

SOME RESULTS ON GRONWALL-BELLMAN INTEGRAL INEQUALITIES

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ABSTRACT. In this paper, we establish some basic integral inequalities of Gronwall-Bellman type and its generalizations involving function of one independent variable which provides explicit bounds on unknown functions of integral equations. An example to illustrate possible applications of our result is included.

1. INTRODUCTION

The Gronwall-Bellman type integral inequalities (see[4]): If the functions $g(t)$ and $u(t)$ are non negative functions for $t \geq 0$, and if $c \geq 0$ satisfying

$$u(t) \leq c + \int_0^t g(s)u(s)ds, \quad t \geq 0$$

implies

$$u(t) \leq c \exp \left(\int_0^t g(s)ds \right), \quad t \geq 0.$$

It serves as a necessary tool for the study of the theory of differential and integral equations. In differential analysis, the Gronwall's inequality is also called the Gronwall's lemma, it allows one to bound a function that is known to satisfy a certain differential or integral inequality by the solution of the corresponding differential or integral equation. The Gronwall's lemma is used as a fundamental estimate to bound quantities which depend on time. For more results on the Gronwall's lemma and its various applications (see[1-19]).

2. MAIN RESULTS

First, we state and prove the following interesting lemma.

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Lemma 2.1. *If $u(t)$, $g(t)$ and $k(t)$ be non-negative continuous functions defined for $I = [0, \infty)$ and, if $n(t)$ be non-decreasing function on $0 \leq t < T$. If*

$$u(t) \leq k(t) + n(t) \int_0^t g(s)u(s)ds, \quad \forall t \in I. \quad (2.1)$$

Then

$$u(t) \leq k(0) \exp \left\{ k(t) - k(0) + n(t) \int_0^t g(s)u(s)ds - \int_0^t n(s)g(s)u(s)ds + \int_0^t n(s)g(s)ds \right\}, \quad \forall t \in I. \quad (2.2)$$

Proof. Define a function $m(t) > 1$ in (2.1) by

$$m(t) = k(t) + n(t) \int_0^t g(s)u(s)ds. \quad (2.3)$$

Differentiating (2.3) with respect to t

$$m'(t) = k'(t) + n'(t) \int_0^t g(s)u(s)ds + n(t)g(t)u(t). \quad (2.4)$$

Divide through by $m(t)$ to have

$$\frac{m'(t)}{m(t)} \leq k'(t) + n'(t) \int_0^t g(s)u(s)ds + n(t)g(t). \quad (2.5)$$

Integrating (2.5) from 0 to t to get

$$\log \frac{m(t)}{m(0)} \leq k(t) - k(0) + n(t) \int_0^t g(s)u(s)ds - \int_0^t n(s)g(s)u(s)ds + \int_0^t n(s)g(s)ds$$

which implies

$$\frac{m(t)}{m(0)} \leq \exp \left\{ k(t) - k(0) + n(t) \int_0^t g(s)u(s)ds - \int_0^t n(s)g(s)u(s)ds + \int_0^t n(s)g(s)ds \right\}.$$

Observe from (2.3) that $m(0) = k(0)$. Then

$$m(t) \leq k(0) \exp \left\{ k(t) - k(0) + n(t) \int_0^t g(s)u(s)ds - \int_0^t n(s)g(s)u(s)ds + \int_0^t n(s)g(s)ds \right\}. \quad (2.6)$$

This completes the proof. $\square$

Now, we prove our main result.

Theorem 2.2. *Let $u(t)$, $f(t)$, $g(t)$ and $k(t)$ be non-negative continuous functions defined for $I = [0, \infty)$, and let $n(t)$ be non-decreasing function on $0 \leq t < T$. If*

$$u(t) \leq k(t) + n(t) \left[\int_0^t f(s)u(s)ds + \int_0^t f(s) \left(\int_0^s g(p)u(p)dp \right) ds \right], \quad \forall t \in I. \quad (2.7)$$

Then

$$u(t) \leq k(t) + k(0)n(t) \int_0^t f(s) \exp \left\{ k(s) - k(0) + \int_0^s (f(p) + g(p)) dp \right\} ds, \quad \forall t \in I. \quad (2.8)$$

Proof. Define a function $m(t)$ in (2.7) by

$$m(t) = k(t) + n(t) \left[\int_0^t f(s)u(s)ds + \int_0^t f(s) \left(\int_0^s g(p)u(p)dp \right) ds \right]. \quad (2.9)$$

Clearly, $m(t) > 1$. Also, we know from (2.9) that

$$u(t) \leq m(t). \quad (2.10)$$

Differentiating (2.9) with respect to t , we have

$$m'(t) = k'(t) + n(t)f(t) \left(u(t) + \int_0^t g(s)u(s)ds \right) + n(t) \int_0^t f(t) \left(u(s) + \int_0^s g(p)u(p)dp \right) ds. \quad (2.11)$$

That is,

$$m'(t) = k'(t) + n(t) + n'(t) \int_0^t f(s)v(s)ds \quad (2.12)$$

where

$$v(t) = m(t) + \int_0^t g(s)u(s)ds. \quad (2.13)$$

Differentiating (2.13) with respect to t , to get

$$\frac{v'(t)}{v(t)} \leq k'(t) + n(t) \left(f(t) + \int_0^t [f(s) + g(s)]ds \right). \quad (2.14)$$

