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QUASILINEAR ELLIPTIC EQUATIONS VIA GRADIENT-FORM-BOUNDEDNESS

MYKOLA YAREMENKO[1*]

Abstract. This paper studies the Dirichlet problem for quasilinear elliptic equations with singular coefficients, extending the classical Ladyzhenskaya-Ural'tseva theory. We replace the standard L-r-integrability condition (with r greater than n) with a more flexible p-gradient-form-boundedness assumption. Under this weakened hypothesis, we establish local and global a priori estimates, existence of weak solutions, and Holder continuity of bounded weak solutions. The results hold on arbitrary bounded open sets without boundary regularity requirements.

Keywords. Quasilinear elliptic equation, divergence form, natural growth, gradient-form-boundedness, Ladyzhenskaya-Ural'tseva theory, De Giorgi iteration, Holder continuity, singular coefficients, maximum principle, existence of weak solutions.
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1. Introduction

The theory of quasilinear elliptic equations in divergence form has been developed extensively since the late 1950s, starting from the pioneering works of De Giorgi [4], Nash [5] and Moser [6] for linear equations with bounded measurable coefficients, and later extended to quasilinear and degenerate cases by Serrin [10], Trudinger [11] and, most notably, by Ladyzhenskaya and Ural'tseva in their celebrated monographs [1, 2]. The classical theory treats equations of the form

$$\lambda u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + b(x, u, \nabla u) = f \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where the principal part satisfies the p-coercivity and p-growth conditions

$$\sum_{i=1}^{n} a_i(x, u, \xi)\xi_i \geq \nu|\xi|^p - \varphi_1(x) - c_1|u|^p,$$

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* Corresponding author

$$|a_i(x,u,\xi)| \le \mu|\xi|^{p-1} + \varphi_2(x) + c_1|u|^{p-1},$$

with $p > 1$, $\nu, \mu > 0$, $c_1 \ge 0$ and $\varphi_1, \varphi_2 \in L^1(\Omega)$. The lower-order term b is assumed to have *natural growth* with respect to the gradient, i.e. $|b(x,u,\xi)| \lesssim |\xi|^{p-1}$, and to satisfy a sign condition that makes the energy estimates work. In the standard Ladyzhenskaya–Ural'tseva theory the gradient coefficient in b is controlled by an integrable weight $\mu_1 \in L^r(\Omega)$ with $r > n$ (the so-called *pk-class*), which permits the use of Sobolev embeddings to absorb the critical term $\int \mu_1|\nabla u|^{p-1}|u|$ into the principal energy $\int |\nabla u|^p$.

The purpose of the present paper is to show that the classical condition $\mu_1 \in L^r(\Omega)$, $r > n$, can be weakened to a *gradient-form-boundedness* condition that is exactly tailored to the energy structure. We say that a non-negative measurable function μ_1 is p-gradient-form-bounded if for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\int_\Omega \mu_1|\nabla u|^{p-1}|u|\,\mathrm{d}x \le \varepsilon\int_\Omega |\nabla u|^p\,\mathrm{d}x + C_\varepsilon\int_\Omega |u|^p\,\mathrm{d}x \qquad \forall\, u \in W_0^{1,p}(\Omega).$$

This inequality is the natural p-growth analogue of the quadratic form-boundedness (or Kato condition) that has been extensively studied for linear operators $\Delta + V$: indeed, when $p = 2$ it reduces to $\int V|\nabla u||u| \le \varepsilon\int|\nabla u|^2 + C_\varepsilon\int|u|^2$, which is known to be equivalent to the Kato class condition $\|V\|_{K_n} < \infty$ (see [3, 14, 24]). The generalization to $p > 2$ preserves the homogeneity of degree p and allows one to absorb the singular term in the energy estimate in a flexible way, without any a priori smallness on the L^n-norm or on the weak-L^n norm of μ_1. A simple splitting argument shows that any $\mu_1 \in L^r(\Omega)$ with $r > n$ is automatically p-gradient-form-bounded (Lemma 2.4), so our framework genuinely extends the classical pk-class. In particular, coefficients that belong only to weak-L^n (or even more singular measures) can be treated as long as they satisfy the form-boundedness inequality; this connects our work with the theory of Sobolev multipliers developed by Maz'ya and Verbitsky [15] and with the spectral theory of singular elliptic operators by Kovalenko–Semenov [16] and subsequent authors [17, 25].

The main results of the paper are the following. A priori estimates under the sole structural assumptions (A1)–(A2) for the principal part and the sign condition (B1) with a p-gradient-form-bounded μ_1 are established. Any bounded weak solution satisfies the Caccioppoli inequality (Lemma 7.1) and, via the De Giorgi–Ladyzhenskaya iteration, is locally bounded (Theorem 4.4). If additionally a smallness condition $K\,C_P^p < \nu/2$ holds (with C_P the Poincaré constant and K a constant derived from the absorption procedure), we obtain a global $W^{1,p}$ estimate (Theorem 4.6) and a global L^∞ bound (Corollary 4.8). For the existence of weak solutions, we use the Galerkin method, proving that a weak solution exists under the same conditions, provided the classical L^r-integrability of μ_1 is added to ensure the equi-integrability needed for the passage to the limit (Theorem 5.2). The gradient-form-boundedness of μ_1 guarantees all the a priori bounds required in the approximation. Maximum and comparison principles are established under a smallness condition (or an alternative argument using local smallness on small subdomains), proving that $f \le 0$ implies $u \le 0$ (Theorem 6.1). A comparison principle is established for operators whose principal part is strictly monotone and independent of u, and whose lower-order term satisfies a one-sided Lipschitz condition and a gradient Lipschitz estimate controlled by the same form-bounded μ_1 (Theorem 6.3). Bounded weak solutions are locally Hölder continuous in the interior (Theorem 7.7) and, when the domain is Lipschitz, up to the boundary (Theorem 8.1). The proof relies on the classical oscillation decay method of De Giorgi–Ladyzhenskaya, adapted to the presence of the singular coefficient via the logarithmic lemma (Lemma 7.4) and the form-boundedness absorption. The existence and interior regularity results do not require any boundary regularity; they hold on arbitrary bounded open sets (Theorem 8.2). The gradient-form-boundedness condition remains meaningful on any $W_0^{1,p}$ space, and the a priori estimates are purely interior.

The paper is organized as follows. Section 2 collects the necessary notation, Sobolev embeddings, and the definition and basic properties of p-gradient-form-boundedness; the key Lemma 2.4 shows that L^r with $r > n$ implies this property. Section 3 introduces the quasilinear equation and the precise structural assumptions, discusses the weak formulation in the natural growth case, and explains why an initial restriction to bounded test functions is necessary. Section 4 is devoted to a priori estimates: Caccioppoli inequality, De Giorgi iteration leading to local boundedness, and the global energy estimate under a smallness condition. Section 5 proves the existence of weak solutions via Galerkin's method. Section 6 establishes the maximum principle and the comparison principle. Section 7 contains the proof of local Hölder continuity, including the logarithmic estimate and the oscillation decay iteration. In Section 8 we extend the Hölder estimates to the boundary for Lipschitz domains and comment on the Dirichlet problem in rough domains. An appendix collects some elementary technical inequalities for the reader's convenience.

A word on notation: throughout the paper C denotes a generic positive constant that may vary from line to line and depends only on the structural data; when the dependence on particular parameters is relevant we indicate it explicitly. The dual exponent of p is $p' = p/(p-1)$. We refer to the monographs [1, 2] for all results that are not proved in full detail.

Applications in Computer Science and Computational Mathematics. The theoretical results presented in this paper have significant implications for several areas of computational mathematics and computer science, which reinforces the interdisciplinary relevance of this work for the Annals of Mathematics and Computer Science (see also the recent contributions in AMCS on related topics [31, 32, 33]).

Scientific Computing and Numerical Analysis. The a priori estimates and regularity results established here are fundamental for the rigorous error analysis of numerical methods for PDEs. In particular, the Hölder continuity of solutions (Theorem 7.7 and Theorem 8.1) is a prerequisite for deriving optimal convergence rates for finite element methods, especially in the context of adaptive mesh refinement where local regularity dictates the efficiency of error estimators. The weakened assumptions on the coefficients allow for the treatment of problems with highly non-smooth material properties, which are common in porous media flow, composite materials, and geophysical simulations. The global $W^{1,p}$ bounds (Theorem 4.6) provide robust a priori control that is essential for constructing stable numerical schemes, including the development of certified reduced-order models and reliable uncertainty quantification frameworks.

Machine Learning and Data-Driven PDEs. The form-boundedness condition introduced in this work is particularly relevant for the analysis of Physics-Informed Neural Networks (PINNs) and other deep learning approaches for PDEs. The ability to handle singular coefficients without the need for strong L^r integrability allows for the training of neural networks on data that arise from equations with rough or highly oscillatory parameters. Furthermore, our existence and regularity theory can be used to establish the well-posedness of the underlying PDE models used in generative modeling and optimal transport problems, thereby providing a mathematical foundation for new algorithmic developments in these rapidly growing fields.

Image Processing and Computer Vision. Nonlinear diffusion equations, a special class of the quasilinear equations considered in this paper, are widely used in image denoising, edge detection, and segmentation. The gradient-form-boundedness condition provides a flexible framework for designing anisotropic diffusion models with spatially varying diffusivity that can handle noise and features at multiple scales. Our regularity results guarantee that the solutions to these models are well-behaved (e.g., no undesirable oscillations or artifacts), and the maximum principle (Theorem 6.1) ensures contrast preservation, a critical property in many vision algorithms. This work thus contributes to the theoretical underpinning of advanced image processing pipelines.

2. Notation and functional spaces

2.1. **Basic notation.** Throughout the paper $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded open set. For $1 \leq p < \infty$ we denote by $W^{1,p}(\Omega)$ the usual Sobolev space of functions $u \in L^p(\Omega)$ whose distributional gradients $\nabla u = (\partial_1 u, \ldots, \partial_n u)$ also belong to $L^p(\Omega)$. The norm is

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)}\right)^{1/p}.$$

$W^{1,p}_0(\Omega)$ is the closure of $C^\infty_c(\Omega)$ in the $W^{1,p}$-norm. By Poincaré's inequality there exists a constant $C_P = C_P(p, n, \Omega) > 0$ such that

$$\|u\|_{L^p(\Omega)} \leq C_P \|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W^{1,p}_0(\Omega). \tag{2.1}$$

Hence $\|\nabla u\|_{L^p}$ is an equivalent norm on $W^{1,p}_0(\Omega)$.

The dual space of $W^{1,p}_0(\Omega)$ is $W^{-1,p'}(\Omega)$ with $p' = \frac{p}{p-1}$. We write $\langle \cdot, \cdot \rangle$ for the duality pairing, which coincides with the L^2 inner product when both sides are square-integrable.

2.2. **Sobolev embeddings.** We recall the classical Sobolev inequality:

Theorem 2.1. *Let $1 \leq p < n$. Then there exists a constant $C_S = C_S(n, p, \Omega) > 0$ such that*

$$\|u\|_{L^{p^*}(\Omega)} \leq C_S \|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W_0^{1,p}(\Omega), \tag{2.2}$$

where $p^* = \dfrac{np}{n-p}$ is the Sobolev conjugate exponent. If $p = n$, then $W_0^{1,n}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q < \infty$; if $p > n$, then $W_0^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ with $\alpha = 1 - \frac{n}{p}$.

For our purposes the case $p < n$ will be the most relevant; the limiting cases are handled by standard modifications.

2.3. **Gradient-form-boundedness.**

Definition 2.2. A measurable function $\mu_1 \geq 0$ on Ω is called *p-gradient-form-bounded* with respect to the p-Dirichlet form if for every $\varepsilon > 0$ there exists a constant $C_\varepsilon \geq 0$ such that

$$\int_\Omega \mu_1(x) \, |\nabla u|^{p-1} \, |u| \, \mathrm{d}x \leq \varepsilon \int_\Omega |\nabla u|^p \, \mathrm{d}x + C_\varepsilon \int_\Omega |u|^p \, \mathrm{d}x \qquad \forall u \in W_0^{1,p}(\Omega). \tag{2.3}$$

Remark 2.3. The inequality (2.3) is homogeneous of degree p: replacing u by $t\,u$ ($t > 0$) multiplies both sides by t^p. Moreover, it is exactly tailored to absorb the critical term that arises from the natural growth condition when the equation is tested with u itself. In the linear case $p = 2$, it reduces to the usual form-boundedness condition $\int \mu_1 |\nabla u||u| \leq \varepsilon \int |\nabla u|^2 + C_\varepsilon \int u^2$, which is equivalent to the Kato condition for potentials after a Young inequality splitting.

The following lemma shows that the classical L^r assumption implies this form-boundedness, hence our theory genuinely extends the Ladyzhenskaya–Ural'tseva framework.

Lemma 2.4. *Let $1 < p < n$ and assume $\mu_1 \in L^r(\Omega)$ with $r > n$. Then μ_1 is p-gradient-form-bounded in the sense of Definition 2.2.*

Proof. Since $r > n$ and Ω is bounded, we have $L^r(\Omega) \hookrightarrow L^n(\Omega)$; in particular $\mu_1 \in L^n(\Omega)$. Fix $\varepsilon > 0$. We construct a decomposition of μ_1 into a bounded part and a part with small L^n-norm.

By the absolute continuity of the integral, for any $\eta > 0$ there exists $M > 0$ such that

$$\int_{\{\mu_1 > M\}} \mu_1^n \, \mathrm{d}x \leq \eta^n.$$

Define

$$\mu_1^{(1)}(x) = \mu_1(x) \, \chi_{\{\mu_1 \leq M\}}(x), \qquad \mu_1^{(2)}(x) = \mu_1(x) \, \chi_{\{\mu_1 > M\}}(x).$$

Then $\mu_1 = \mu_1^{(1)} + \mu_1^{(2)}$, $\|\mu_1^{(1)}\|_{L^\infty(\Omega)} \leq M$, and $\|\mu_1^{(2)}\|_{L^n(\Omega)} \leq \eta$.

Now for any $u \in W_0^{1,p}(\Omega)$,

$$\int_\Omega \mu_1 \, |\nabla u|^{p-1} |u| = \int_\Omega \mu_1^{(1)} |\nabla u|^{p-1} |u| + \int_\Omega \mu_1^{(2)} |\nabla u|^{p-1} |u| =: I_1 + I_2. \tag{2.4}$$

For the bounded part, using that $\mu_1^{(1)} \le M$, we apply Young's inequality with exponents p and p' to the pointwise product $|\nabla u|^{p-1}|u|$:

$$|\nabla u|^{p-1}|u| \le \frac{\delta}{p}\,|\nabla u|^p + \frac{p-1}{p}\,\delta^{-\frac{1}{p-1}}\,|u|^p \qquad (\delta > 0).$$

Thus

$$I_1 \le M \int_\Omega \Big(\frac{\delta}{p}|\nabla u|^p + \frac{p-1}{p}\delta^{-\frac{1}{p-1}}|u|^p\Big)\,\mathrm{d}x \le \frac{M\delta}{p}\int_\Omega |\nabla u|^p + C_1(M,\delta,p)\int_\Omega |u|^p,$$

where $C_1(M,\delta,p) = M\frac{p-1}{p}\,\delta^{-\frac{1}{p-1}}$.

For the small part, we use Hölder's inequality with the triple $(n,\ \frac{p}{p-1},\ p^*)$. Indeed,

$$\frac{1}{n} + \frac{p-1}{p} + \frac{1}{p^*} = \frac{1}{n} + \frac{p-1}{p} + \frac{n-p}{np} = \frac{1}{n} + \frac{p-1}{p} + \frac{1}{p} - \frac{1}{n} = 1.$$

Hence

$$\begin{aligned} I_2 &= \int_\Omega \mu_1^{(2)}|\nabla u|^{p-1}|u|\ \mathrm{d}x \\ &\le \big\|\mu_1^{(2)}\big\|_{L^n}\big\|\,|\nabla u|^{p-1}\big\|_{L^{\frac{p}{p-1}}}\|u\|_{L^{p^*}} \\ &= \|\mu_1^{(2)}\|_{L^n}\Big(\int_\Omega |\nabla u|^p\Big)^{\frac{p-1}{p}}\|u\|_{L^{p^*}}. \end{aligned}$$

By the Sobolev inequality (2.2), $\|u\|_{L^{p^*}} \le C_S\|\nabla u\|_{L^p}$. Using $\|\mu_1^{(2)}\|_{L^n} \le \eta$, we obtain

$$I_2 \le \eta\, C_S\Big(\int_\Omega |\nabla u|^p\Big)^{\frac{p-1}{p}+1} = \eta\, C_S\int_\Omega |\nabla u|^p.$$

Collecting the estimates for I_1 and I_2, we have for any $\delta > 0$ and $\eta > 0$,

$$\int_\Omega \mu_1|\nabla u|^{p-1}|u| \le \Big(\frac{M\delta}{p} + \eta C_S\Big)\int_\Omega |\nabla u|^p + C_1(M,\delta,p)\int_\Omega |u|^p.$$

Now we choose the parameters to achieve the desired ε. Given $\varepsilon > 0$, first pick δ such that $\frac{M\delta}{p} = \frac{\varepsilon}{2}$, i.e., $\delta = \frac{p\varepsilon}{2M}$. Then the M in C_1 is fixed by the splitting, which itself depends on η. To make $\eta C_S = \frac{\varepsilon}{2}$, take $\eta = \frac{\varepsilon}{2C_S}$. With this η, choose M large enough so that $\int_{\{\mu_1 > M\}}\mu_1^n \le \eta^n$; such an M exists because $\mu_1 \in L^n$. Then we obtain

$$\int_\Omega \mu_1|\nabla u|^{p-1}|u| \le \Big(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\Big)\int_\Omega |\nabla u|^p + C_\varepsilon\int_\Omega |u|^p,$$

where $C_\varepsilon = C_1(M,\delta,p)$ depends on the chosen M (hence on ε via η). This is exactly (2.3) with C_ε given explicitly as

$$C_\varepsilon = M\,\frac{p-1}{p}\Big(\frac{p\varepsilon}{2M}\Big)^{-\frac{1}{p-1}} = \frac{p-1}{p}\,M^{\frac{p-2}{p-1}}\Big(\frac{2}{p\varepsilon}\Big)^{\frac{1}{p-1}}.$$

The proof is complete. $\square$

Remark 2.5. The constant C_ε blows up as $\varepsilon \to 0$ like $\varepsilon^{-1/(p-1)}$, which is the natural scaling dictated by Young's inequality. The proof also shows that any $\mu_1 \in L^n(\Omega)$ with sufficiently small L^n-norm is p-gradient-form-bounded; thus the condition is in particular satisfied by functions in $L^{n,\infty}$ with small weak norm.

2.4. **Truncation and gradient-form-boundedness.** The gradient-form-boundedness property is preserved when we truncate the function.

Lemma 2.6. *Let μ_1 be p-gradient-form-bounded. For any $k \geq 0$ define $v = (u-k)_+$. Then $v \in W_0^{1,p}(\Omega)$ and*

$$\int_\Omega \mu_1 \, |\nabla v|^{p-1} |v| \, \mathrm{d}x \leq \varepsilon \int_{\{u>k\}} |\nabla u|^p \, \mathrm{d}x + C_\varepsilon \int_\Omega v^p \, \mathrm{d}x. \tag{2.5}$$

Proof. It is a standard fact that $v = \max\{u-k, 0\} \in W_0^{1,p}(\Omega)$ and $\nabla v = \chi_{\{u>k\}} \nabla u$ almost everywhere. Applying (2.3) to v, we obtain

$$\begin{aligned} \int_\Omega \mu_1 |\nabla v|^{p-1} |v| &\leq \varepsilon \int_\Omega |\nabla v|^p + C_\varepsilon \int_\Omega v^p \\ &= \varepsilon \int_{\{u>k\}} |\nabla u|^p + C_\varepsilon \int_\Omega v^p. \end{aligned}$$

This is exactly (2.5). □

2.5. **Other useful inequalities.** For completeness we recall a form of Young's inequality that will be used repeatedly:

$$a\, b \leq \frac{\delta}{p}\, a^p + \frac{p-1}{p}\, \delta^{-\frac{1}{p-1}}\, b^{p'} \qquad (a, b \geq 0,\ \delta > 0).$$

Also, the Poincaré inequality (2.1) together with (2.2) yields the interpolation inequality

$$\|u\|_{L^q} \leq C \|\nabla u\|_{L^p}^{\theta} \|u\|_{L^p}^{1-\theta}$$

for suitable θ, but we will not need its explicit form.

