambda n} \geq (2[\frac{\mu}{2}])_\lambda \frac{2n(n+1)}{3}$, $(\gamma)_{\alpha n} \geq \gamma^{\alpha n}$ ($n \in \mathbb{N}$) on (2.13), we have the following:

$$\begin{aligned}
\left| z g''_{p,b,c,\xi}(z) \right| &\leq \left| \frac{3(-c)}{4(2[\frac{\mu}{2}])_\lambda \gamma^\alpha} \sum_{n=1}^{\infty} \frac{(-c)^{n-1}}{4^{n-1}(\gamma^\alpha)^{n-1}} \right| \\
&\leq \frac{3|c|}{4(2[\frac{\mu}{2}])_\lambda \gamma^\alpha} \sum_{n=1}^{\infty} \left(\frac{|c|}{4\gamma^\alpha} \right)^{n-1} \\
&= \frac{3|c|}{4(2[\frac{\mu}{2}])_\lambda (4\gamma^\alpha - |c|)}, \quad \left(\gamma^\alpha > \frac{|c|}{4} \right).
\end{aligned} \tag{2.15}$$

Combining equation (2.14) and (2.15) yields

$$\left| \frac{z g''_{p,b,c,\xi}(z)}{g'_{p,b,c,\xi}(z)} \right| = \frac{3|c|}{(2[\frac{\mu}{2}])_\lambda (4\gamma^\alpha - |c|) - 2|c|}, \quad \left[\gamma^\alpha > \frac{(1 + (2[\frac{\mu}{2}])_\lambda)|c|}{4(2[\frac{\mu}{2}])_\lambda} \right].$$

Hence, by Lemma 1.5, then $\delta_{p,b,c,\xi}(z) \in U \subset V$. $\square$

Remark 2.7. By suitably varying the parameters involved such that $\xi = 1$, $\lambda = \alpha = 1$, and $\mu = \frac{3}{2}$ is arbitrary, so that $\gamma = p + \frac{b+2}{2}$ and $(\frac{\mu}{2})_\lambda = (\frac{3}{4})_1 \equiv \frac{3}{4}$. Then, we have the following:

Corollary 2.8. *Let $q > \frac{19|c|}{12}$, then $V_{L,b,c} \in U \subset V$.*

When $\alpha = p$ and $\gamma = \frac{p}{L} + \frac{b+2}{2} = q$.

This is the Theorem 3 in [1].

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