Applying, Lemma (2.1) in (2.14) to have

$$v(t) = k(0) \exp \left\{ k(t) - k(0) + \int_0^t [f(s) + g(s)]ds \right\}. \quad (2.15)$$

Integrating (2.12) with respect to t and substitute (2.15), we have

$$m'(t) \leq k'(t) + k(0) \frac{d}{dt} \left[n(t) \int_0^t f(s) \exp \left\{ k(s) - k(0) + \int_0^s [f(p) + g(p)]dp \right\} ds \right]. \quad (2.16)$$

Integrating both sides of the (2.16) from 0 to t , we get

$$m(t) - m(0) \leq k(t) - k(0) + k(0)n(t) \int_0^t f(s) \exp \left\{ k(s) - k(0) + \int_0^s [f(p) + g(p)]dp \right\} ds.$$

And, since $m(0) = k(0)$. Then

$$m(t) \leq k(t) + k(0)n(t) \int_0^t f(s) \exp \left\{ k(s) - k(0) + \int_0^s [f(p) + g(p)] dp \right\} ds. \quad (2.17)$$

This completes the proof. $\square$

Let $n(t) = 1$ in Theorem 2.4, we have the following corollary:

Corollary 2.3. *Let $u(t)$, $f(t)$, $g(t)$ and $k(t)$ be non-negative continuous functions defined for $I = [0, \infty)$. If*

$$u(t) \leq k(t) + \int_0^t f(s)u(s)ds + \int_0^t f(s) \left(\int_0^s g(p)u(p)dp \right) ds, \quad \forall t \in I. \quad (2.18)$$

Then

$$u(t) \leq k(t) + k(0) \int_0^t f(s) \exp \left\{ k(s) - k(0) + \int_0^s [f(p) + g(p)] dp \right\} ds, \quad \forall t \in I. \quad (2.19)$$

Theorem 2.4. *Let $u(t)$, $f(t)$, $g(t)$ and $k(t)$ be non-negative continuous functions defined for $I = [0, \infty)$ and let $n(t)$ be non-decreasing function on $0 \leq t < T$. If*

$$u(t) \leq k(t) + n(t) \left[\int_0^t f(s)u(s)ds + \int_0^t f(s) \left(\int_0^s f(\tau) \left(\int_0^\tau g(p)u(p)dp \right) d\tau \right) ds \right], \quad \forall t \in I. \quad (2.20)$$

Then

$$\begin{aligned} u(t) \leq k(t) + \int_0^t f(s) [k(s) + k(0)n(s) \int_0^s f(\tau) \exp(k(\tau) - k(0)) \\ + n(\tau) \int_0^\tau [f(p) + g(p)] dp d\tau] ds, \quad \forall t \in I. \end{aligned} \quad (2.21)$$

The proof of Theorem 2.4 follows easily from Theorem 2.1

Let $n(t) = 1$ in Theorem 2.2, we have the following corollary:

Corollary 2.5. *Let $u(t)$, $f(t)$, $g(t)$ and $k(t)$ be non-negative continuous functions defined for $I = [0, \infty)$. If*

$$u(t) \leq k(t) + \int_0^t f(s)u(s)ds + \int_0^t f(s) \left(\int_0^s f(\tau) \left(\int_0^\tau g(p)u(p)dp \right) d\tau \right) ds, \quad \forall t \in I. \quad (2.22)$$

Then

$$u(t) \leq k(t) + \int_0^t f(s) \left[k(s) + k(0) \int_0^s f(\tau) \exp \left(k(\tau) - k(0) + \int_0^\tau [f(p) + g(p)] dp \right) d\tau \right] ds, \quad \forall t \in I. \quad (2.23)$$

3. APPLICATIONS

Let us consider the bound for the solution of variant Volterra integral equation of the form

$$x(t) = f(t) + n(t) \int_0^t p(t, s)g(t, x(s), Tx(s))ds \quad (3.1)$$

where x , f and g are elements of $\mathbb{R}^n$, $p(t, s)$ is a square matrix, $g \in C[1 \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$ and $x \in C[1, \mathbb{R}^n]$ and T be a continuous operator such that T maps $C[I]$ into $C[I]$.

Define

$$|p(t, s)| \leq 1 \quad (3.2)$$

and

$$|g(t, x, y)| \leq f(t)[|x| + |y|], \quad t \in I. \quad (3.3)$$

Also, let

$$|f(t)| \leq k(t), \quad \text{where } k(t) > 1, \quad (3.4)$$

$$|Tx(t)| \leq \int_0^t g(s)|x(s)|ds, \quad t \in I. \quad (3.5)$$

Then

$$|x(t)| \leq k(t) + k(0)n(t) \int_0^t f(s) \exp \left(k(s) - k(0) + \int_0^s [f(p) + g(p)]dp \right) ds, \quad \forall t \in I. \quad (3.6)$$

Proof. Taking the absolute value of both sides of (3.1), we get

$$|x(t)| \leq |f(t)| + n(t) \int_0^t |p(t, s)g(t, x(s), Tx(s))|ds. \quad (3.7)$$

Substituting (3.2), (3.3), (3.4), (3.5) and (3.6) in (3.7), to get

$$|x(t)| \leq k(t) + n(t) \int_0^t f(s)|x(s)|ds + \int_0^t f(s) \left(\int_0^s g(p)|x(p)|dp \right) ds, \quad \forall t \in I. \quad (3.8)$$

The rest of this proof will be the same as the proof of Theorem 2.4 with suitable modifications. $\square$

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