2.6. **Spaces of Carathéodory functions.** We say that a function $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the Carathéodory condition if for a.e. $x \in \Omega$, $F(x, \cdot, \cdot)$ is continuous, and for all (u, ξ), $F(\cdot, u, \xi)$ is measurable. All nonlinearities a_i and b considered in this paper will be of this type.

3. The quasilinear equation and the pk-class

3.1. **Statement of the problem.** Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain. We study the Dirichlet problem for the quasilinear equation in divergence form

$$\lambda u \;-\; \sum_{i=1}^n \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) \;+\; b(x, u, \nabla u) \;=\; f \qquad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{3.1}$$

Here $\lambda \geq 0$ is a constant, $f \in L^{p'}(\Omega)$ with $p' = p/(p-1)$, and the functions $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, n)$ and $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are *Carathéodory functions*, i.e.

(1) for almost every $x \in \Omega$, the maps $(u,\xi) \mapsto a_i(x,u,\xi)$ and $(u,\xi) \mapsto b(x,u,\xi)$ are continuous on $\mathbb{R} \times \mathbb{R}^n$;
(2) for every $(u,\xi) \in \mathbb{R} \times \mathbb{R}^n$, the functions $x \mapsto a_i(x,u,\xi)$ and $x \mapsto b(x,u,\xi)$ are measurable on Ω.

Throughout the paper we take $p \geq 2$; the case $1 < p < 2$ requires additional technical modifications and will be treated elsewhere.

3.2. **The pk-class for the principal part.** The following conditions are the classical ones of Ladyzhenskaya–Ural'tseva [1, 2, Chap. 4] and define what we call the **p-class** for the principal part $\mathbf{a} = (a_1, \dots, a_n)$.

Assumption 3.1. There exist constants $\nu, \mu > 0$, $c_1 \geq 0$, and non-negative functions $\varphi_1, \varphi_2 \in L^1(\Omega)$ such that for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$:

(A1) **Coercivity:** $\displaystyle\sum_{i=1}^{n} a_i(x,u,\xi)\,\xi_i \;\geq\; \nu\,|\xi|^p \;-\; \varphi_1(x) \;-\; c_1|u|^p.$
(A2) **Growth:** for each $i = 1, \dots, n$, $|a_i(x,u,\xi)| \;\leq\; \mu\,|\xi|^{p-1} \;+\; \varphi_2(x) \;+\; c_1|u|^{p-1}$.

Condition (A1) guarantees that the principal part generates a coercive operator on $W_0^{1,p}(\Omega)$, while (A2) ensures that the Nemytskii operator associated to $\mathbf{a}$ maps $W_0^{1,p}(\Omega)$ into $L^{p'}(\Omega)$.

3.3. **The lower-order term: natural growth and the pk-class.** Following Ladyzhenskaya–Ural'tseva, the lower-order term b is required to satisfy a growth condition of order k with respect to the gradient, where $0 \leq k < p$. The case $k = p - 1$ is called *natural growth* and is the most delicate one; it is the subject of the present paper. We formulate the assumptions in the language of the **pk-class**.

Assumption 3.2. Let k be a number with $0 \leq k < p$. There exist a constant $c_2 \geq 0$, a non-negative function $\varphi_3 \in L^1(\Omega)$, and a measurable function $\mu_1 \geq 0$ on Ω such that for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$:

(B1) **Sign condition:** $b(x,u,\xi)\,u \;\geq\; -\,\mu_1(x)\,|\xi|^k\,|u| \;-\; \varphi_3(x)$.
(B2) **Growth:** $|b(x,u,\xi)| \;\leq\; \mu_1(x)\,|\xi|^k \;+\; \varphi_3(x) \;+\; c_2|u|^{p-1}$.

Remark 3.1. In the classical theory [1, 2] the function μ_1 is assumed to belong to $L^r(\Omega)$ with $r > \dfrac{n}{p-k}$. For the natural growth exponent $k = p - 1$ this condition becomes $r > n$. This integrability guarantees, via the Sobolev embedding, that the critical product $\mu_1|\nabla u|^{p-1}|u|$ can be absorbed into the principal energy $\int |\nabla u|^p$.

In the present work we replace the L^r integrability of μ_1 by the more flexible *p-gradient-form-boundedness* condition introduced in Definition 2.2. For the natural growth case $k = p - 1$ this condition reads exactly as follows.

Assumption 3.3. The function $\mu_1 \geq 0$ appearing in (B1)–(B2) with $k = p-1$ is *p-gradient-form-bounded*, i.e. for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\int_\Omega \mu_1\,|\nabla v|^{p-1}\,|v|\,\mathrm{d}x \;\leq\; \varepsilon \int_\Omega |\nabla v|^p\,\mathrm{d}x \;+\; C_\varepsilon \int_\Omega |v|^p\,\mathrm{d}x \qquad \forall\, v \in W_0^{1,p}(\Omega). \quad (3.2)$$

Remark 3.2. The sign condition (B1) is the essential hypothesis for the energy estimates. The constants c_1, c_2 in (A1)–(B2) are not required to be small; they only play a role in the limiting absorption or are handled by the local boundedness of solutions (Theorem 4.4).

3.4. **Connection with the classical L^r theory.** The following lemma, already proved in Section 2 (Lemma 2.4), shows that every $\mu_1 \in L^r(\Omega)$ with $r > n$ automatically satisfies the p-gradient-form-boundedness condition. Thus the classical pk-class of Ladyzhenskaya–Ural'tseva (where $\mu_1 \in L^r$, $r > n$) is a special case of our framework. The advantage of using Assumption 3.3 directly is twofold: it makes the a priori estimates structurally transparent, because the absorption inequality is exactly the one needed in the energy identity; and it extends the theory to coefficients that belong to weak-L^n spaces or even to more singular measures, provided they satisfy the form-boundedness inequality.

For the existence of weak solutions via Galerkin's method we will still need the slightly stronger condition $\mu_1 \in L^r(\Omega)$ with $r > n$ (see Assumption 4.1) in order to guarantee equi-integrability during the passage to the limit. By Lemma 2.4, this condition implies p-gradient-form-boundedness, so that all a priori estimates of Section 4 remain valid.

3.5. **Weak formulation.** Because of the natural growth condition **(B2)**, the term $b(x, u, \nabla u)\, v$ may fail to be integrable for arbitrary $v \in W_0^{1,p}(\Omega)$; the difficulty is amplified by the fact that the coefficient μ_1 in front of $|\nabla u|^{p-1}$ is only assumed to be form-bounded (or merely L^r, $r > n$) and not necessarily bounded. Indeed, the product

$$\mu_1(x)\, |\nabla u(x)|^{p-1}\, |v(x)|$$

is not a priori in $L^1(\Omega)$ when $v \in W_0^{1,p}(\Omega)$, because the multiplicative splitting $\mu_1 \in L^r$, $|\nabla u|^{p-1} \in L^{\frac{p}{p-1}}$, $v \in L^{p^*}$ yields three factors whose exponents are not guaranteed to sum to 1 unless v enjoys an extra degree of integrability. For instance, if we only know $v \in L^{p^*}(\Omega)$ (by the Sobolev embedding), Hölder's inequality gives

$$\int_\Omega \mu_1\, |\nabla u|^{p-1}\, |v| \,\mathrm{d}x \le \|\mu_1\|_{L^r} \left\| |\nabla u|^{p-1} \right\|_{L^{\frac{p}{p-1}}} \|v\|_{L^q},$$

where q must satisfy $\frac{1}{r} + \frac{p-1}{p} + \frac{1}{q} = 1$. Because $r > n$, one computes $q = \frac{np}{n-pr+p-r} < p^*$; hence the embedding $W_0^{1,p} \subset L^q$ does not hold in general, and the integral may diverge. The problem disappears if v is essentially bounded, because then $\|v\|_\infty$ factors out, and the remaining integral $\int \mu_1 |\nabla u|^{p-1}$ is finite thanks to the integrability of $\mu_1^{p'}$ (Lemma A.3) and the bound on ∇u in L^p. This observation forces us to restrict the space of admissible test functions to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ at the initial stage.

Consequently, we introduce the following notion of weak solution.

Definition 3.3. A function $u \in W_0^{1,p}(\Omega)$ is called a *weak solution* of equation (3.1) if $b(x, u, \nabla u)$ is measurable and for every test function $v \in W_0^{1,p}(\Omega) \cap$

$L^\infty(\Omega)$ the products $a_i(x,u,\nabla u)\,\partial_i v$ and $b(x,u,\nabla u)\,v$ are integrable and

$$\int_\Omega \Big(\sum_{i=1}^n a_i(x,u,\nabla u)\,\partial_i v + b(x,u,\nabla u)\,v + \lambda u\,v\Big)\,\mathrm{d}x = \int_\Omega f\,v\,\mathrm{d}x. \tag{3.3}$$

Remark 3.4. In the Galerkin approximation of Section 5 we work with finite-dimensional subspaces V_m spanned by smooth, compactly supported functions; in particular, every element of V_m is bounded. Hence the definition is perfectly suited for the construction of approximate solutions. The uniform estimates of Section 4 will prove that any weak solution (and indeed the Galerkin approximations themselves) is locally bounded, which retrospectively justifies the choice of test functions.

Remark 3.5. If the lower-order term grows strictly sub-naturally, i.e. $|b(x,u,\xi)| \le \mu_1(x)|\xi|^k + \dots$ with $k < p-1$, then $b(x,u,\nabla u)\,v$ is automatically integrable for any $v \in W_0^{1,p}(\Omega)$ because one may apply Hölder's inequality with the exponents $\frac{p}{k}$, $\frac{p^*}{p-1-k}$ and obtain a convergent product. In that case the restriction to bounded test functions is unnecessary. The natural growth $k = p-1$ is the borderline where this restriction becomes essential, and it is precisely the case studied in this paper.

Once the local boundedness of solutions is established (Theorem 4.4), we can enlarge the class of admissible test functions to the whole space $W_0^{1,p}(\Omega)$ by a standard truncation argument. Let us sketch this enlargement now, even though it depends on results that will be proved in Section 4, to highlight the logic of the theory.

Proposition 3.6. *Assume that $u \in W_0^{1,p}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ is a weak solution in the sense of Definition 3.3. Then equation (3.3) holds for every $v \in W_0^{1,p}(\Omega)$.*

Proof. We only need to verify that the integral involving b is well defined for unbounded v. Fix an arbitrary $v \in W_0^{1,p}(\Omega)$ and for each $m \in \mathbb{N}$ define the truncation $T_m(s) = \max\{-m, \min\{s,m\}\}$ and set $v_m = T_m(v)$. Then $v_m \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $v_m \to v$ strongly in $W_0^{1,p}(\Omega)$, and $\nabla v_m = \chi_{\{|v|<m\}} \nabla v$ a.e. Inserting v_m into (3.3) yields

$$\int_\Omega \Big(\sum_i a_i(x,u,\nabla u)\,\partial_i v_m + b(x,u,\nabla u)\,v_m + \lambda u\,v_m\Big)\,\mathrm{d}x = \int_\Omega f\,v_m\,\mathrm{d}x.$$

The terms involving a_i and λu pass to the limit easily by dominated convergence or strong convergence of v_m and the growth conditions on a_i. The right-hand side converges because $f \in L^{p'}$ and $v_m \to v$ strongly in L^p. The critical term is $I_m = \int_\Omega b(x,u,\nabla u)\,v_m\,\mathrm{d}x$. From the growth condition (B2) we have

$$|b(x,u,\nabla u)| \le \mu_1(x)|\nabla u|^{p-1} + \varphi_3(x) + c_2|u|^{p-1}.$$

Since u is locally bounded, the function $c_2|u|^{p-1}$ is in L^∞, so its product with v_m converges by dominated convergence. The term $\varphi_3 v_m$ converges in L^1 because $\varphi_3 \in L^1$ and $v_m \to v$ weakly-$*$ in L^∞ (or simply using that $|v_m| \le |v|$ and dominated convergence on the support of φ_3). Thus the only obstacle is the convergence of

$$J_m = \int_\Omega \mu_1(x)|\nabla u|^{p-1}\, v_m \,\mathrm{d}x.$$

Because u is bounded and belongs to $W_0^{1,p}(\Omega)$, the function $g(x) = \mu_1(x)|\nabla u(x)|^{p-1}$ belongs to $L^1(\Omega)$; indeed,

$$\int_\Omega g \,\mathrm{d}x \le \|\mu_1\|_{L^r}\|\nabla u\|_p^{p-1} < \infty$$

by the integrability properties of μ_1 and ∇u (note that $r > n$ implies $p-1 \le p'$ as shown in Lemma A.3). Since $g \in L^1$ and $v_m \to v$ a.e. with $|v_m| \le |v|$, the dominated convergence theorem implies $J_m \to \int_\Omega g\, v \,\mathrm{d}x$. Hence the full integral involving b converges to $\int_\Omega b(x,u,\nabla u)\, v \,\mathrm{d}x$. Passing to the limit in the identity for v_m proves (3.3) for the unbounded test function v. □

Remark 3.7. The argument above crucially uses the fact that u is locally bounded, which guarantees that the singular term $\mu_1|\nabla u|^{p-1}$ is globally integrable and can serve as a dominating function. Without local boundedness, the form-boundedness condition (3.2) alone does not imply $\mu_1|\nabla u|^{p-1} \in L^1(\Omega)$ for a general $W_0^{1,p}$ function; therefore the restriction to bounded test functions is not merely a technicality but a necessary feature of the correct weak formulation in the natural growth case with singular coefficients.

4. A priori estimates

In this section we work under the structural assumptions 3.1 and 3.2, together with the classical integrability condition

Assumption 4.1. $\mu_1 \in L^r(\Omega)$ with $r > n$ (and $p < n$).

By Lemma 2.4 this implies that μ_1 is p-gradient-form-bounded. For the a priori estimate we further assume that the constant c_1 in (A1) is zero, i.e. the principal part is coercive without a lower-order $|u|^p$ term:

$$\sum_{i=1}^n a_i(x,u,\xi)\,\xi_i \ge \nu|\xi|^p - \varphi_1(x).$$

All estimates are first proved for bounded weak solutions; they extend to limits by density and lower semicontinuity.

4.1. Energy identity and absorption of the weighted gradient term.

Taking $v = u$ in the weak formulation (3.3) and using the coercivity and the sign condition (B1) we obtain

$$\nu\int_\Omega |\nabla u|^p \,\mathrm{d}x + \lambda\int_\Omega u^2 \,\mathrm{d}x \le J + \Phi + \int_\Omega |f||u| \,\mathrm{d}x, \tag{4.1}$$

where $\Phi = \|\varphi_1 + \varphi_3\|_{L^1}$ and $J = \int_\Omega \mu_1|\nabla u|^{p-1}|u| \,\mathrm{d}x$.

Lemma 4.1. *Let $\mu_1 \in L^r(\Omega)$ with $r > n$. For every $\varepsilon > 0$ there exists a constant $\widehat{C}_\varepsilon > 0$ such that*

$$J \leq \varepsilon \|\nabla u\|_p^p + \widehat{C}_\varepsilon \|u\|_p^p \qquad \forall u \in W_0^{1,p}(\Omega).$$

Proof. This is exactly Lemma 2.4, whose proof is reproduced in the Appendix. □

We now choose $\varepsilon = \nu/2$. Inserting the lemma into (4.1) gives

$$\frac{\nu}{2}\|\nabla u\|_p^p + \lambda \|u\|_2^2 \leq K\|u\|_p^p + \Phi + \|f\|_{p'}\|u\|_p, \tag{4.2}$$

with $K = \widehat{C}_{\nu/2}$.

4.2. **Global gradient bound under a smallness condition.** To close the estimate we use Poincaré's inequality $\|u\|_p \leq C_P\|\nabla u\|_p$. Then from (4.2),

$$\frac{\nu}{2}\|\nabla u\|_p^p \leq KC_P^p\|\nabla u\|_p^p + \Phi + \|f\|_{p'}C_P\|\nabla u\|_p.$$

If $KC_P^p < \nu/2$, we can absorb the first term on the right-hand side and obtain a uniform bound on $\|\nabla u\|_p$ depending only on the data. Explicitly,

$$\left(\frac{\nu}{2} - KC_P^p\right)\|\nabla u\|_p^p \leq \Phi + \frac{\nu/2 - KC_P^p}{2}\|\nabla u\|_p^p + C\|f\|_{p'}^{p'},$$

which yields $\|\nabla u\|_p \leq C(1 + \|f\|_{p'})$. The condition $KC_P^p < \nu/2$ is a compatibility requirement between the absorption constant K (which depends on $\|\mu_1\|_{L^r}$ and the domain) and the Poincaré constant. It is satisfied, for instance, if the L^n-norm of μ_1 is sufficiently small, or if the domain is such that C_P is small. In the general case, one may first work on a small ball where C_P can be made arbitrarily small, and then use a covering argument to obtain a global bound.

4.3. **Caccioppoli inequality.**

Lemma 4.2. *Let u be a weak solution of (3.1) under Assumptions 3.1 and 3.2. For any $k \in \mathbb{R}$ and any $0 < \rho < R$ with $B_R \Subset \Omega$, let $\eta \in C_c^\infty(B_R)$ satisfy $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_ρ, and $|\nabla \eta| \leq \frac{2}{R-\rho}$. Then*

$$\begin{aligned}\int_{B_\rho \cap \{u>k\}} |\nabla u|^p \,\mathrm{d}x \leq C\biggl(\frac{1}{(R-\rho)^p}\int_{B_R\cap\{u>k\}} (u-k)^p \,\mathrm{d}x \\ + \int_{B_R\cap\{u>k\}} |u|^p \,\mathrm{d}x + |B_R \cap \{u>k\}|\biggr),\end{aligned} \tag{4.3}$$

where $C > 0$ depends only on p, ν, μ, the form-boundedness constants $\varepsilon, C_\varepsilon$ from (2.3), and the data $\varphi_1, \varphi_2, \varphi_3, f$. (For the De Giorgi iteration the term $\int |u|^p$ is harmless because u will eventually be bounded; it can be absorbed into $|A_{k,R}|$ after an L^∞ estimate is known.)

Proof. The idea is to test the equation with a truncated version of $w := (u - k)_+$, multiplied by η^p, and then use the structural conditions together with the gradient-form-boundedness of μ_1. We first assume $k \geq 0$; the case $k < 0$ follows by applying the same argument to $-u$.

For $M > 0$ set $w_M := \min\{w, M\}$. Since $w \in W_0^{1,p}(\Omega)$, we have $w_M \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\nabla w_M = \chi_{\{0<w<M\}} \nabla w$ a.e. Define

$$v_M := \eta^p w_M.$$

Then $v_M \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and is supported in B_R, so it is an admissible test function in (3.3).

Using v_M in (3.3) gives

$$\begin{aligned}\int_{B_R} \Big(\sum_{i=1}^n a_i(x, u, \nabla u)\, \partial_i v_M + b(x, u, \nabla u)\, v_M + \lambda u v_M \Big)\, \mathrm{d}x \\ = \int_{B_R} f v_M \, \mathrm{d}x.\end{aligned} \tag{1}$$

We estimate each term from below on the set $A_{k,R} := B_R \cap \{u > k\}$, where $w > 0$ and $\nabla u = \nabla w$. Note that v_M vanishes outside $A_{k,R}$.

Compute $\partial_i v_M = \eta^p \partial_i w_M + p\eta^{p-1} w_M \partial_i \eta$. By **(A1)** and the fact that $\nabla u = \nabla w$ on $A_{k,R}$,

$$\begin{aligned}\sum_{i=1}^n a_i(x, u, \nabla u)\, \partial_i v_M \geq \eta^p \Big(\nu |\nabla w|^p - \varphi_1 - c_1 |u|^p \Big) \\ + p\eta^{p-1} w_M \sum_{i=1}^n a_i\, \partial_i \eta.\end{aligned} \tag{2}$$

For the cut-off term we use **(A2)** and Young's inequality:

$$\begin{aligned}\Big| p\eta^{p-1} w_M \sum_{i=1}^n a_i\, \partial_i \eta \Big| &\leq p\mu\, \eta^{p-1} |\nabla \eta|\, w_M\, |\nabla u|^{p-1} \\ &\leq \frac{\nu}{4}\, \eta^p |\nabla u|^p + C(\nu, \mu, p)\, |\nabla \eta|^p w_M^p.\end{aligned} \tag{3}$$

Because $w_M \leq w$, the last term is bounded by $C|\nabla \eta|^p w^p$.

We exploit the sign condition **(B1)** with a trick for the shifted function. On $A_{k,R}$ we have $u = w + k \geq k \geq 0$, hence $u > 0$ and $|u| = u$. From $b\, u \geq -\mu_1 |\nabla u|^{p-1} u - \varphi_3$ we obtain

$$b \geq -\mu_1 |\nabla u|^{p-1} - \frac{\varphi_3}{u} \qquad \text{a.e. on } A_{k,R}.$$

Multiplying by $v_M = \eta^p w_M \leq \eta^p w$ and using $w \leq u$,

$$\begin{aligned} b\, v_M &\geq -\eta^p \mu_1 |\nabla u|^{p-1} w_M - \eta^p \varphi_3 \frac{w_M}{u} \\ &\geq -\eta^p \mu_1 |\nabla u|^{p-1} w_M - \eta^p \varphi_3.\end{aligned} \tag{4}$$

The term $-\eta^p \varphi_3$ will eventually produce a contribution $\int_{A_{k,R}} \varphi_3 \, \mathrm{d}x$.

The gradient-form-boundedness condition (2.3) applied to the function w_M (after extending it by zero outside $A_{k,R}$) gives, for an arbitrary $\varepsilon > 0$,

$$\int_{A_{k,R}} \mu_1 |\nabla w_M|^{p-1} w_M \, \mathrm{d}x \leq \varepsilon \int_{A_{k,R}} |\nabla w_M|^p \, \mathrm{d}x + C_\varepsilon \int_{A_{k,R}} w_M^p \, \mathrm{d}x.$$

Since $\nabla w_M = \nabla u$ on $\{0 < w < M\}$ and 0 elsewhere, we have $|\nabla w_M| \le |\nabla u|$ pointwise. Moreover $w_M \le w$. Thus

$$\int_{A_{k,R}} \mu_1 |\nabla u|^{p-1} w_M \, dx \le \varepsilon \int_{A_{k,R}} |\nabla u|^p \, dx + C_\varepsilon \int_{A_{k,R}} w^p \, dx. \tag{5}$$

(Here ε will be chosen as $\varepsilon = \nu/4$.)

The $\lambda u v_M$ term is non-negative because $u \ge 0$ on $A_{k,R}$ and $v_M \ge 0$; therefore it can be omitted from the lower bound. The right-hand side $\int f v_M$ is estimated by

$$\int_{A_{k,R}} |f| \eta^p w_M \, dx \le \|f\|_{L^{p'}} \Big(\int_{A_{k,R}} w_M^p \Big)^{1/p} \le \frac{1}{2} \int_{A_{k,R}} w_M^p + \frac{1}{2} \|f\|_{L^{p'}}^{p'}.$$

For the data terms we collect

$$D := \int_{A_{k,R}} \big(\eta^p \varphi_1 + \eta^p \varphi_3 + |f v_M| \big) \, dx \le \int_{A_{k,R}} \big(\varphi_1 + \varphi_3 + \frac{1}{2} w^p + \frac{1}{2} |f|^{p'} \big) \, dx.$$

Insert (2)–(5) into (1) and drop the non-negative λ term. Using $\eta \equiv 1$ on B_ρ and $|\nabla \eta| \le 2/(R-\rho)$, we obtain

$$\begin{aligned} \nu \int_{A_{k,\rho}} |\nabla u|^p \, dx &\le \frac{\nu}{2} \int_{A_{k,R}} \eta^p |\nabla u|^p \, dx + C \int_{A_{k,R}} \big(|\nabla \eta|^p w^p + w^p \big) \, dx \\ &\quad + \int_{A_{k,R}} \big(c_1 \eta^p |u|^p + \varphi_1 + \varphi_3 \big) \, dx + \frac{1}{2} \|f\|_{p'}^{p'}. \end{aligned} \tag{4.4}$$

The term $\frac{\nu}{2} \int \eta^p |\nabla u|^p$ can be absorbed by the left-hand side after noting that $\eta \le 1$ and $\int_{A_{k,\rho}} |\nabla u|^p \le \int_{A_{k,R}} \eta^p |\nabla u|^p$. After renaming constants we get

$$\int_{A_{k,\rho}} |\nabla u|^p \, dx \le C \Big(\frac{1}{(R-\rho)^p} \int_{A_{k,R}} w^p \, dx + \int_{A_{k,R}} (|u|^p + \varphi_1 + \varphi_3) \, dx + \|f\|_{p'}^{p'} \Big).$$

Because $\varphi_1, \varphi_3 \in L^1$, we can bound their integrals over $A_{k,R}$ by $\max\{\|\varphi_1\|_1, \|\varphi_3\|_1\}$, but to fit the form of (4.3) we simply absorb the constant data into $C|A_{k,R}|$ by using the trivial estimate $\int_{A_{k,R}} (\varphi_1 + \varphi_3) \le \|\varphi_1 + \varphi_3\|_\infty |A_{k,R}|$ provided φ_1, φ_3 are bounded. Even if they are only L^1, in the De Giorgi iteration one first proves a local estimate and the additive constant is harmless; we keep the term $|A_{k,R}|$ as a placeholder for the data, understanding that the constant C may also depend on the L^1 norms. The term $\int_{A_{k,R}} |u|^p$ cannot be discarded yet, but after the boundedness result it will be controlled. The inequality above is exactly (4.3) (with an extra $\|f\|_{p'}^{p'}$ that can be absorbed into $|A_{k,R}|$ if $f \in L^\infty$; otherwise it remains as an additive constant, which again does not affect the iteration).

All estimates are independent of M. Letting $M \to \infty$, the monotone convergence theorem yields the same inequality for $w = (u-k)_+$, completing the proof for $k \ge 0$. If $k < 0$, we apply the result to $-u$ and the threshold $-k$. □

4.4. **Local boundedness via De Giorgi iteration.** We prove that weak solutions are locally bounded. The proof follows the classical iteration scheme of De Giorgi (see [1, Chap. 4, §6]) adapted to the gradient-form-boundedness framework.

Lemma 4.3. *Let $\{y_n\}_{n=0}^{\infty}$ be a sequence of non-negative numbers such that for some constants $C_0, b > 1$ and $\delta > 0$,*

$$y_{n+1} \le C_0\, b^n\, y_n^{1+\delta}, \qquad n = 0, 1, 2, \dots \tag{4.5}$$

If $y_0 \le C_0^{-1/\delta} b^{-1/\delta^2}$, then $y_n \to 0$ as $n \to \infty$.

Proof. By induction one obtains $\ln y_n \le (1+\delta)^n \big(\ln y_0 + \frac{\ln(C_0 b^{-1/\delta})}{\delta}\big) - \frac{\ln(C_0 b^{-1/\delta})}{\delta}$. The condition on y_0 makes the bracket negative, hence $y_n \to 0$. See [1, Lemma 4.3] for the explicit calculation. □

Theorem 4.4. *Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of (3.1) under Assumptions 3.1 and 3.2. For every ball $B_{2R} \Subset \Omega$,*

$$\operatorname{ess\,sup}_{B_R} |u| \le C\Big(R^{-n/p}\|u\|_{L^p(B_{2R})} + K\Big), \tag{4.6}$$

where $C > 0$ depends only on p, n, ν, μ, the form-boundedness constants $\varepsilon, C_\varepsilon$, and the data φ_i, f, while K is a constant depending on the same quantities but independent of u.

Proof. It is enough to estimate the positive part u_+; the negative part is handled by applying the same argument to $-u$. Fix a ball $B_{2R} \Subset \Omega$ and set

$$\mu(R) = \|u_+\|_{L^p(B_{2R})}. \tag{4.7}$$

We shall show that there exists a constant $d > 0$, depending on $\mu(R)$ and the structural data, such that $u \le d$ a.e. in B_R. The final estimate will then express d in the required form.

Define for $n = 0, 1, 2, \dots$

$$\rho_n = R + \frac{R}{2^n}, \qquad k_n = d\big(1 - 2^{-n-1}\big), \tag{4.8}$$

where $d \ge 1$ is a number to be chosen later. Then $\rho_0 = 2R$, $\rho_n \searrow R$, $k_0 = d/2$, $k_n \nearrow d$, and

$$k_{n+1} - k_n = d\, 2^{-n-2}, \qquad \rho_n - \rho_{n+1} = R\, 2^{-n-1}. \tag{4.9}$$

For each n set

$$A_n = B_{\rho_n} \cap \{u > k_n\}, \qquad y_n = \int_{A_n} (u - k_n)^p \,\mathrm{d}x. \tag{4.10}$$

Clearly $y_{n+1} \le y_n$ and $|A_{n+1}| \le |A_n|$.

Applying Lemma 4.2 with $k = k_n$, $\rho = \rho_{n+1}$, $R = \rho_n$, and noting that the cut-off function η can be chosen to satisfy the required bounds because $\rho_{n+1} < \rho_n$, we obtain

$$\int_{A_{k_n,\rho_{n+1}}} |\nabla u|^p \,\mathrm{d}x \le C\Big(\frac{1}{(\rho_n - \rho_{n+1})^p} \int_{A_n} (u - k_n)^p \,\mathrm{d}x + (k_n^p + 1)|A_n|\Big), \tag{4.11}$$

where C depends only on the structural constants (the terms $\int_{A_{k,R}} |u|^p$ and $|A_{k,R}|$ from Lemma 4.2 have been replaced by the larger quantity $(k_n^p+1)|A_n|$ plus the integral, which is exactly y_n; the data terms φ_i, f are bounded by a constant K_0 that can be absorbed into $|A_n|$ after possibly enlarging C). Using $\rho_n - \rho_{n+1} = R\,2^{-n-1}$ we write

$$\int_{A_{k_n,\rho_{n+1}}} |\nabla u|^p \, \mathrm{d}x \le C_1 2^{(n+1)p} R^{-p} y_n + C_1(k_n^p+1)|A_n|, \tag{4.12}$$

with $C_1 = C$.

For every $n \ge 0$ we have $A_{n+1} \subset A_{k_n,\rho_{n+1}} \subset A_n$. By Chebyshev's inequality,

$$|A_{n+1}| \le \frac{1}{(k_{n+1}-k_n)^p} \int_{A_{k_n,\rho_{n+1}}} (u-k_n)^p \, \mathrm{d}x \le \frac{1}{(d\,2^{-n-2})^p}\, y_n = d^{-p}\, 2^{(n+2)p}\, y_n. \tag{4.13}$$

The same argument with n replaced by $n-1$ (for $n \ge 1$) gives

$$|A_n| \le d^{-p}\, 2^{(n+1)p}\, y_{n-1}. \tag{4.14}$$

Moreover, since $k_n \le d$, we may use the rough bound $k_n^p + 1 \le d^p + 1 \le 2d^p$ (because $d \ge 1$).

Let $v_n = (u-k_n)_+$. Then v_n is supported in $A_{k_n,\rho_{n+1}}$, and the Sobolev embedding together with Hölder's inequality yields (see [1, (2.11)])

$$\|v_n\|^p_{L^p(B_{\rho_{n+1}})} \le C_S^p\, |A_{k_n,\rho_{n+1}}|^{p/n} \int_{A_{k_n,\rho_{n+1}}} |\nabla v_n|^p \, \mathrm{d}x. \tag{4.15}$$

Because $v_n = u - k_n$ on $A_{k_n,\rho_{n+1}}$ and $\nabla v_n = \nabla u$ there, we obtain

$$\int_{A_{k_n,\rho_{n+1}}} (u-k_n)^p \, \mathrm{d}x \le C_S^p\, |A_n|^{p/n} \int_{A_{k_n,\rho_{n+1}}} |\nabla u|^p \, \mathrm{d}x. \tag{4.16}$$

The left-hand side is not smaller than y_{n+1}, because $A_{n+1} \subset A_{k_n,\rho_{n+1}}$ and $u - k_{n+1} \le u - k_n$ on that set; hence

$$y_{n+1} \le C_S^p\, |A_n|^{p/n} \int_{A_{k_n,\rho_{n+1}}} |\nabla u|^p \, \mathrm{d}x. \tag{4.17}$$

Insert (4.12) into (4.17):

$$\begin{aligned} y_{n+1} &\le C_S^p\, |A_n|^{p/n} \Big(C_1 2^{(n+1)p} R^{-p} y_n + C_1(k_n^p+1)|A_n| \Big) \\ &\le C_2 |A_n|^{p/n} \Big(2^{(n+1)p} R^{-p} y_n + d^p |A_n| \Big), \end{aligned} \tag{4.18}$$

with $C_2 = C_S^p C_1$. Now use the bound (4.14) for $|A_n|$: $|A_n| \le d^{-p} 2^{(n+1)p} y_{n-1}$. Therefore

$$|A_n|^{p/n} \le \big(d^{-p} 2^{(n+1)p}\big)^{p/n} y_{n-1}^{p/n} = d^{-p^2/n} 2^{(n+1)p^2/n} y_{n-1}^{p/n}, \tag{4.19}$$

$$|A_n|^{1+p/n} \le d^{-p(1+p/n)} 2^{(n+1)p(1+p/n)} y_{n-1}^{1+p/n} \le d^{-p} 2^{(n+1)p} y_{n-1}^{1+p/n} \quad (\text{since } d \ge 1 \text{ and } n \ge 0). \tag{4.20}$$

Substituting these into (4.18) yields, for $n \geq 1$,

$$\begin{aligned} y_{n+1} &\leq C_2\, d^{-p^2/n} 2^{(n+1)p^2/n} y_{n-1}^{p/n} \Big(2^{(n+1)p} R^{-p} y_n + d^p \cdot d^{-p} 2^{(n+1)p} y_{n-1} \Big) \\ &= C_2\, d^{-p^2/n} 2^{(n+1)p^2/n} 2^{(n+1)p} y_{n-1}^{p/n} \big(R^{-p} y_n + y_{n-1} \big). \end{aligned} \tag{4.21}$$

Since $y_n \leq y_{n-1}$, we have $R^{-p} y_n + y_{n-1} \leq (R^{-p}+1) y_{n-1}$. Moreover $y_{n-1}^{p/n} \leq y_{n-1}^{p/n}$. Hence, for $n \geq 1$,

$$y_{n+1} \leq C_3\, d^{-p^2/n}\, b^n\, y_{n-1}^{1+p/n}, \tag{4.22}$$

where $C_3 = C_2(R^{-p}+1)$ and $b = 2^{p+p^2}$ (absorbing the factor $2^{p^2/n}$ into b^n for $n \geq 1$).

The recurrence (4.22) links y_{n+1} to y_{n-1}. Consider the even and odd subsequences. For $m \geq 0$, set $z_m = y_{2m}$. Then from (4.22) with $n = 2m+1$ we obtain

$$y_{2m+2} \leq C_3\, d^{-p^2/(2m+1)}\, b^{2m+1}\, y_{2m}^{1+p/(2m+1)}.$$

Because $d \geq 1$, $d^{-p^2/(2m+1)} \leq d^{-p^2/n_0}$ for some fixed n_0 when m is small, and for $m \geq 1$ we can absorb the d-power into the constant by choosing d large. To simplify, we may restrict to $m \geq m_0$ with m_0 such that the exponent $p/(2m+1)$ is small. A cleaner way is to note that the sequence y_n is non-increasing, so the convergence of the even subsequence implies that of the whole sequence. We apply Lemma 4.3 to z_m with the worst-case exponents $\delta = p/n$ (since $p/(2m+1) \leq p/n$ for $2m+1 \geq n$). After choosing n sufficiently large, we may assume the recurrence holds with a uniform $\delta > 0$ and a constant $C_4 = C_3 d^{-\alpha}$ for some $\alpha > 0$. Explicitly, for all $m \geq 1$,

$$z_{m+1} \leq C_4\, (b^2)^m\, z_m^{1+\delta}, \qquad \delta = \frac{p}{n}.$$

By Lemma 4.3, $z_m \to 0$ provided $z_0 \leq C_4^{-1/\delta} (b^2)^{-1/\delta^2}$.

Now $z_0 = y_0 = \int_{B_{2R} \cap \{u > d/2\}} (u - d/2)^p \,\mathrm{d}x \leq \int_{B_{2R}} u_+^p \,\mathrm{d}x = \mu(R)^p$. Thus we can guarantee the smallness condition by choosing d so large that

$$\mu(R)^p \leq C_4^{-1/\delta} b^{-2/\delta^2} \simeq C_5\, d^{\alpha/\delta}.$$

Because $\alpha = p^2/n$, $\delta = p/n$, we have $\alpha/\delta = p$. Hence the condition is

$$\mu(R)^p \leq C_6\, d^p \quad \Longleftrightarrow \quad d \geq C_7\, \mu(R).$$

Taking into account the R^{-p} factors hidden in C_3 (which contain $(R^{-p}+1)$), a more precise tracking gives $d \geq C_8 \left(R^{-n/p} \mu(R) + \text{data} \right)$.

Consequently, there exists a constant d satisfying the above bound and such that $z_m \to 0$, i.e. $y_n \to 0$. In particular $y_n \to 0$, which forces $|A_\infty| = |B_R \cap \{u > d\}| = 0$. Hence $u \leq d$ a.e. in B_R. The estimate (4.6) follows by repeating the argument for $-u$ and taking the maximum of the two bounds. The additive constant K accounts for the data φ_i, f that were treated as bounded terms. $\square$

4.5. **Global $W^{1,p}$ estimate under a smallness condition.** We now derive a global gradient bound for weak solutions, assuming a compatibility condition between the absorption constant and the Poincaré constant. The argument proceeds directly from the weak formulation without assuming beforehand that u is bounded; the critical step is the justification that u itself can be used as a test function.

Lemma 4.5. *Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of (3.1) under Assumptions 3.1 and 3.2 with the additional integrability condition $\mu_1 \in L^r(\Omega)$, $r > n$. Then $b(x,u,\nabla u)\, u \in L^1(\Omega)$ and*

$$\int_\Omega \Big(\sum_{i=1}^n a_i(x,u,\nabla u)\, \partial_i u + b(x,u,\nabla u)\, u + \lambda u^2 \Big)\, \mathrm{d}x = \int_\Omega f u \, \mathrm{d}x. \tag{4.23}$$

Proof. For $m \in \mathbb{N}$ define $T_m(s) = \max\{-m, \min\{s, m\}\}$ and $u_m = T_m(u)$. Then $u_m \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $u_m \to u$ strongly in $W_0^{1,p}(\Omega)$, and $\nabla u_m = \chi_{\{|u|<m\}} \nabla u$ a.e. From the weak formulation (3.3) with test function $v = u_m$ we obtain

$$\int_\Omega \Big(\sum_i a_i(x,u,\nabla u)\, \partial_i u_m + b(x,u,\nabla u)\, u_m + \lambda u u_m \Big)\, \mathrm{d}x = \int_\Omega f u_m \, \mathrm{d}x. \tag{*}$$

We now pass to the limit $m \to \infty$. The term $\lambda u u_m$ converges to λu^2 by dominated convergence because $u \in L^2$ (recall $W_0^{1,p} \subset L^2$ for $p \geq 2$; if $p < 2$ a different embedding is used, but we assume $p \geq 2$ as stated in Section 3). The right-hand side converges because $f \in L^{p'}$ and $u_m \to u$ strongly in L^p.

For the principal part, $\partial_i u_m = \chi_{\{|u|<m\}} \partial_i u$. Hence

$$\sum_i a_i(x,u,\nabla u)\, \partial_i u_m = \sum_i a_i(x,u,\nabla u)\, \partial_i u \ \text{ on } \{|u| < m\},$$

and 0 elsewhere. By coercivity (A1) and the fact that the integrand is non-negative almost everywhere (after absorbing the φ_1 and c_1 terms), the sequence is non-negative and increasing. The monotone convergence theorem gives

$$\int_\Omega \sum_i a_i(x,u,\nabla u)\, \partial_i u_m \, \mathrm{d}x \longrightarrow \int_\Omega \sum_i a_i(x,u,\nabla u)\, \partial_i u \, \mathrm{d}x \quad \text{as } m \to \infty,$$

and the limit is finite because $u \in W_0^{1,p}$ and the growth (A2) guarantees the integral is bounded by $\mu \|\nabla u\|_p^p + \dots$.

The most delicate term is the one involving b. Using **(B1)** and **(B2)** we have the pointwise bound

$$|b(x,u,\nabla u)\, u_m| \leq \mu_1(x) |\nabla u|^{p-1} |u| + \varphi_3(x) + c_2 |u|^{p-1} |u|.$$

Lemma 4.1 (weighted absorption) shows that $\mu_1 |\nabla u|^{p-1} |u| \in L^1(\Omega)$, while $\varphi_3 \in L^1$ and $c_2 |u|^p \in L^1$ by Sobolev embedding. Therefore the sequence $b(x,u,\nabla u)\, u_m$ is dominated by an L^1 function. Moreover, $u_m \to u$ a.e., so the pointwise limit is $b(x,u,\nabla u)\, u$. By the dominated convergence theorem,

$$\int_\Omega b(x,u,\nabla u)\, u_m \, \mathrm{d}x \longrightarrow \int_\Omega b(x,u,\nabla u)\, u \, \mathrm{d}x,$$

and the limit is finite. Passing to the limit in (*) proves the identity. □

Theorem 4.6. *Under the hypotheses of Lemma 4.5, assume that the constant $K := C_{\nu/2}$ from Lemma 4.1 satisfies the smallness condition*

$$K\,C_P^p < \frac{\nu}{2}, \tag{4.24}$$

where C_P is the Poincaré constant from (2.1). Then every weak solution $u \in W_0^{1,p}(\Omega)$ satisfies

$$\|u\|_{W_0^{1,p}(\Omega)} \le C\big(1 + \|f\|_{L^{p'}(\Omega)}\big), \tag{4.25}$$

with $C > 0$ depending only on ν, μ, p, n, Ω, the form-boundedness data of μ_1, and the L^1 norms of φ_1, φ_3.

Proof. Starting from the energy identity (4.23), we use **(A1)** and **(B1)** together with $\lambda \ge 0$ to obtain

$$\nu \int_\Omega |\nabla u|^p \,\mathrm{d}x \le \int_\Omega \big(\mu_1 |\nabla u|^{p-1}|u| + \varphi_1 + c_1|u|^p + \varphi_3\big)\,\mathrm{d}x + \int_\Omega f u \,\mathrm{d}x.$$

(The c_1 term from (A1) is added to the right; it will be handled later.) Applying Lemma 4.1 with $\varepsilon = \nu/2$ gives

$$\int_\Omega \mu_1 |\nabla u|^{p-1}|u|\,\mathrm{d}x \le \frac{\nu}{2}\|\nabla u\|_p^p + K\|u\|_p^p,$$

where $K = C_{\nu/2}$. Consequently,

$$\frac{\nu}{2}\|\nabla u\|_p^p \le K\|u\|_p^p + c_1\|u\|_p^p + \|\varphi_1 + \varphi_3\|_{L^1} + \|f\|_{p'}\|u\|_p.$$

Using the Poincaré inequality $\|u\|_p \le C_P\|\nabla u\|_p$, we obtain

$$\frac{\nu}{2}\|\nabla u\|_p^p \le (K + c_1)C_P^p\|\nabla u\|_p^p + \|\Phi\|_{L^1} + \|f\|_{p'}C_P\|\nabla u\|_p,$$

where $\Phi = \varphi_1 + \varphi_3$. The smallness assumption (4.24) implies $KC_P^p < \nu/2$; even if $c_1 > 0$ we can absorb it by a smallness condition on c_1 or by noting that the L^∞ estimate from Theorem 4.4 makes the $c_1\|u\|_p^p$ term harmless after a further absorption. To keep the presentation concise we assume $c_1 = 0$; the general case can be handled with an L^∞ estimate (see Remark 4.7). Hence

$$\Big(\frac{\nu}{2} - KC_P^p\Big)\|\nabla u\|_p^p \le \|\Phi\|_{L^1} + \|f\|_{p'}C_P\|\nabla u\|_p.$$

By Young's inequality,

$$\|f\|_{p'}C_P\|\nabla u\|_p \le \frac{1}{p}\Big(\frac{\nu}{2} - KC_P^p\Big)\|\nabla u\|_p^p + C\Big(\frac{\nu}{2} - KC_P^p\Big)^{-\frac{1}{p-1}}\big(C_P\|f\|_{p'}\big)^{p'}.$$

Absorbing the gradient term into the left-hand side yields

$$\frac{p-1}{p}\Big(\frac{\nu}{2} - KC_P^p\Big)\|\nabla u\|_p^p \le \|\Phi\|_{L^1} + C'\|f\|_{p'}^{p'}.$$

Thus $\|\nabla u\|_p$ is bounded by a constant depending only on the data. Finally, $\|u\|_{W_0^{1,p}}^p = \|\nabla u\|_p^p + \|u\|_p^p \le (1 + C_P^p)\|\nabla u\|_p^p$, which proves (4.25). □

Remark 4.7. If $c_1 > 0$ in **(A1)**, the term $c_1\|u\|_p^p$ is not directly absorbable without an additional smallness hypothesis on c_1. However, one first proves a local boundedness theorem (Theorem 4.4) which holds without any smallness condition and provides an L^∞ bound. With that bound, $\|u\|_p^p \le |\Omega|\,\|u\|_\infty^p$, and the term $c_1\|u\|_p^p$ becomes a harmless constant that can be added to the right-hand side. Therefore the global $W^{1,p}$ estimate remains valid for any $c_1 \ge 0$.

Corollary 4.8. *Under the same conditions, the weak solution u is globally bounded and satisfies*

$$\|u\|_{L^\infty(\Omega)} \le C\big(1 + \|f\|_{p'}\big),$$

with C depending on the structural data. This follows by combining the local boundedness estimate of Theorem 4.4 with the $W^{1,p}$ estimate and a standard covering of $\overline{\Omega}$ if the boundary is sufficiently regular, or by a direct global De Giorgi iteration on Ω when a boundary Caccioppoli inequality is available.

5. Existence of weak solutions

In this section we prove the existence of a weak solution to (3.1) under the structural assumptions 3.1 and 3.2, together with the classical integrability condition

Assumption 5.1. $\mu_1 \in L^r(\Omega)$ with $r > n$ (and $p < n$).

By Lemma 2.4 this implies that μ_1 is p-gradient-form-bounded, so all the estimates of Section 4 are available.

We use the Galerkin method as presented in [1]. The main points are the uniform estimates, the strong convergence of gradients obtained via the Minty–Browder trick, and the equi-integrability argument for the lower-order term.

5.1. Galerkin approximation.

Let $\{\phi_j\}_{j=1}^\infty \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a sequence of linearly independent functions whose finite linear spans $V_m = \operatorname{span}\{\phi_1, \dots, \phi_m\}$ are dense in $W_0^{1,p}(\Omega)$. For each m we look for an approximate solution $u_m = \sum_{j=1}^m c_{m,j}\phi_j \in V_m$ satisfying

$$\int_\Omega \Big(\sum_i a_i(x, u_m, \nabla u_m)\,\partial_i v + b(x, u_m, \nabla u_m)\,v + \lambda u_m v\Big)\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x \qquad \forall v \in V_m. \tag{5.1}$$

Lemma 5.1. *For every m there exists at least one $u_m \in V_m$ satisfying* (5.1).

Proof. Fix $m \in \mathbb{N}$ and equip $\mathbb{R}^m$ with the Euclidean norm $|c|$. Identify the finite-dimensional space $V_m = \operatorname{span}\{\phi_1, \dots, \phi_m\}$ with $\mathbb{R}^m$ via $c = (c_1, \dots, c_m) \leftrightarrow u = \sum_{j=1}^m c_j\phi_j$. Because each ϕ_j belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, every $u \in V_m$ is bounded and its gradient is in L^p.

For $c \in \mathbb{R}^m$ set $u = \sum_{j=1}^m c_j\phi_j$ and define

$$F_j(c) = \int_\Omega \Big(\sum_{i=1}^n a_i(x, u, \nabla u)\,\partial_i\phi_j + b(x, u, \nabla u)\,\phi_j + \lambda u\phi_j - f\phi_j\Big)\,\mathrm{d}x, \qquad j = 1, \dots, m. \tag{5.2}$$

We must check that each integral is finite. From the growth conditions **(A2)** and **(B2)** we have pointwise bounds

$$\Big|\sum_i a_i(x,u,\nabla u)\,\partial_i\phi_j\Big| \le C_\phi\big(\mu|\nabla u|^{p-1}+\varphi_2+c_1|u|^{p-1}\big),$$

$$|b(x,u,\nabla u)\,\phi_j| \le \|\phi_j\|_\infty\big(\mu_1|\nabla u|^{p-1}+\varphi_3+c_2|u|^{p-1}\big).$$

Since $u\in V_m\subset L^\infty(\Omega)$ and $\nabla u\in L^p(\Omega)$, the terms $|u|^{p-1}$ and $|\nabla u|^{p-1}$ belong to $L^{p'}(\Omega)$. Moreover, $\mu_1\in L^r(\Omega)$ with $r>n$ implies $\mu_1^{p'}\in L^1(\Omega)$ (Lemma A.3), so by Hölder's inequality $\mu_1|\nabla u|^{p-1}$ is integrable. The data $\varphi_2,\varphi_3\in L^1$, and $f\in L^{p'}$. Hence all integrals in (5.2) are absolutely convergent.

Let $c^{(k)}\to c$ in $\mathbb{R}^m$ and set $u_k=\sum_j c_j^{(k)}\phi_j$, $u=\sum_j c_j\phi_j$. Then $u_k\to u$ uniformly on Ω and $\nabla u_k\to\nabla u$ in $L^p(\Omega;\mathbb{R}^n)$. After passing to a subsequence we may assume pointwise convergence a.e. as well. Using the Carathéodory property, for a.e. $x\in\Omega$,

$$a_i(x,u_k(x),\nabla u_k(x))\to a_i(x,u(x),\nabla u(x)),\quad b(x,u_k(x),\nabla u_k(x))\to b(x,u(x),\nabla u(x)).$$

The growth conditions provide dominating functions in L^1 (modulo the equi-integrability of $\mu_1|\nabla u_k|^{p-1}$, which follows from the strong L^p convergence of the gradients and the absolute continuity of the integral of $\mu_1^{p'}$). By the dominated convergence theorem, each $F_j(c^{(k)})$ converges to $F_j(c)$. Hence F is continuous.

Compute the scalar product of $F(c)$ with c:

$$F(c)\cdot c=\sum_{j=1}^m c_jF_j(c)=\int_\Omega\Big(\sum_{i=1}^n a_i(x,u,\nabla u)\,\partial_i u+b(x,u,\nabla u)\,u+\lambda u^2-fu\Big)\,\mathrm{d}x.$$

Invoke the structural assumptions. Using **(A1)** (coercivity) and **(B1)** (sign condition) we obtain

$$F(c)\cdot c\ge \nu\int_\Omega|\nabla u|^p\,\mathrm{d}x-\int_\Omega\mu_1|\nabla u|^{p-1}|u|\,\mathrm{d}x-\int_\Omega\varphi_1\,\mathrm{d}x-c_1\int_\Omega|u|^p\,\mathrm{d}x-\int_\Omega\varphi_3\,\mathrm{d}x+\lambda\int_\Omega u^2\,\mathrm{d}x-\int_\Omega fu\,\mathrm{d}x.$$

Apply the weighted absorption inequality of Lemma 4.1 with $\varepsilon=\nu/2$; recall that this gives a constant $K=C_{\nu/2}$ such that

$$\int_\Omega\mu_1|\nabla u|^{p-1}|u|\,\mathrm{d}x\le\frac{\nu}{2}\int_\Omega|\nabla u|^p\,\mathrm{d}x+K\int_\Omega|u|^p\,\mathrm{d}x.$$

Setting $\Phi=\|\varphi_1+\varphi_3\|_{L^1}$ and using $\lambda\ge0$, we arrive at

$$F(c)\cdot c\ge\frac{\nu}{2}\|\nabla u\|_p^p-(K+c_1)\|u\|_p^p-\Phi-\|f\|_{p'}\|u\|_p. \tag{5.3}$$

Now we use the smallness condition assumed in this section (cf. Theorem 4.6):

$$K\,C_P^p<\frac{\nu}{2}, \tag{5.4}$$

where C_P is the Poincaré constant of Ω. (If $c_1>0$ we also need $c_1C_P^p<\nu/2$, which can be arranged either by an additional smallness hypothesis or by an initial L^∞ estimate; for simplicity we take $c_1 = 0$.) By the Poincaré inequality, $\|u\|_p^p \le C_P^p \|\nabla u\|_p^p$, so

$$\frac{\nu}{2}\|\nabla u\|_p^p - K\|u\|_p^p \ge \Big(\frac{\nu}{2} - KC_P^p\Big)\|\nabla u\|_p^p =: \alpha\|\nabla u\|_p^p,$$

with $\alpha > 0$. Consequently,

$$F(c)\cdot c \ge \alpha\|\nabla u\|_p^p - \Phi - \|f\|_{p'}\|u\|_p.$$

By the equivalence of all norms on the finite-dimensional space V_m, there exists a constant $C_m > 0$ such that $\|\nabla u\|_p \ge C_m\|u\|_{W_0^{1,p}}$. Using Young's inequality,

$$\|f\|_{p'}\|u\|_p \le \|f\|_{p'}C_P\|\nabla u\|_p \le \frac{\alpha}{2}\|\nabla u\|_p^p + C(\alpha, C_P)\|f\|_{p'}^{p'}.$$

Inserting this above gives

$$F(c)\cdot c \ge \frac{\alpha}{2}\|\nabla u\|_p^p - \Phi - C\|f\|_{p'}^{p'}.$$

Because $\|\nabla u\|_p \ge C_m\|u\|_{W_0^{1,p}}$ and $\|u\|_{W_0^{1,p}} \to \infty$ as $|c| \to \infty$ (all norms on $\mathbb{R}^m$ being equivalent), the right-hand side tends to $+\infty$ as $|c| \to \infty$. Hence there exists $R > 0$ such that for all c with $|c| = R$, $F(c)\cdot c > 0$.

The continuous map F satisfies $F(c)\cdot c > 0$ on the sphere $\partial B_R(0)$. By Brouwer's fixed point theorem (or its consequence for zeros of vector fields), there exists a point $c^* \in B_R(0)$ with $F(c^*) = 0$. Setting $u_m = \sum_{j=1}^m c_j^*\phi_j$ yields exactly (5.1). This completes the proof. □

5.2. **Uniform estimates for the Galerkin approximations.** We now derive a priori bounds for the Galerkin solutions that are independent of the discretisation parameter m. The main obstacle is that the test function required for the Caccioppoli inequality does not necessarily belong to the finite-dimensional space V_m. To overcome this, we first regularise the lower-order term and then extend the class of admissible test functions by a density argument exactly as in [1, Lemma 4.5].

For a fixed $\varepsilon > 0$ define the bounded lower-order term

$$b_\varepsilon(x,u,\xi) = \frac{b(x,u,\xi)}{1+\varepsilon\,|b(x,u,\xi)|}.$$

Then b_ε is a Carathéodory function that satisfies the same growth and sign conditions **(B1)**, **(B2)** as b (with the same μ_1, φ_3, c_2) and, in addition, $|b_\varepsilon| \le 1/\varepsilon$. Consequently, for any $u \in W_0^{1,p}(\Omega)$, the map $v \mapsto \int_\Omega b_\varepsilon(x,u,\nabla u)v$ is a bounded linear functional on $L^p(\Omega)\cap L^\infty(\Omega)$; in particular, the integrals are finite for all test functions in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

We apply the Galerkin method to the regularised equation

$$\lambda u - \sum_{i=1}^n \frac{\partial}{\partial x_i}a_i(x,u,\nabla u) + b_\varepsilon(x,u,\nabla u) = f \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

By Lemma 5.1 there exists, for every m, a function $u_{m,\varepsilon} \in V_m$ satisfying

$$\int_\Omega \Big(\sum_i a_i(x, u_{m,\varepsilon}, \nabla u_{m,\varepsilon})\, \partial_i v + b_\varepsilon(x, u_{m,\varepsilon}, \nabla u_{m,\varepsilon})\, v + \lambda u_{m,\varepsilon} v - f v \Big)\, dx = 0 \qquad \forall v \in V_m. \tag{5.5}$$

Because the lower-order term is bounded and the functions ϕ_j are smooth, a standard argument (see [1, Lemma 4.5]) shows that (5.5) actually holds for every $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. The proof consists of approximating such a v by functions from V_m in the weak topology of $W_0^{1,p}(\Omega)$ and the weak-$*$ topology of $L^\infty(\Omega)$, and then passing to the limit using the boundedness of b_ε. Henceforth we may freely use any bounded test function in the estimates below.

Testing (5.5) with $v = u_{m,\varepsilon} \in V_m$ (which is admissible without further justification) and using the coercivity **(A1)** together with the sign condition **(B1)** (which is also satisfied by b_ε) gives exactly the energy inequality

$$\nu \int_\Omega |\nabla u_{m,\varepsilon}|^p\, dx \le \int_\Omega \mu_1 |\nabla u_{m,\varepsilon}|^{p-1} |u_{m,\varepsilon}|\, dx + \int_\Omega (\varphi_1 + \varphi_3)\, dx + c_1 \int_\Omega |u_{m,\varepsilon}|^p\, dx + \int_\Omega |f|\, |u_{m,\varepsilon}|\, dx.$$

The right-hand side term involving μ_1 is controlled by the weighted absorption inequality (Lemma 4.1) with $\varepsilon = \nu/2$:

$$\int_\Omega \mu_1 |\nabla u_{m,\varepsilon}|^{p-1} |u_{m,\varepsilon}|\, dx \le \frac{\nu}{2} \int_\Omega |\nabla u_{m,\varepsilon}|^p\, dx + K \int_\Omega |u_{m,\varepsilon}|^p\, dx,$$

where $K = C_{\nu/2}$ depends only on the structural data. Substituting this and rearranging yields

$$\frac{\nu}{2} \int_\Omega |\nabla u_{m,\varepsilon}|^p\, dx \le (K + c_1) \int_\Omega |u_{m,\varepsilon}|^p\, dx + \|\varphi_1 + \varphi_3\|_{L^1} + \|f\|_{L^{p'}} \|u_{m,\varepsilon}\|_{L^p}.$$

Now apply the Poincaré inequality $\|u_{m,\varepsilon}\|_p \le C_P \|\nabla u_{m,\varepsilon}\|_p$. We assume the smallness condition

$$K C_P^p < \frac{\nu}{2}. \tag{5.6}$$

If $c_1 > 0$ we can either absorb it via an additional smallness assumption or, alternatively, first prove a local L^∞ bound (which in this regularised setting is obtained with the already known $W^{1,p}$ estimate – see Step 3) and then return to handle c_1. For simplicity of exposition we set $c_1 = 0$; the general case is treated in Remark 4.7. Thus

$$\frac{\nu}{2} \|\nabla u_{m,\varepsilon}\|_p^p \le K C_P^p \|\nabla u_{m,\varepsilon}\|_p^p + \|\Phi\|_{L^1} + C_P \|f\|_{p'} \|\nabla u_{m,\varepsilon}\|_p.$$

Absorbing the gradient terms using (5.6) and Young's inequality yields, after elementary manipulations,

$$\|\nabla u_{m,\varepsilon}\|_p^p \le C\big(1 + \|f\|_{p'}^{p'}\big),$$

with a constant C independent of m and ε. By the Poincaré inequality we obtain a uniform bound in $W_0^{1,p}(\Omega)$.

Now that the sequence $\{u_{m,\varepsilon}\}$ is bounded in $W_0^{1,p}(\Omega)$, its L^p norms are uniformly bounded. We prove a local L^∞ bound via the De Giorgi iteration. Fix a ball $B_{2R} \Subset \Omega$ and, for $k \ge 0$, set $w = (u_{m,\varepsilon} - k)_+$. Because the Galerkin equations hold for all bounded test functions, we may insert $v = \eta^p w$ with a cut-off function

$\eta \in C_c^\infty(B_R)$ into (5.5). Repeating the proof of Lemma 4.2 with b replaced by b_ε (the estimates are identical since b_ε satisfies the same bounds) we obtain the Caccioppoli inequality

$$\int_{A_{k,\rho}} |\nabla u_{m,\varepsilon}|^p\, dx \le C\biggl(\frac{1}{(R-\rho)^p}\int_{A_{k,R}} (u_{m,\varepsilon}-k)^p\, dx + |A_{k,R}|\biggr),$$

where $A_{k,r} = B_r \cap \{u_{m,\varepsilon} > k\}$ and C depends only on the structural data. This is exactly the inequality required for the De Giorgi class $DG_p(\Omega)$.

Applying the De Giorgi iteration (Theorem 4.4) to each function $u_{m,\varepsilon}$ – which only uses the Caccioppoli inequality and the Sobolev embedding, and whose constants are independent of the particular function – we obtain, for any ball $B_{2R} \Subset \Omega$,

$$\operatorname{ess\,sup}_{B_R} |u_{m,\varepsilon}| \le C\Bigl(R^{-n/p}\|u_{m,\varepsilon}\|_{L^p(B_{2R})} + 1\Bigr).$$

The L^p norm is already uniformly bounded, hence

$$\|u_{m,\varepsilon}\|_{L^\infty(B_R)} \le C'$$

with C' independent of m, ε. A standard covering of $\overline{\Omega}$ by a finite number of balls (using the Lipschitz regularity of the boundary for boundary balls, or a reflection argument) gives a global uniform bound

$$\|u_{m,\varepsilon}\|_{L^\infty(\Omega)} \le M,$$

where M does not depend on m or ε.

For each fixed $\varepsilon > 0$, the sequence $\{u_{m,\varepsilon}\}_{m=1}^\infty$ is bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. By the Rellich–Kondrachov theorem we can extract a subsequence (still denoted by m) such that

$$u_{m,\varepsilon} \rightharpoonup u_\varepsilon \quad \text{weakly in } W_0^{1,p}(\Omega), \qquad u_{m,\varepsilon} \to u_\varepsilon \quad \text{strongly in } L^p(\Omega) \text{ and a.e. in } \Omega.$$

The uniform L^∞ bound implies that the limit u_ε is also essentially bounded.

The estimates above are independent of ε. Therefore, letting $\varepsilon \to 0$ along a further subsequence, we obtain a function $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ which is the limit of u_ε in the same topologies. The next subsections are devoted to showing that u is a weak solution of the original problem (3.1) and that it inherits the uniform bounds.

5.3. **Strong convergence of the gradients.** We prove that $\nabla u_m \to \nabla u$ strongly in $L^p(\Omega;\mathbb{R}^n)$. From the weak formulation for u_m and the uniform bound, $\mathbf{a}(x, u_m, \nabla u_m)$ is bounded in $L^{p'}(\Omega;\mathbb{R}^n)$; hence, for a further subsequence,

$$\mathbf{a}(x, u_m, \nabla u_m) \rightharpoonup \chi \quad \text{weakly in } L^{p'}.$$

Choosing $v = u_m - u$ (which belongs to V_m for sufficiently large m after a density argument, or more precisely using the Minty–Browder trick) in the limit, one obtains

$$\limsup_{m\to\infty} \int_\Omega \mathbf{a}(x, u_m, \nabla u_m) \cdot \nabla(u_m - u)\, \mathrm{d}x \le 0.$$

Together with the monotonicity of $\mathbf{a}$ (implied by (A1)–(A2) after a standard modification) or directly by the pseudomonotonicity of the operator, this inequality forces

$$\int_\Omega \mathbf{a}(x, u_m, \nabla u_m) \cdot \nabla u_m \, \mathrm{d}x \to \int_\Omega \chi \cdot \nabla u \, \mathrm{d}x,$$

and then the strict monotonicity yields $\nabla u_m \to \nabla u$ strongly in L^p. (For a detailed proof we refer to [1, Chapter 4, §1].) Consequently, up to a subsequence, $\nabla u_m \to \nabla u$ a.e. in Ω.

5.4. **Passage to the limit in the lower-order term.** For a fixed test function $v = \phi_k \in L^\infty(\Omega)$ we must show

$$\int_\Omega b(x, u_m, \nabla u_m)\, \phi_k \, \mathrm{d}x \to \int_\Omega b(x, u, \nabla u)\, \phi_k \, \mathrm{d}x.$$

From the growth condition (B2),

$$|b(x, u_m, \nabla u_m)| \le \mu_1 |\nabla u_m|^{p-1} + \varphi_3 + c_2 |u_m|^{p-1}. \tag{5.7}$$

The term $c_2|u_m|^{p-1}\phi_k$ converges strongly in L^1 because $u_m \to u$ in L^p. The term φ_3 is fixed.

For the singular term we prove $\mu_1|\nabla u_m|^{p-1} \to \mu_1|\nabla u|^{p-1}$ strongly in $L^1(\Omega)$. Because $\|\nabla u_m\|_p \le C$, Hölder's inequality with $\frac{1}{r} + \frac{p-1}{p} + \frac{1}{q} = 1$ (where $q < p^*$) gives

$$\int_\Omega \mu_1 |\nabla u_m|^{p-1} \, \mathrm{d}x \le \|\mu_1\|_r \|\nabla u_m\|_p^{p-1} |\Omega|^{1/q} \le C.$$

For any measurable $E \subset \Omega$,

$$\int_E \mu_1 |\nabla u_m|^{p-1} \, \mathrm{d}x \le \|\mu_1\|_{L^r(E)} \|\nabla u_m\|_p^{p-1} |E|^{1/q}.$$

Since $\mu_1 \in L^r$, its integral over small sets is uniformly small; hence the sequence $\{\mu_1|\nabla u_m|^{p-1}\}$ is equi-integrable. We already have $\nabla u_m \to \nabla u$ a.e., so the pointwise limit is $\mu_1|\nabla u|^{p-1}$. Vitali's theorem implies strong convergence in L^1.

Now from (5.7) and the pointwise convergence of $u_m, \nabla u_m$, the Carathéodory property gives $b(x, u_m, \nabla u_m) \to b(x, u, \nabla u)$ a.e. Because the dominating function $\mu_1|\nabla u_m|^{p-1} + \varphi_3 + c_2|u_m|^{p-1}$ converges strongly in L^1, the dominated convergence theorem yields the desired limit.

5.5. **Conclusion.** For every $v = \phi_k$,

$$\int_\Omega \Big(\chi \cdot \nabla v + b(x, u, \nabla u)\, v + \lambda u v\Big) \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x.$$

By pseudomonotonicity $\chi = \mathbf{a}(x, u, \nabla u)$. Density of $\{\phi_k\}$ in $W_0^{1,p} \cap L^\infty$ implies u is a weak solution.

Theorem 5.2. *Under Assumptions 3.1, 3.2 and 5.1 there exists a weak solution $u \in W_0^{1,p}(\Omega)$ of (3.1).*

6. Maximum and comparison principles

Throughout this section $u \in W_0^{1,p}(\Omega)$ is a weak solution of (3.1) under Assumptions 3.1, 3.2 and the classical condition $\mu_1 \in L^r(\Omega)$, $r > n$. The local boundedness of u has already been established in Theorem 4.4; we use it here without further comment.

6.1. **Weak maximum principle.** We prove that weak solutions satisfy a maximum principle under appropriate sign conditions on the data and a smallness assumption. The result is classical and follows the arguments of Ladyzhenskaya–Ural'tseva [1, Theorem 6.1] adapted to our gradient-form-boundedness framework.

Theorem 6.1. *In addition to Assumptions 3.1 and 3.2 (with $\varphi_1 = \varphi_3 \equiv 0$, $c_1 = 0$) and the integrability condition $\mu_1 \in L^r(\Omega)$, $r > n$, suppose that $f \le 0$ a.e. in Ω and that the smallness condition*

$$K\, C_P^p < \frac{\nu}{2}, \tag{6.1}$$

holds, where $K = C_{\nu/2}$ is the constant from Lemma 4.1. Then every weak solution u of (3.1) satisfies $u \le 0$ a.e. in Ω.

Proof. Set $v = u_+ = \max\{u, 0\}$. Since $u \in W_0^{1,p}(\Omega)$, we have $v \in W_0^{1,p}(\Omega)$ and $\nabla v = \chi_{\{u>0\}} \nabla u$ a.e. The energy identity of Lemma 4.5 guarantees that $b(x,u,\nabla u)\, u \in L^1(\Omega)$, hence the product $b(x,u,\nabla u)\, v$ is integrable. Consequently v is an admissible test function in the weak formulation (3.3) (the right-hand side $\int fv$ is well defined because $f \in L^{p'}$ and $v \in L^p$). Using v as test function we obtain

$$\int_{\{u>0\}} \Big(\sum_{i=1}^n a_i(x,u,\nabla u)\, \partial_i u + b(x,u,\nabla u)\, u + \lambda u^2 \Big)\, \mathrm{d}x = \int_{\{u>0\}} f u \, \mathrm{d}x. \tag{6.2}$$

Because $f \le 0$, the right-hand side of (6.2) is non-positive and can be dropped. On the set $\{u > 0\}$ we employ the structural conditions. From **(A1)** with $\varphi_1 \equiv 0$, $c_1 = 0$ we have

$$\sum_{i=1}^n a_i(x,u,\nabla u)\, \partial_i u \ge \nu |\nabla u|^p. \tag{6.3}$$

From **(B1)** with $\varphi_3 \equiv 0$ we obtain

$$b(x,u,\nabla u)\, u \ge -\mu_1(x) |\nabla u|^{p-1} u. \tag{6.4}$$

The term λu^2 is non-negative and will also be dropped from the estimate (it only helps). Substituting (6.3)–(6.4) into (6.2) and using $f \le 0$, $\lambda \ge 0$ gives

$$\nu \int_{\{u>0\}} |\nabla u|^p \, \mathrm{d}x \le \int_{\{u>0\}} \mu_1 |\nabla u|^{p-1} u \, \mathrm{d}x. \tag{6.5}$$

The right-hand side of (6.5) involves exactly the weighted product controlled by the gradient-form-boundedness condition. Applying Lemma 4.1 to the function

$v = u_+$ (which belongs to $W_0^{1,p}(\Omega)$) with $\varepsilon = \nu/2$, we obtain

$$\int_\Omega \mu_1 |\nabla v|^{p-1} v \, \mathrm{d}x \le \frac{\nu}{2} \int_\Omega |\nabla v|^p \, \mathrm{d}x + K \int_\Omega v^p \, \mathrm{d}x, \tag{6.6}$$

where $K = C_{\nu/2}$. Since $\nabla v = \chi_{\{u>0\}} \nabla u$ and $v = u$ on $\{u > 0\}$, this is exactly

$$\int_{\{u>0\}} \mu_1 |\nabla u|^{p-1} u \, \mathrm{d}x \le \frac{\nu}{2} \int_{\{u>0\}} |\nabla u|^p \, \mathrm{d}x + K \int_{\{u>0\}} u^p \, \mathrm{d}x. \tag{6.7}$$

Insert (6.7) into (6.5) and absorb the gradient term:

$$\frac{\nu}{2} \int_{\{u>0\}} |\nabla u|^p \, \mathrm{d}x \le K \int_{\{u>0\}} u^p \, \mathrm{d}x.$$

By the Poincaré inequality (2.1) applied to $v = u_+$,

$$\int_{\{u>0\}} u^p \, \mathrm{d}x = \int_\Omega v^p \, \mathrm{d}x \le C_P^p \int_\Omega |\nabla v|^p \, \mathrm{d}x = C_P^p \int_{\{u>0\}} |\nabla u|^p \, \mathrm{d}x.$$

Consequently,

$$\frac{\nu}{2} \int_{\{u>0\}} |\nabla u|^p \, \mathrm{d}x \le K C_P^p \int_{\{u>0\}} |\nabla u|^p \, \mathrm{d}x,$$

that is,

$$\Big(\frac{\nu}{2} - K C_P^p \Big) \int_{\{u>0\}} |\nabla u|^p \, \mathrm{d}x \le 0.$$

By the smallness condition (6.1) the factor in parentheses is strictly positive. Hence

$$\int_{\{u>0\}} |\nabla u|^p \, \mathrm{d}x = 0.$$

This forces $|\nabla u| = 0$ a.e. on $\{u > 0\}$, and therefore u is constant on each connected component of $\{u > 0\}$. Because $u = 0$ on $\partial\Omega$ in the sense of traces and $\{u > 0\}$ is open, the only constant compatible with the boundary condition is 0. Thus $|\{u > 0\}| = 0$, i.e. $u \le 0$ a.e. in Ω. □

Remark 6.2. The additional hypothesis $b(x, u, 0)u \ge 0$ for $u \ge 0$, which is often included in classical formulations, is not needed here because it is automatically satisfied when $\varphi_3 = 0$; indeed, from **(B1)** with $\xi = 0$ we have $b(x, u, 0)\, u \ge -\varphi_3(x) = 0$. The proof also shows that the smallness condition (6.1) can be replaced by a local smallness condition on sufficiently small subdomains, which would then give a local maximum principle; the global version follows by covering arguments if the domain can be exhausted by sets where the smallness holds.

6.2. **Comparison principle.** In this subsection we prove that bounded weak solutions can be ordered if the data are suitably related. To obtain a clean statement we supplement the general assumptions (A1)–(A2) and (B1)–(B2) with the following additional hypotheses.

Assumption 6.1. (C1) **Principal part independent of u.** $a_i(x,u,\xi) = a_i(x,\xi)$ for $i = 1,\dots,n$; the vector field $\mathbf{a}(x,\xi)$ is strictly monotone in ξ and satisfies the uniform p-ellipticity condition

$$\big(\mathbf{a}(x,\xi) - \mathbf{a}(x,\eta)\big)\cdot(\xi-\eta) \geq \nu_0|\xi-\eta|^p \qquad \forall \xi,\eta\in\mathbb{R}^n, \tag{6.8}$$

with a constant $\nu_0 > 0$.

(C2) **One-sided Lipschitz condition for b.** There exists a constant $L \geq 0$ such that for a.e. $x\in\Omega$, all $u_1,u_2\in\mathbb{R}$ and all $\xi\in\mathbb{R}^n$,

$$\big(b(x,u_1,\xi) - b(x,u_2,\xi)\big)(u_1-u_2) \geq -L|u_1-u_2|^p. \tag{6.9}$$

(C3) **Gradient dependence of b is controlled.** For a.e. x, all u and all ξ,η,

$$|b(x,u,\xi) - b(x,u,\eta)| \leq \mu_1(x)|\xi-\eta|^{p-1}, \tag{6.10}$$

where μ_1 is the same p-gradient-form-bounded function that appears in (B1)–(B2).

Condition (6.10) is compatible with the natural growth (B2) and holds for many concrete models (for instance when $b(x,u,\xi) = \mu_1(x)|\xi|^{p-2}\xi + b_0(x,u)$). Together with the gradient-form-boundedness of μ_1 it will allow us to absorb the gradient discrepancy into the elliptic term.

Theorem 6.3. *Let Assumptions 3.1, 3.2 and 6.1 hold, let $f_1,f_2\in L^{p'}(\Omega)$ and let $u_1,u_2\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ be weak solutions of*

$$\lambda u - \sum_{i=1}^n \frac{\partial}{\partial x_i}a_i(x,\nabla u) + b(x,u,\nabla u) = f_1, \qquad \lambda u - \sum_{i=1}^n \frac{\partial}{\partial x_i}a_i(x,\nabla u) + b(x,u,\nabla u) = f_2,$$

respectively, with $f_1 \leq f_2$ a.e. in Ω and $u_1\leq u_2$ on $\partial\Omega$ in the sense of traces. Then $u_1\leq u_2$ a.e. in Ω.

Proof. Set $w = (u_1-u_2)_+$. Because u_1,u_2 are bounded and have the same boundary values, $w\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. It is an admissible test function for both weak formulations. Subtract the two equations and test with w:

$$\int_\Omega \big(\mathbf{a}(x,\nabla u_1) - \mathbf{a}(x,\nabla u_2)\big)\cdot\nabla w + \big(b(x,u_1,\nabla u_1) - b(x,u_2,\nabla u_2)\big)w + \lambda w^2\,dx = \int_\Omega (f_1-f_2)w\,dx. \tag{6.11}$$

The right-hand side is ≤ 0 because $f_1\leq f_2$ and $w\geq 0$. The term λw^2 is non-negative and will be dropped. All integrals are restricted to the set $\Omega_+ = \{x\in\Omega : w(x) > 0\} = \{u_1 > u_2\}$; on this set $\nabla w = \nabla u_1 - \nabla u_2$.

From (6.8) we have

$$\big(\mathbf{a}(x,\nabla u_1) - \mathbf{a}(x,\nabla u_2)\big)\cdot\nabla w \geq \nu_0|\nabla w|^p. \tag{6.12}$$

Write

$$b(x,u_1,\nabla u_1) - b(x,u_2,\nabla u_2) = \big[b(x,u_1,\nabla u_1) - b(x,u_1,\nabla u_2)\big] + \big[b(x,u_1,\nabla u_2) - b(x,u_2,\nabla u_2)\big].$$

By (6.9) the second bracket satisfies

$$\big(b(x,u_1,\nabla u_2)-b(x,u_2,\nabla u_2)\big)w \ge -Lw^p.$$

For the first bracket we use (6.10) and Young's inequality:

$$\begin{aligned}\big|\big[b(x,u_1,\nabla u_1)-b(x,u_1,\nabla u_2)\big]w\big| &\le \mu_1(x)|\nabla w|^{p-1}w\\ &= \mu_1(x)\big(|\nabla w|^{p-1}w\big).\end{aligned}$$

Because μ_1 is p-gradient-form-bounded (Definition 2.2), for any $\varepsilon>0$ there exists C_ε such that

$$\int_\Omega \mu_1|\nabla w|^{p-1}w\,dx \le \varepsilon\int_\Omega |\nabla w|^p\,dx + C_\varepsilon\int_\Omega w^p\,dx.$$

Choose $\varepsilon=\frac{\nu_0}{2}$. Then

$$\int_{\Omega_+}\big[b(x,u_1,\nabla u_1)-b(x,u_1,\nabla u_2)\big]w\,dx \ge -\frac{\nu_0}{2}\int_\Omega|\nabla w|^p - C_{\nu_0/2}\int_\Omega w^p.$$

Combining the two parts we obtain

$$\int_{\Omega_+}\big(b(x,u_1,\nabla u_1)-b(x,u_2,\nabla u_2)\big)w\,dx \ge -\frac{\nu_0}{2}\int_\Omega|\nabla w|^p - \big(C_{\nu_0/2}+L\big)\int_\Omega w^p. \tag{6.13}$$

Insert (6.12) and (6.13) into (6.11) and drop the non-positive right-hand side as well as λw^2:

$$\nu_0\int_\Omega|\nabla w|^p-\frac{\nu_0}{2}\int_\Omega|\nabla w|^p-\big(C_{\nu_0/2}+L\big)\int_\Omega w^p\le 0.$$

Hence

$$\frac{\nu_0}{2}\int_\Omega|\nabla w|^p\le\big(C_{\nu_0/2}+L\big)\int_\Omega w^p. \tag{6.14}$$

Because $w\in W_0^{1,p}(\Omega)$, the Poincaré inequality $\|w\|_p\le C_P\|\nabla w\|_p$ yields

$$\frac{\nu_0}{2}\int_\Omega|\nabla w|^p\le\big(C_{\nu_0/2}+L\big)C_P^p\int_\Omega|\nabla w|^p.$$

If $(C_{\nu_0/2}+L)C_P^p<\frac{\nu_0}{2}$, we immediately obtain $\int|\nabla w|^p=0$, so $w=0$ a.e. In the general case we can avoid a smallness condition by a standard iteration argument: from (6.14) we have $\|\nabla w\|_p^p\le C\|w\|_p^p$. Using the interpolation inequality (or the fact that w is bounded and the L^p norm can be absorbed after repeated application of the estimate on smaller subdomains) one concludes that $w=0$. A detailed proof of this last step can be found in [1, Theorem 2.2]. Therefore $u_1\le u_2$ a.e. in Ω. □

Remark 6.4. Condition (6.10) is the natural requirement that makes the gradient-form-boundedness applicable to the difference of the lower-order terms. If b does not depend on ∇u, the constant $\nu_0/2$ in the absorption step can be taken equal to 0, and the proof simplifies considerably. The smallness condition in the last step is not necessary if one combines the estimate with the boundedness of u_1,u_2 and a suitable iteration on the measure of $\{w>0\}$; we refer to [1] for the complete argument.

7. Hölder continuity of solutions

In this section we prove that bounded weak solutions of (3.1) are locally Hölder continuous. The proof follows the De Giorgi–Ladyzhenskaya iteration scheme (see [1, Chap. 4,§6]). The key ingredient is the Caccioppoli inequality already established in Lemma 4.2 (or the simpler version Lemma 7.1 below). The boundedness of u (Theorem 4.4) will be used throughout.

Let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution under Assumptions 3.1, 3.2 and the classical condition $\mu_1 \in L^r(\Omega)$ with $r > n$ (Assumption 4.1). We fix a ball $B_{2R} \Subset \Omega$ and work with the truncated functions $w = (u-k)_+$ for $k \geq 0$. The estimates for $(u-k)_-$ are completely analogous.

7.1. **Caccioppoli inequality on level sets.** Let $0 < \rho < R$ and let $\eta \in C_c^\infty(B_R)$ satisfy $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_ρ, and $|\nabla \eta| \leq \frac{2}{R-\rho}$. Set $A_{k,r} = B_r \cap \{u > k\}$.

Lemma 7.1. *There exists a constant $C > 0$, depending only on the structural data, such that*

$$\int_{A_{k,\rho}} |\nabla w|^p \, dx \leq C \left(\frac{1}{(R-\rho)^p} \int_{A_{k,R}} w^p \, dx + |A_{k,R}| \right).$$

Proof. Let u be a weak solution of (3.1) under Assumptions 3.1 and 3.2. By Theorem 4.4 the function u is locally bounded; in particular, on the ball $B_R \Subset \Omega$ we have $|u| \leq M_R$ for some constant $M_R > 0$. Set $w = (u-k)_+$ and $A_{k,\rho} = B_\rho \cap \{u > k\}$.

Take $\eta \in C_c^\infty(B_R)$ with the properties stated in the lemma. The function $v = \eta^p w$ belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, hence it is an admissible test function in the weak formulation (3.3). Substituting v into (3.3) yields

$$\begin{aligned} \int_{A_{k,R}} \Big(\eta^p \sum_{i=1}^n a_i(x,u,\nabla u)\, \partial_i w + p\eta^{p-1} w \sum_{i=1}^n a_i(x,u,\nabla u)\, \partial_i \eta \\ + \eta^p b(x,u,\nabla u)\, w + \lambda \eta^p u w \Big) \, \mathrm{d}x = \int_{A_{k,R}} f \eta^p w \, \mathrm{d}x. \end{aligned} \tag{1}$$

Note that the integrals are restricted to $A_{k,R}$ because w and ∇w vanish outside this set. Also, on $A_{k,R}$ we have $u \geq k \geq 0$ (we consider $k \geq 0$; the case $k < 0$ is analogous), so the λ term is non-negative and may be discarded from the inequality.

From **(A1)** and the fact that $\nabla u = \nabla w$ on $A_{k,R}$ we obtain

$$\sum_{i=1}^n a_i(x,u,\nabla u)\, \partial_i w \geq \nu |\nabla w|^p - \varphi_1 - c_1 |u|^p. \tag{7.1}$$

The terms φ_1 and $c_1|u|^p$ will be controlled later using the boundedness of u and the integrability of φ_1.

By **(A2)** and Young's inequality,

$$\Big|p\eta^{p-1}w\sum_{i=1}^{n}a_i\,\partial_i\eta\Big| \le p\mu\,\eta^{p-1}|\nabla\eta|\,w\,|\nabla u|^{p-1}$$
$$\le \frac{\nu}{4}\,\eta^p|\nabla u|^p + C(\nu,\mu,p)\,|\nabla\eta|^p w^p. \qquad (2)$$

Since $\nabla u = \nabla w$ on $A_{k,R}$, the gradient term on the right is $\frac{\nu}{4}\eta^p|\nabla w|^p$.

The sign condition **(B1)** gives, on $A_{k,R}$,

$$b(x,u,\nabla u)\,w \ge -\mu_1|\nabla u|^{p-1}w - \varphi_3. \qquad (7.2)$$

For the singular part we apply the gradient-form-boundedness condition to w itself (note that w is supported in $A_{k,R}$ and $|\nabla w| = |\nabla u|$ there). For any $\varepsilon > 0$,

$$\int_{A_{k,R}}\eta^p\mu_1|\nabla w|^{p-1}w\,\mathrm{d}x \le \varepsilon\int_{A_{k,R}}\eta^p|\nabla w|^p\,\mathrm{d}x + C_\varepsilon\int_{A_{k,R}}\eta^p w^p\,\mathrm{d}x. \qquad (7.3)$$

Choose $\varepsilon = \nu/4$ and denote $C_{\nu/4}$ by K_1.

Using Hölder's inequality,

$$\int_{A_{k,R}}|f|\eta^p w\,\mathrm{d}x \le \|f\|_{L^{p'}}\Big(\int_{A_{k,R}}w^p\,\mathrm{d}x\Big)^{1/p} \le \frac{1}{2}\int_{A_{k,R}}w^p\,\mathrm{d}x + \frac{1}{2}\|f\|_{p'}^{p'}.$$

The terms φ_1, φ_3, and $c_1|u|^p$ are bounded: on $A_{k,R}$, $|u| \le M_R$, so $|u|^p \le M_R^p$. Therefore

$$\int_{A_{k,R}}\eta^p\big(\varphi_1+\varphi_3+c_1|u|^p\big)\,\mathrm{d}x \le \big(\|\varphi_1\|_{L^1}+\|\varphi_3\|_{L^1}+c_1M_R^p|\Omega|\big) =: C_{\mathrm{data}}.$$

Insert (7.1)–(7.3) into (1) and use the bounds above. Discarding the non-negative λ term, we obtain

$$\nu\int_{A_{k,R}}\eta^p|\nabla w|^p\,\mathrm{d}x \le \frac{\nu}{2}\int_{A_{k,R}}\eta^p|\nabla w|^p\,\mathrm{d}x + C\int_{A_{k,R}}|\nabla\eta|^p w^p\,\mathrm{d}x$$
$$+ K_1\int_{A_{k,R}}\eta^p w^p\,\mathrm{d}x + C'_{\mathrm{data}}. \qquad (7.4)$$

The constant C'_{data} depends on C_{data}, $\|f\|_{p'}$, and the constants from the previous inequalities. Subtracting the first term on the right and using $\eta \le 1$ gives

$$\frac{\nu}{2}\int_{A_{k,R}}\eta^p|\nabla w|^p\,\mathrm{d}x \le C\int_{A_{k,R}}|\nabla\eta|^p w^p\,\mathrm{d}x + C\int_{A_{k,R}}w^p\,\mathrm{d}x + C'_{\mathrm{data}}.$$

By the properties of η, $|\nabla\eta| \le 2/(R-\rho)$ and $\eta \equiv 1$ on B_ρ. Therefore

$$\int_{A_{k,\rho}}|\nabla w|^p\,\mathrm{d}x \le \frac{C}{(R-\rho)^p}\int_{A_{k,R}}w^p\,\mathrm{d}x + C\int_{A_{k,R}}w^p\,\mathrm{d}x + C'_{\mathrm{data}}.$$

Because $w \le u_+$ and u is bounded, $w \le M_R$. Hence $w^p \le M_R^p$, and the second integral is bounded by $M_R^p|A_{k,R}|$. The term C'_{data} can also be bounded by a constant times $|A_{k,R}|$ (after adding $|A_{k,R}|$ to the right if necessary), leading to

$$\int_{A_{k,\rho}}|\nabla w|^p\,\mathrm{d}x \le C\left(\frac{1}{(R-\rho)^p}\int_{A_{k,R}}w^p\,\mathrm{d}x + |A_{k,R}|\right),$$

where the new constant C depends on p, ν, μ, the form-boundedness constants, and the data. This completes the proof. $\square$

7.2. **De Giorgi class and the iteration lemma.** The Caccioppoli inequality of Lemma 7.1 shows that every weak solution belongs to the *De Giorgi class* $DG_p(\Omega)$: for every $k \geq 0$ and $0 < \rho < R$ with $B_R \Subset \Omega$,

$$\int_{A_{k,\rho}} |\nabla u|^p \, dx \leq C\left(\frac{1}{(R-\rho)^p} \int_{A_{k,R}} (u-k)^p \, dx + |A_{k,R}|\right),$$

where the constant C depends only on the structural data. From this inequality the classical De Giorgi iteration yields first a relation between the measures of level sets and then, via an elementary lemma on sequences, the local boundedness of u. The core of the iteration is the following lemma, which is a purely analytic fact about sequences.

Lemma 7.2. *Let $\{y_n\}_{n=0}^{\infty}$ be a sequence of non-negative numbers and assume that there exist constants $C_0 > 0$, $b > 1$ and $\delta > 0$ such that*

$$y_{n+1} \leq C_0 \, b^n \, y_n^{1+\delta}, \qquad n = 0, 1, 2, \ldots \tag{7.5}$$

If the initial term satisfies

$$y_0 \leq C_0^{-1/\delta} \, b^{-1/\delta^2}, \tag{7.6}$$

then $y_n \to 0$ as $n \to \infty$.

Proof. The proof is standard and can be found in [1, Lemma 4.3]. It consists of setting $z_n = y_n C_0^{1/\delta} b^{n/\delta^2}$, which reduces the recurrence to $z_{n+1} \leq z_n^{1+\delta}$. If $z_0 \leq 1$, then $z_n \to 0$, and the conclusion follows. The condition (7.6) is precisely $z_0 \leq 1$. $\square$

Remark 7.3. The iteration lemma is the engine behind the local boundedness theorem. To apply it one constructs sequences of radii $\rho_n = R + \frac{R}{2^n}$, thresholds $k_n = d\big(1-2^{-n-1}\big)$, and sets $y_n = \int_{A_{k_n,\rho_n}} (u-k_n)^p dx$. The Caccioppoli and Sobolev inequalities provide a recurrence of the form $y_{n+1} \leq C b^n y_n^{1+\delta}$ with $\delta = p/n$ (see [1, Theorem 6.1]). The smallness condition on y_0 translates into d being sufficiently large, and the lemma forces $y_n \to 0$, hence $|A_{d,R}| = 0$. This proves local boundedness.

7.3. **Logarithmic estimate.** For the local regularity analysis we assume, without loss of generality, that the lower-order terms in (A1)–(B2) satisfy

$$\varphi_1 = \varphi_3 \equiv 0, \qquad c_1 = c_2 = 0, \qquad f \equiv 0,$$

while the singular coefficient μ_1 remains only p-gradient-form-bounded. (Once the Hölder estimate is proved under these conditions, the general case follows by a standard perturbation argument, because all additional terms are either bounded or in suitable Lebesgue spaces and contribute harmless additive constants.) We also use the local boundedness of u already established in Theorem 4.4.

Lemma 7.4. *Let u be a bounded weak solution of (3.1) in a ball $B_{2R} \Subset \Omega$ under the reduced structural assumptions above. Set $M = \sup_{B_{2R}} u$. For any $k \geq 0$ and $\varepsilon > 0$ define $w_\varepsilon = (u-k)_+ + \varepsilon$. There exists a constant $C > 0$, depending only on p, ν, μ and $\|\mu_1^{p'}\|_{L^1(B_{2R})}$, such that for every cut-off function $\eta \in C_c^\infty(B_R)$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_ρ and $|\nabla \eta| \leq \frac{2}{R-\rho}$,*

$$\int_{B_\rho} \left|\nabla \log w_\varepsilon\right|^p dx \leq C\Big(\frac{R^n}{(R-\rho)^p} + 1\Big).$$

The constant is independent of k and ε; passing to the limit $\varepsilon \to 0^+$ gives the same estimate for the function $(u-k)_+$ on the set where it is positive.

Proof. Set $w = (u-k)_+$ and $w_\varepsilon = w + \varepsilon$. Because u is bounded, $M < \infty$. For fixed $\varepsilon > 0$ define

$$v = \eta^p \big(w_\varepsilon^{1-p} - (M+\varepsilon)^{1-p}\big).$$

The function v is non-negative, belongs to $W_0^{1,p}(B_R) \cap L^\infty(\Omega)$, and is therefore an admissible test function in the weak formulation (3.3). A straightforward calculation gives

$$\nabla v = \eta^p (1-p) w_\varepsilon^{-p} \nabla w + p\eta^{p-1}\big(w_\varepsilon^{1-p} - (M+\varepsilon)^{1-p}\big)\nabla \eta.$$

All integrals are supported on $A_{k,R} = B_R \cap \{u > k\}$, where $\nabla w = \nabla u$.

Substituting v into (3.3) and using $f = 0$, $\lambda = 0$ we obtain

$$(1-p)\int_{A_{k,R}} \eta^p w_\varepsilon^{-p} \sum_{i=1}^n a_i \partial_i w + p\int_{A_{k,R}} \eta^{p-1}\big(w_\varepsilon^{1-p} - (M+\varepsilon)^{1-p}\big)\sum_{i=1}^n a_i \partial_i \eta + \int_{A_{k,R}} b\, v = 0. \tag{7.7}$$

Multiplying by -1 and rearranging,

$$(p-1)\int_{A_{k,R}} \eta^p w_\varepsilon^{-p} \sum_{i=1}^n a_i \partial_i w = -p\int_{A_{k,R}} \eta^{p-1}\big(w_\varepsilon^{1-p} - (M+\varepsilon)^{1-p}\big)\sum_{i=1}^n a_i \partial_i \eta - \int_{A_{k,R}} b\, v. \tag{7.8}$$

From the coercivity (A1) with $\varphi_1 = c_1 = 0$ we have

$$\sum_{i=1}^n a_i(x,u,\nabla u)\partial_i w \geq \nu |\nabla w|^p = \nu\, w_\varepsilon^p |\nabla \log w_\varepsilon|^p.$$

Hence the left-hand side of (7.8) satisfies

$$\text{LHS} \geq (p-1)\nu \int_{A_{k,R}} \eta^p |\nabla \log w_\varepsilon|^p\, dx.$$

Using (A2) and the fact that the bracket is bounded by w_ε^{1-p}, we have

$$\Big|p\int \eta^{p-1}\big(w_\varepsilon^{1-p} - (M+\varepsilon)^{1-p}\big)\sum_i a_i \partial_i \eta\Big| \leq p\mu \int_{A_{k,R}} \eta^{p-1} |\nabla \eta|\, w_\varepsilon^{1-p} |\nabla w|^{p-1}.$$

Since $w_\varepsilon^{1-p}|\nabla w|^{p-1} = |\nabla \log w_\varepsilon|^{p-1}$, Young's inequality yields, for any $\delta > 0$,

$$p\mu\, \eta^{p-1} |\nabla \eta|\, |\nabla \log w_\varepsilon|^{p-1} \leq \delta\, \eta^p |\nabla \log w_\varepsilon|^p + C(\delta, p, \mu)\, |\nabla \eta|^p.$$

We choose $\delta = \frac{(p-1)\nu}{4}$.

From the growth condition (B2) with $\varphi_3 = c_2 = 0$ we have $|b| \le \mu_1 |\nabla u|^{p-1}$. Therefore,

$$\Big|\int_{A_{k,R}} b\, v\Big| \le \int_{A_{k,R}} \mu_1 |\nabla w|^{p-1}\, \eta^p w_\varepsilon^{1-p} = \int_{A_{k,R}} \eta^p \mu_1 |\nabla \log w_\varepsilon|^{p-1}.$$

Applying Young's inequality again,

$$\mu_1 |\nabla \log w_\varepsilon|^{p-1} \le \delta\, \eta^p |\nabla \log w_\varepsilon|^p + C(\delta)\, \mu_1^{p'},$$

with the same δ as above.

Insert the estimates above into (7.8). The terms containing $|\nabla \log w_\varepsilon|^p$ on the right amount to at most $2\delta \int \eta^p |\nabla \log w_\varepsilon|^p = \frac{(p-1)\nu}{2} \int \eta^p |\nabla \log w_\varepsilon|^p$. Moving this to the left yields

$$\frac{(p-1)\nu}{2} \int_{A_{k,R}} \eta^p |\nabla \log w_\varepsilon|^p\, dx \le C \int_{B_R} |\nabla \eta|^p\, dx + C \int_{B_R} \mu_1^{p'}\, dx.$$

By the properties of η, the first integral is bounded by $CR^n/(R-\rho)^p$. The second integral is finite by Lemma A.3 and is bounded independently of k, ε. Hence

$$\int_{B_\rho} |\nabla \log w_\varepsilon|^p\, dx \le C\Big(\frac{R^n}{(R-\rho)^p} + 1\Big).$$

The right-hand side does not depend on ε or k, so we may let $\varepsilon \to 0^+$ by Fatou's lemma to obtain the same estimate for the limit. □

7.4. **Decay of oscillation.** In this section we combine the Caccioppoli inequality (Lemma 7.1) and the logarithmic estimate (Lemma 7.4) to prove that the oscillation of a bounded weak solution decays geometrically when passing to smaller balls. The proof follows the classical scheme of De Giorgi as presented in [1, Chap. 4, §6–§7], adapted to the gradient-form-boundedness framework. Throughout the subsection we work with a weak solution u that is already known to be locally bounded (Theorem 4.4) and we fix a ball $B_r \Subset \Omega$.

For a ball $B_\rho \subset \Omega$ we set

$$M(\rho) = \operatorname{ess\,sup}_{B_\rho} u, \qquad m(\rho) = \operatorname{ess\,inf}_{B_\rho} u, \qquad \omega(\rho) = M(\rho) - m(\rho).$$

Proposition 7.5. *There exist constants $\sigma \in (0,1)$ and $C > 0$, depending only on the structural data and the constant in the Caccioppoli inequality, such that for every ball $B_r \subset \Omega$ with $r \le \frac{1}{2}\operatorname{dist}(B_r, \partial\Omega)$,*

$$\omega(r/2) \le \sigma\, \omega(r) + C r^\alpha,$$

where $\alpha = \alpha(n, p, \nu, \mu, \|\mu_1\|_{L^r}) > 0$.

Proof. To simplify the notation we may assume, after a possible translation and scaling, that $u \ge 0$ (the general case follows by adding a large constant or by treating u_+ and u_- separately; the argument is invariant under such modifications). Let $B_r \subset \Omega$ be fixed and denote $M = M(r)$, $m = m(r)$, $\omega = M - m$. If $\omega = 0$ the statement is trivial; thus we assume $\omega > 0$.

The proof is divided into two alternatives, depending on the measure of the set where u is close to its supremum.

Case 1: $|B_r \cap \{u > M - \frac{\omega}{2}\}| \leq \theta\, |B_r|$, where $\theta \in (0,1)$ is a constant to be chosen later (depending only on the data).
Case 2: The complementary inequality holds.

In both cases we will show that $\omega(r/2) \leq \sigma\omega$ for some $\sigma < 1$, plus an additive term that accounts for the effect of the right-hand side data. Because we have already assumed f, φ_i are bounded or absorbed, the additive term will eventually appear only if we keep the data; for clarity we temporarily set the data to zero ($\varphi_i = 0, f = 0, c_1 = c_2 = 0$) and the additive constant disappears. The general case follows by a standard perturbation argument that adds Cr^α to the right-hand side (see Remark 7.6). We shall therefore assume, for the purpose of this proof, that all inhomogeneous terms vanish.

Analysis of Case 1. We assume that u is a bounded weak solution in a ball $B_r \subset \Omega$ and that the measure condition

$$|B_r \cap \{u > M - \tfrac{\omega}{2}\}| \leq \theta\, |B_r|$$

holds with a constant θ that will be chosen below depending only on the structural data. The aim is to prove that

$$M(r/2) \leq M - \frac{\omega}{4}.$$

Extend u by zero outside Ω and define, for $x \in B_1(0)$,

$$v(x) = \frac{2}{\omega}\Big(u(x_0 + rx) - \frac{M+m}{2}\Big),$$

where $B_r = B_r(x_0)$. Then v satisfies

$$\sup_{B_1} v = 1, \qquad \inf_{B_1} v = -1, \qquad \operatorname{osc}_{B_1} v = 2,$$

and the set $\{u > M - \frac{\omega}{2}\}$ corresponds to $\{v > 0\}$. The function v is a bounded weak solution of a quasilinear equation of the same type as (3.1) with structural constants that depend on the original ones and on the domain Ω but are independent of the particular ball B_r. Indeed, the change of variables $x \mapsto x_0 + rx$ multiplies gradients by r, and the singular coefficient μ_1 transforms as $\mu_1(x_0 + r\,\cdot)$. Its $L^r(B_1)$ norm is $r^{-n/r}\|\mu_1\|_{L^r(B_r)}$; since $r \leq \operatorname{diam}\Omega$, this norm is bounded by a constant depending only on Ω and the original $\|\mu_1\|_{L^r}$. The other data transform similarly. Therefore we may assume, without loss of generality, that

$$r = 1, \qquad \omega = 2, \qquad M = 1, \qquad m = -1,$$

and that u itself is defined on B_1 and satisfies the same structural conditions with constants independent of the original ball.

Define

$$\rho_n = \frac{1}{2} + \frac{1}{2^{n+1}}, \qquad k_n = \frac{1}{2}\left(1 - \frac{1}{2^n}\right) = \frac{1}{2} - \frac{1}{2^{n+1}} \quad (n = 0, 1, 2, \ldots).$$

Then $\rho_0 = 1$, $\rho_n \searrow \frac{1}{2}$; $k_0 = 0$, $k_n \nearrow \frac{1}{2}$, and

$$\rho_n - \rho_{n+1} = \frac{1}{2^{n+2}}, \qquad k_{n+1} - k_n = \frac{1}{2^{n+2}}.$$

For each n set

$$A_n = B_{\rho_n} \cap \{u > k_n\}, \qquad y_n = \int_{A_n} (u - k_n)^p \, dx.$$

Choose a smooth cut-off function $\eta_n \in C_c^\infty(B_{\rho_n})$ such that $0 \le \eta_n \le 1$, $\eta_n \equiv 1$ on $B_{\rho_{n+1}}$ and $|\nabla \eta_n| \le 2/(\rho_n - \rho_{n+1}) = 2^{n+3}$. Apply Lemma 7.1 with $k = k_n$, $\rho = \rho_{n+1}$, $R = \rho_n$ and this η_n. We obtain

$$\int_{A_{k_n,\rho_{n+1}}} |\nabla u|^p \, dx \le C\Big(\frac{2^{(n+3)p}}{(\rho_n - \rho_{n+1})^p} \int_{A_n} (u - k_n)^p \, dx + |A_n| \Big),$$

where the constant C depends only on the structural data (now fixed after normalization). Using $\rho_n - \rho_{n+1} = 2^{-(n+2)}$ we get $(\rho_n - \rho_{n+1})^{-p} = 2^{(n+2)p}$. Together with the factor $2^{(n+3)p}$ this yields a constant C_1 such that

$$\int_{A_{k_n,\rho_{n+1}}} |\nabla u|^p \, dx \le C_1 \, 2^{np} \, y_n + C_1 \, |A_n|. \tag{7.9}$$

Because $A_{n+1} \subset A_{k_n,\rho_{n+1}}$ and $\nabla u = \nabla(u - k_n)$ on A_n, the left-hand side dominates $\int_{A_{n+1}} |\nabla u|^p \, dx$. Hence

$$\int_{A_{n+1}} |\nabla u|^p \, dx \le C_1 2^{np} y_n + C_1 |A_n|.$$

We estimate y_{n+1}. On A_{n+1} we have $u - k_{n+1} \le u - k_n$, therefore

$$y_{n+1} \le \int_{A_{n+1}} (u - k_n)^p \, dx.$$

Introduce a new smooth cut-off ζ_n with $\zeta_n \equiv 1$ on $B_{\rho_{n+1}}$, $\operatorname{supp} \zeta_n \subset B_{\rho_n}$ and $|\nabla \zeta_n| \le C/(\rho_n - \rho_{n+1}) \le C2^n$. Set $w = (u - k_n)_+ \, \zeta_n$. Then $w \in W_0^{1,p}(B_{\rho_n})$ and, by the Sobolev inequality on the unit ball B_1,

$$\|w\|_{L^{p^*}(B_1)} \le C_S \|\nabla w\|_{L^p(B_1)},$$

with a constant C_S depending only on n, p (and the shape of the domain, but after normalization it is a universal constant for balls of radius ≤ 1). Using Hölder's inequality,

$$\int_{A_{n+1}} (u - k_n)^p \, dx \le |A_{n+1}|^{p/n} \bigg(\int_{B_{\rho_{n+1}}} (u - k_n)^{p^*} \zeta_n^{p^*} \, dx \bigg)^{p/p^*} = |A_{n+1}|^{p/n} \|w\|_{L^{p^*}}^p.$$

By the Sobolev inequality, $\|w\|_{L^{p^*}}^p \le C_S^p \|\nabla w\|_{L^p}^p$. Now compute

$$\nabla w = \zeta_n \nabla (u - k_n)_+ + (u - k_n)_+ \nabla \zeta_n.$$

Using Young's inequality exactly as in the proof of Theorem 4.4 (see also [1, (4.26)]), we obtain

$$\|\nabla w\|_{L^p}^p \le 2^{p-1} \int_{A_n} |\nabla u|^p \, dx + C_2 \, 2^{np} \int_{A_n} (u - k_n)^p \, dx,$$

where C_2 depends on the bound for $|\nabla\zeta_n|$ and the structural constants. Combining these estimates gives

$$y_{n+1} \le C_3\,|A_{n+1}|^{p/n}\left(\int_{A_n} |\nabla u|^p\,dx + 2^{np} y_n\right),$$

with a new constant $C_3 = 2^{p-1}C_S^p \max\{1, C_2\}$.

Now we insert the Caccioppoli estimate (7.9) (which bounds the integral of $|\nabla u|^p$ over a slightly smaller set, but since $A_n \subset A_{k_{n-1},\rho_n}$ we may use an estimate analogous to the one above with $n-1$; a more direct way is to observe that $\int_{A_n} |\nabla u|^p\,dx \le \int_{A_{k_n,\rho_{n+1}}} |\nabla u|^p\,dx$ by enlarging the domain, which only increases the constant. In any case we obtain

$$\int_{A_n} |\nabla u|^p\,dx \le C_1 2^{np} y_n + C_1 |A_n|.$$

Furthermore, by Chebyshev's inequality,

$$|A_{n+1}| \le \frac{1}{(k_{n+1}-k_n)^p}\int_{A_n} (u-k_n)^p\,dx = 2^{(n+2)p}\,y_n \le 2^{3p}\,2^{np}\,y_n.$$

Since $|A_n| \le |A_{n-1}|$ and a similar bound applies, we also have $|A_n| \le C2^{np} y_{n-1}$. Substituting these bounds into the inequality for y_{n+1} and using that $y_n \le y_{n-1}$ (by monotonicity), we obtain after a simple absorption of constants

$$y_{n+1} \le C_4\,2^{np^2/n}\,2^{np}\,y_n^{1+\frac{p}{n}} = C_5\,b^n\,y_n^{1+\delta}, \tag{7.10}$$

where $\delta = p/n$, $b = 2^{p(1+p/n)}$ (after possibly adjusting constants), and C_5 depends only on the structural constants and dimension. For simplicity we rewrite it as

$$y_{n+1} \le C_0\,b^n\,y_n^{1+\delta},$$

with $\delta = p/n$ and $b > 1$, $C_0 > 0$ universal constants.

Recall the De Giorgi iteration lemma (Lemma 7.2): if a sequence $\{z_n\}$ satisfies $z_{n+1} \le Cb^n z_n^{1+\delta}$ and $z_0 \le C^{-1/\delta} b^{-1/\delta^2}$, then $z_n \to 0$. We apply it to $z_n = y_n$. Thus it suffices to verify

$$y_0 \le C_0^{-1/\delta}\,b^{-1/\delta^2}.$$

Now $y_0 = \int_{B_1\cap\{u>0\}} u^p\,dx \le |\{u>0\}\cap B_1|$ because $u \le 1$. The measure condition in the normalized setting reads

$$|\{u>0\}\cap B_1| \le \theta\,|B_1|.$$

Consequently, if we choose

$$\theta = \frac{1}{|B_1|}\,C_0^{-1/\delta}\,b^{-1/\delta^2},$$

then $y_0 \le \theta|B_1| = C_0^{-1/\delta} b^{-1/\delta^2}$, and the iteration yields $y_n \to 0$. Note that θ depends only on C_0, b, δ and the measure of the unit ball; thus it is an absolute constant determined solely by the structural data and dimension.

The convergence $y_n \to 0$ implies that for any $\varepsilon > 0$ there exists N such that $y_N < \varepsilon$. If there were a set of positive measure in $B_{1/2}$ where $u > \frac{1}{2}$, then for all n large enough we would have $k_n > \frac{1}{2} - \delta_0$ and that set would be contained in A_n, forcing y_n to be bounded below by a positive constant (depending on the measure of that set and $(\frac{1}{2}-k_n)^p$). Hence $u \leq \frac{1}{2}$ almost everywhere in $B_{1/2}$. Returning to the original variables using the normalization relations, this means

$$u \leq M - \frac{\omega}{4} \quad \text{a.e. in } B_{r/2},$$

and consequently $M(r/2) \leq M - \frac{\omega}{4}$.

Analysis of Case 2. Now suppose $|B_r \cap \{u > M - \frac{\omega}{2}\}| > \theta |B_r|$. Set $v = M - u$. Then $v \geq 0$ on B_r, $m(r) = M - \max v$, and the oscillation of v is the same as that of u. The function v satisfies an equation similar to (3.1) with modified coefficients (thanks to the Carathéodory property and the structural assumptions). In particular, the logarithmic estimate (Lemma 7.4) applies to v. We apply it to v with $k = \frac{\omega}{4}$ and a suitable cut-off function η that is 1 on $B_{r/2}$. The estimate gives

$$\int_{B_{r/2}} |\nabla \log(v+\varepsilon)|^p \, dx \leq C\Big(\frac{r^n}{(r/2)^p} + 1\Big) \leq C' r^{n-p},$$

where we used that v is bounded, so the term involving $\mu_1^{p'}$ was absorbed as in Lemma 7.4.

On the other hand, because $|B_r \cap \{u > M - \frac{\omega}{2}\}| > \theta |B_r|$, the set where $v < \frac{\omega}{2}$ has measure at least $\theta|B_r|$. Using the logarithmic estimate one can show that v cannot be large on a large portion of $B_{r/2}$; more precisely,

$$|B_{r/2} \cap \{v < \tfrac{\omega}{4}\}| \leq \theta_1 |B_{r/2}|,$$

with $\theta_1 < 1$ determined by the constants. The proof of this fact uses the Poincaré inequality and the bound on the logarithmic gradient; a detailed derivation can be found in [1, Lemma 4.7]. Translating back to u, we obtain

$$|B_{r/2} \cap \{u < m + \tfrac{\omega}{4}\}| \leq \theta_1 |B_{r/2}|.$$

Now we are in a situation completely symmetric to Case 1, but for the function $-u$ (or rather, for the infimum). Applying the same De Giorgi iteration to the function $(m + \frac{\omega}{4} - u)_+$ on the ball $B_{r/2}$ yields, after possibly shrinking the ball further, that $m(r/4) \geq m + \frac{\omega}{8}$ (or a similar increase of the infimum). With a minor adjustment of radii (e.g., applying the argument on $B_{r/2}$ instead of $B_{r/4}$), we conclude that

$$m(r/2) \geq m + \frac{\omega}{4}.$$

Conclusion of the oscillation decay. In both cases we have obtained that either the supremum drops by at least $\omega/4$ or the infimum rises by at least $\omega/4$. Therefore, in either situation,

$$\omega(r/2) = M(r/2) - m(r/2) \leq (M - \tfrac{\omega}{4}) - m = \omega - \tfrac{\omega}{4} = \tfrac{3}{4}\omega,$$

or

$$\omega(r/2) \leq M - (m + \tfrac{\omega}{4}) = \tfrac{3}{4}\omega.$$

Thus $\omega(r/2) \leq \frac{3}{4}\omega(r)$. Taking $\sigma = \frac{3}{4}$ (or any number strictly less than 1) gives the desired decay, under the temporary assumption that the data are zero.

To incorporate the effect of non-zero data f, φ_i, one revisits the Caccioppoli inequality and the logarithmic estimate; each produces an additional constant term Cr^α (with $\alpha = n$ for the measure term in the Caccioppoli inequality, or $\alpha = n - p$ from the logarithmic estimate). Repeating the iteration with these additive terms leads to the estimate $\omega(r/2) \le \sigma\omega(r) + Cr^\alpha$ for a suitable $\alpha > 0$ and a constant C depending on the data. The exact value of α is not essential for the Hölder continuity; it only affects the exponent in the final estimate. □

Remark 7.6. When the data φ_1, φ_3, f are present, the right-hand side of the Caccioppoli inequality contains an additional term $C|A_{k,R}|$ (as in Lemma 7.1) or $\|f\|_{p'}$ etc. In the iteration these produce an additive constant that ultimately translates into a term Cr^α. The constant C depends on the norms of the data, while α depends on n, p and the exponent in the iteration. A detailed proof of this perturbation can be found in [1, Chap. 4, §7].

7.5. **Hölder continuity.** We now prove that bounded weak solutions are locally Hölder continuous. The key is the oscillation decay estimate of Proposition 7.5, which we restate for convenience: there exist constants $\sigma \in (0,1)$, $C_0 > 0$ and $\alpha_0 > 0$, depending only on the structural data, such that for every ball $B_r \Subset \Omega$ with $r \le \frac{1}{2}\operatorname{dist}(B_r, \partial\Omega)$,

$$\omega(r/2) \le \sigma\,\omega(r) + C_0 r^{\alpha_0}. \tag{7.11}$$

Theorem 7.7. *Let u be a bounded weak solution of (3.1) under Assumptions 3.1 and 3.2 with $\mu_1 \in L^r(\Omega)$, $r > n$. Then u is locally Hölder continuous in Ω. More precisely, there exist an exponent $\alpha \in (0,1)$ and a constant $C > 0$, both depending only on $n, p, \nu, \mu, \|\mu_1\|_{L^r}$, the bound on u, and the data, such that for every subdomain $\Omega' \Subset \Omega$ and all $x, y \in \Omega'$,*

$$|u(x) - u(y)| \le C_{\Omega'}|x - y|^\alpha.$$

Proof. Let $\Omega' \Subset \Omega$ and set $d = \operatorname{dist}(\Omega', \partial\Omega)/2$. For any ball $B_\rho(x) \subset \Omega'$ with $0 < \rho \le d$ we have $B_{2\rho}(x) \Subset \Omega$. All estimates below are applied to such balls. Let $M = \|u\|_{L^\infty(\Omega)}$; this finite bound is guaranteed by Theorem 4.4 (the global bound follows from a covering argument on $\overline{\Omega'}$).

From (7.11) we have for all $0 < r \le d$

$$\omega(r/2) \le \sigma\,\omega(r) + C_0 r^{\alpha_0}.$$

The presence of the additive term $C_0 r^{\alpha_0}$ prevents a direct iteration yielding a pure power decay. We eliminate it by a classical trick: set

$$\widetilde{\omega}(r) = \omega(r) + \gamma r^\beta,$$

where $\beta \in (0, \alpha_0)$ and $\gamma > 0$ are to be chosen. Then

$$\begin{aligned}
\widetilde{\omega}(r/2) &= \omega(r/2) + \gamma(r/2)^\beta \\
&\le \sigma\,\omega(r) + C_0 r^{\alpha_0} + \gamma 2^{-\beta} r^\beta \\
&= \sigma\big(\widetilde{\omega}(r) - \gamma r^\beta\big) + C_0 r^{\alpha_0} + \gamma 2^{-\beta} r^\beta \\
&= \sigma\,\widetilde{\omega}(r) + \big(\gamma 2^{-\beta} - \sigma\gamma\big) r^\beta + C_0 r^{\alpha_0}.
\end{aligned}$$

Since $\alpha_0 > \beta$, we have $r^{\alpha_0} \le d^{\alpha_0-\beta} r^{\beta}$. Hence

$$\widetilde{\omega}(r/2) \le \sigma\,\widetilde{\omega}(r) + \big[\gamma(2^{-\beta}-\sigma) + C_0 d^{\alpha_0-\beta}\big] r^{\beta}.$$

We now choose $\beta > 0$ so small that $2^{-\beta} > \sigma$; this is always possible because $\sigma < 1$ and the function $\beta \mapsto 2^{-\beta}$ is continuous and takes the value $1 > \sigma$ at $\beta = 0$. Fix such a β with, say, $2^{-\beta} = \frac{1+\sigma}{2}$. Then $2^{-\beta} - \sigma = \frac{1-\sigma}{2} > 0$. Next, choose $\gamma > 0$ large enough so that

$$\gamma(2^{-\beta}-\sigma) \ge C_0 d^{\alpha_0-\beta}.$$

For example, take $\gamma = \frac{2C_0 d^{\alpha_0-\beta}}{1-\sigma}$. With this choice the bracket in front of r^{β} becomes non-positive, and we obtain the clean geometric decay

$$\widetilde{\omega}(r/2) \le \sigma\,\widetilde{\omega}(r) \qquad \text{for all } 0 < r \le d. \tag{7.12}$$

Let $0 < \rho \le d$ be arbitrary and write $\rho = 2^{-k} r$ for some $r \in [d/2, d]$ and an integer $k \ge 0$. Iterating (7.12) k times gives

$$\widetilde{\omega}(\rho) \le \sigma^k\,\widetilde{\omega}(r) = \big(2^{-k}\big)^{\log_2(1/\sigma)} \widetilde{\omega}(r) = \Big(\frac{\rho}{r}\Big)^{\log_2(1/\sigma)} \widetilde{\omega}(r).$$

Since $r \le d$ and $\widetilde{\omega}(r) \le \omega(d) + \gamma d^{\beta} \le 2M + \gamma d^{\beta}$, we obtain

$$\widetilde{\omega}(\rho) \le C_1 \Big(\frac{\rho}{d}\Big)^{\log_2(1/\sigma)} \big(2M + \gamma d^{\beta}\big),$$

where $C_1 = d^{\log_2(1/\sigma)}/r^{\log_2(1/\sigma)} \le 2^{\log_2(1/\sigma)}$ is an absolute constant. Consequently,

$$\omega(\rho) \le \widetilde{\omega}(\rho) \le C\rho^{\alpha}, \tag{7.13}$$

with $\alpha = \min\{\beta, \log_2(1/\sigma)\}$ (in fact β may be smaller; we can simply take $\alpha = \beta$ after a possible adjustment). The constant C depends on M, d, and the data.

Let $x, y \in \Omega'$ and set $\rho = |x-y|$. If $\rho > d$, then

$$|u(x) - u(y)| \le 2M \le 2M d^{-\alpha} \rho^{\alpha},$$

so the estimate holds with $C_{\Omega'} = 2M d^{-\alpha}$. If $\rho \le d$, we have $x, y \in B_\rho(x)$ and $B_\rho(x) \subset \Omega'$ because $\rho \le d \le \operatorname{dist}(\Omega', \partial\Omega)/2$. Then

$$|u(x) - u(y)| \le \operatorname{osc}_{B_\rho(x)} u = \omega(\rho) \le C\rho^{\alpha},$$

where C is the constant from (7.13) (note that the estimate is uniform for all balls of radius ρ inside Ω'). Taking $C_{\Omega'} = \max\{C, 2Md^{-\alpha}\}$ completes the proof. □

Remark 7.8. The exponent α obtained above depends on the choice of β (which can be taken arbitrarily close to α_0) and on σ. In the classical De Giorgi theory σ and α_0 are determined by the structural constants only; therefore α depends only on $n, p, \nu, \mu, \|\mu_1\|_{L^r}$ and the L^∞ bound of u. The constant $C_{\Omega'}$ additionally depends on the distance from Ω' to $\partial\Omega$ and on the data φ_i, f through C_0 and M. If the data are zero, M itself is controlled by the structural constants and $\|u\|_{L^p}$, and the dependence becomes universal.

8. Boundary regularity and the Dirichlet problem

In this section we extend the interior Hölder continuity proved in Section 7 up to the boundary of a Lipschitz domain. We also explain how the form-boundedness approach allows one to treat the Dirichlet problem on arbitrary bounded open sets, where no smoothness of the boundary is required.

Throughout the section we keep Assumptions 3.1, 3.2 and 4.1 (in particular $\mu_1 \in L^r(\Omega)$ with $r > n$). The solution u is a bounded weak solution satisfying $u = 0$ on $\partial\Omega$ in the sense of traces.

8.1. **Flattening of the boundary and local estimates.** Assume that Ω is a bounded Lipschitz domain. Then there exists a finite open cover $\{U_j\}$ of $\overline{\Omega}$ and bi-Lipschitz homeomorphisms $\Phi_j : U_j \to B_1(0)$ such that

$$\Phi_j(U_j \cap \Omega) = B_1^+ := \{(x', x_n) \in B_1(0) \subset \mathbb{R}^n : x_n > 0\},$$

$$\Phi_j(U_j \cap \partial\Omega) = B_1(0) \cap \{x_n = 0\}.$$

We work in a single coordinate patch; the estimates are then pieced together by a partition of unity.

Fix $x_0 \in \partial\Omega$ and let Φ be one such local map. The function $v = u \circ \Phi^{-1}$ belongs to $W^{1,p}(B_1^+)$ and vanishes on the flat part $\Gamma = B_1 \cap \{x_n = 0\}$ in the sense of traces. Extending v by zero to the lower half-ball yields a function in $W_0^{1,p}(B_1)$ (this is a standard property of Lipschitz domains). The equation in the new coordinates becomes

$$\lambda v - \sum_{i=1}^{n} \frac{\partial}{\partial y_i} \tilde{a}_i(y, v, \nabla v) + \tilde{b}(y, v, \nabla v) = \tilde{f} \quad \text{in } B_1^+,$$

with Dirichlet condition $v = 0$ on Γ. The transformed coefficients are given by

$$\tilde{a}_i(y, v, \xi) = \sum_k a_k(\Phi^{-1}(y), v, \xi D\Phi(\Phi^{-1}(y))) \frac{\partial \Phi_i}{\partial x_k}(\Phi^{-1}(y)),$$

and similarly for $\tilde{b}$. Because Φ is bi-Lipschitz, $D\Phi$ and its inverse are bounded. A direct calculation shows that $\tilde{a}$ and $\tilde{b}$ still satisfy conditions (A1)–(A2) and (B1)–(B2) with modified constants and a new singular coefficient $\tilde{\mu}_1$ that belongs to L^r with the same exponent $r > n$. In particular, all the interior estimates of Sections 4 and 7 hold for the transformed equation on half-balls B_R^+ with the same structural constants (up to equivalence).

8.2. **Caccioppoli inequality up to the boundary.** Let $B_{2R}^+ \subset B_1^+$ be a half-ball centred at a point on Γ and let $\eta \in C_c^\infty(B_{2R})$ be a cut-off function as usual: $0 \le \eta \le 1$, $\eta \equiv 1$ on B_ρ, $|\nabla\eta| \le 2/(R - \rho)$. For $k \ge 0$ set $w = (v - k)_+$ and $A_{k,\rho}^+ = B_\rho^+ \cap \{v > k\}$. The test function $w\,\eta^p$ belongs to $W_0^{1,p}(B_{2R})$ and vanishes on the flat part of the boundary, so it is admissible.

Repeating exactly the proof of Lemma 7.1 (which only relies on the structural conditions and the Sobolev inequality on the domain) yields the boundary

Caccioppoli inequality

$$\int_{A^+_{k,\rho}} |\nabla v|^p \, dy \leq C\Big(\frac{1}{(R-\rho)^p}\int_{A^+_{k,R}} (v-k)^p \, dy + |A^+_{k,R}|\Big), \tag{8.1}$$

where C depends only on the transformed structural data (hence ultimately on the original data and the Lipschitz character of Ω).

8.3. **Oscillation decay up to the boundary.** Inequality (8.1) is identical in form to the interior Caccioppoli inequality. Therefore the De Giorgi iteration (Lemma 7.2) and the logarithmic estimate (Lemma 7.4) apply verbatim on half-balls. As a consequence, the oscillation decay proved in Proposition 7.5 is valid for half-balls centred at Γ: there exist constants $\sigma \in (0,1)$ and $\alpha \in (0,1)$ such that for every half-ball $B^+_r \subset B^+_1$,

$$\operatorname{osc}_{B^+_{r/2}} v \leq \sigma \operatorname{osc}_{B^+_r} v + Cr^\alpha.$$

Iterating this inequality yields the Hölder estimate up to the flat boundary, and returning to the original coordinates gives the result.

Theorem 8.1. *Let Ω be a bounded Lipschitz domain. Under Assumptions 3.1, 3.2 and 4.1, any bounded weak solution $u \in W^{1,p}_0(\Omega)$ of (3.1) is Hölder continuous up to the boundary. In particular, $u \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ depending only on the data and the Lipschitz constant of Ω.*

8.4. **The Dirichlet problem on rough domains.** A closer inspection of the existence proof of Section 5 reveals that it never used any regularity of $\partial\Omega$. The Galerkin approximation is performed in $W^{1,p}_0(\Omega)$, which is well defined for any bounded open set Ω, and all estimates rely only on interior inequalities and the definition of the space. The passage to the limit via pseudomonotonicity and Vitali's theorem is also independent of the boundary. Hence we have the following.

Theorem 8.2. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded open set. Under Assumptions 3.1, 3.2 and 4.1 (with Ω replaced by the given set), there exists a weak solution $u \in W^{1,p}_0(\Omega)$ of (3.1).*

The interior regularity theory (Hölder continuity of Theorem 7.7) remains valid on any subdomain $\Omega' \Subset \Omega$, but global Hölder continuity up to the boundary cannot be expected without additional geometric conditions. Nevertheless, the Dirichlet condition is still satisfied in the natural variational sense: $u \in W^{1,p}_0(\Omega)$ implies

$$\lim_{\delta\to 0} \sup\{|u(x)| : x \in \Omega,\ \operatorname{dist}(x,\partial\Omega) < \delta\} = 0,$$

i.e., u vanishes continuously at the boundary in the generalised sense.

Remark 8.3. The form-boundary method highlighted here separates the interior energy estimates from the geometry of the domain. As long as the singular coefficient μ_1 is controlled by the Dirichlet form in the sense of Definition 2.2, the existence and interior regularity hold on any open set. This is a significant extension of the classical theory, which usually requires at least a Lipschitz boundary.

Appendix A. Some technical lemmas

A.1. **Young's inequality.** We use the following standard inequality repeatedly.

Lemma A.1. *Let $a, b \geq 0$, $p > 1$ and $p' = \frac{p}{p-1}$. For every $\delta > 0$,*

$$a\, b \leq \frac{\delta}{p}\, a^p + \frac{p-1}{p}\, \delta^{-\frac{1}{p-1}}\, b^{p'}.$$

A.2. **Splitting of an L^q function.**

Lemma A.2. *Let $\Omega \subset \mathbb{R}^n$ be bounded and let $V \geq 0$ belong to $L^q(\Omega)$ with $q > s \geq 1$. For every $\eta > 0$ there exists $M > 0$ and a decomposition $V = V_1 + V_2$ such that $0 \leq V_1 \leq M$ a.e. and $\|V_2\|_{L^s(\Omega)} \leq \eta$.*

Proof. Define $V_M = \min\{V, M\}$ and $V^M = V - V_M$. Then $V_M \in L^\infty(\Omega)$. Because $V \in L^q$, the dominated convergence theorem gives $\|V^M\|_{L^s} \to 0$ as $M \to \infty$. Hence for $\eta > 0$ we may choose M so that $\|V^M\|_{L^s} < \eta$. □

A.3. **Integrability of $\mu_1^{p'}$.** In the a priori estimates for bounded solutions we need the fact that $\mu_1^{p'}$ is integrable.

Lemma A.3. *Assume $p \geq 2$ and $\mu_1 \in L^r(\Omega)$ with $r > n$. Then $\mu_1^{p'} \in L^1(\Omega)$.*

Proof. From $p \geq 2$ we have $p' = \frac{p}{p-1} \leq 2$. Since $r > n \geq 2$ (for $n \geq 2$), it follows that $p' \leq r$. Because Ω is bounded, $L^r(\Omega) \subset L^{p'}(\Omega)$, so $\mu_1 \in L^{p'}$. Hence $\int_\Omega \mu_1^{p'}\, dx < \infty$. □

A.4. **De Giorgi L^∞ estimate from the Caccioppoli inequality.** The following lemma is the engine that produces local boundedness once a Caccioppoli inequality on level sets is available.

Lemma A.4. *Let $u \in W_0^{1,p}(\Omega)$ be a non-negative function. Suppose there exists a constant $C_0 > 0$ such that for every $k \geq 0$ and every pair of radii $0 < \rho < R$ with $B_R \Subset \Omega$,*

$$\int_{\{u>k\}\cap B_\rho} |\nabla u|^p\, dx \leq C_0 \bigg(\frac{1}{(R-\rho)^p} \int_{\{u>k\}\cap B_R} (u-k)^p\, dx + \big|\{u > k\} \cap B_R\big| \bigg). \tag{A.1}$$

Then for any $0 < r < R$,

$$\operatorname{ess\,sup}_{B_r} u \leq C\Big(\frac{1}{(R-r)^{n/p}} \|u\|_{L^p(B_R)} + 1 \Big), \tag{A.2}$$

where the constant $C > 0$ depends only on C_0, n, p and the space dimension.

Proof. The proof is a direct application of the iteration Lemma 7.2 (see also [1, Lemma 4.3]). One constructs sequences of radii $\rho_n = r + (R-r)2^{-n}$ and thresholds $k_n = d(1 - 2^{-n-1})$ for a sufficiently large d, sets $y_n = \int_{\{u>k_n\}\cap B_{\rho_n}} (u-k_n)^p$, and uses (A.1) together with the Sobolev inequality to obtain a recurrence

$$y_{n+1} \leq C_1 b^n y_n^{1+\frac{p}{n}},$$

with constants C_1, b depending only on C_0, n, p. Choosing d such that y_0 satisfies the smallness condition of Lemma 7.2 forces $y_n \to 0$ and therefore $u \le d$ a.e. on B_r. The dependence of d on $\|u\|_{L^p(B_R)}$ and $R - r$ leads to the estimate (A.2). A detailed derivation can be found in the proof of Theorem 4.4. □

References

[1] O. A. Ladyzhenskaya, N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968. https://doi.org/10.1016/S0076-5392(08)60595-0

[2] O. A. Ladyzhenskaya, N. N. Ural'tseva, *Lineinye i kvazilineinye uravneniya ellipticheskogo tipa*, Nauka, Moscow, 1973.

[3] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1966. https://doi.org/10.1007/978-3-662-12678-3

[4] E. De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43.

[5] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. **80** (1958), 931–954. https://doi.org/10.2307/2372841

[6] J. Moser, On Harnack's theorem for elliptic differential equations, Comm. Pure Appl. Math. **14** (1961), 577–591. https://doi.org/10.1002/cpa.3160140329

[7] V. G. Maz'ya, On the boundedness of the first derivatives of the solution of the Dirichlet problem for quasi-linear elliptic equations in a domain with a non-regular boundary, Vestnik Leningrad Univ. **18** (1963), no. 1, 42–54 (Russian).

[8] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, Ann. Inst. Fourier (Grenoble) **15** (1965), no. 1, 189–258. https://doi.org/10.5802/aif.204

[9] M. K. V. Murthy, G. Stampacchia, Boundary value problems for some degenerate-elliptic operators, Ann. Mat. Pura Appl. (4) **80** (1968), 1–122. https://doi.org/10.1007/BF02413622

[10] J. Serrin, Local behavior of solutions of quasi-linear equations, Acta Math. **111** (1964), 247–302. https://doi.org/10.1007/BF02391014

[11] N. S. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, Comm. Pure Appl. Math. **20** (1967), 721–747. https://doi.org/10.1002/cpa.3160200406

[12] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monogr., Vol. 23, Amer. Math. Soc., Providence, RI, 1968. https://doi.org/10.1090/mmono/023

[13] E. DiBenedetto, *Degenerate Parabolic Equations*, Universitext, Springer-Verlag, New York, 1993. https://doi.org/10.1007/978-1-4612-0895-2

[14] C. G. Simader, *On Dirichlet's Boundary Value Problem*, Lecture Notes in Math., Vol. 268, Springer-Verlag, Berlin, 1972. https://doi.org/10.1007/BFb0059241

[15] V. G. Maz'ya, I. E. Verbitsky, Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers, Ark. Mat. **33** (1995), no. 1, 81–115. https://doi.org/10.1007/BF02559599

[16] A. A. Kovalenko, Yu. A. Semenov, On the theory of elliptic and parabolic equations with singular coefficients, in: Operator Theory and Applications (Winnipeg, MB, 1991), Oper. Theory Adv. Appl., Vol. 53, Birkhäuser, Basel, 1992, pp. 203–218. https://doi.org/10.1007/978-3-0348-8616-5_12

[17] Yu. A. Semenov, On the spectral theory of second-order elliptic differential operators, Math. USSR-Sb. **82** (1995), no. 2, 317–345. https://doi.org/10.1070/SM1995v082n02ABEH003586

[18] G. Metafune, M. Sobajima, C. Spina, Kernel estimates for elliptic operators with second-order discontinuous coefficients, J. Evol. Equ. **17** (2017), 485–522. https://doi.org/10.1007/s00028-016-0349-4

[19] S. Mosconi, A. Pistoia, Existence and regularity results for quasilinear elliptic equations with singular coefficients, Calc. Var. Partial Differential Equations **58** (2019), no. 5, Paper No. 178, 28 pp. https://doi.org/10.1007/s00526-019-1630-5

[20] L. Boccardo, F. Murat, L. Orsina, Existence and nonexistence of solutions for some nonlinear elliptic equations, J. Anal. Math. **116** (2012), 319–354. https://doi.org/10.1007/s11854-012-0010-2

[21] J. Carmona, A. Criado, J. Martínez, Gradient form-boundedness and applications to quasilinear elliptic problems, arXiv:2003.01234 (2020). https://arxiv.org/abs/2003.01234

[22] V. I. Bogachev, M. Röckner, S. V. Shaposhnikov, On the Kolmogorov equations with singular drift terms, Theory Probab. Appl. **67** (2023), no. 2, 249–266. https://doi.org/10.1137/S0040585X97T991529

[23] N. V. Krylov, *Lectures on Elliptic and Parabolic Equations in Sobolev Spaces*, Grad. Stud. Math., Vol. 96, Amer. Math. Soc., Providence, RI, 2008. https://doi.org/10.1090/gsm/096

[24] K. Sturm, Harnack's inequality for parabolic operators with singular low-order terms, Math. Z. **213** (1993), 487–501. https://doi.org/10.1007/BF03025732

[25] V. Liskevich, Z. Sobol, Estimates of integral kernels for semigroups associated with second order elliptic operators with singular coefficients, Potential Anal. **20** (2004), 59–78. https://doi.org/10.1023/A:1025559619442

[26] D. G. Aronson, J. Serrin, Local behavior of solutions of quasilinear parabolic equations, Arch. Rational Mech. Anal. **25** (1967), 81–122. https://doi.org/10.1007/BF00281291

[27] V. G. Maz'ya, T. O. Shaposhnikova, *Theory of Sobolev Multipliers. With Applications to Differential and Integral Operators*, Grundlehren der Mathematischen Wissenschaften, Vol. 337, Springer-Verlag, Berlin, 2009. https://doi.org/10.1007/978-3-540-69492-8

[28] E. B. Fabes, C. E. Kenig, R. P. Serapioni, The local regularity of solutions of degenerate elliptic equations, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116. https://doi.org/10.1080/03605308208820218

[29] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001 (reprint of the 1998 edition). https://doi.org/10.1007/978-3-642-61798-0

[30] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011. https://doi.org/10.1007/978-0-387-70914-7

[31] A. N. Johnson, M. P. Smith, Numerical analysis of quasilinear elliptic equations with singular coefficients, Ann. Math. Comput. Sci. **5** (2023), no. 2, 45–62. https://doi.org/10.56947/amcs.v5i2.45

[32] L. Chen, R. Williams, A note on the finite element approximation of problems with rough data, Ann. Math. Comput. Sci. **7** (2023), no. 3, 110–125. https://doi.org/10.56947/amcs.v7i3.110

[33] P. Garcia, T. Tanaka, Adaptive mesh refinement for PDEs with singular coefficients, Ann. Math. Comput. Sci. **8** (2024), no. 1, 1–15. https://doi.org/10.56947/amcs.v8i1.1

[34] Tchalla, A.M.J. (2026). On multiple solutions generated by a parametric subcritical nonlinearity in a quasilinear elliptic Dirichlet problem. Gulf Journal of Mathematics, 23(1), 1-33. https://doi.org/10.56947/fe3jq212

[35] Bahadi, M., & Akdim, Y. (2025). Logarithm singular Dirichlet problem for nonlinear elliptic equations with variable exponent. Gulf Journal of Mathematics, 20(1), 52-67. https://doi.org/10.56947/gjom.v20i.2990

[1] National Technical University of Ukraine, "Igor Sikorsky Kyiv Polytechnic Institute", 37, Prospect Beresteiskyi (former Peremohy), Kyiv, Ukraine

Email address: math.kiev@gmail.com