Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 35 (2026) 1-21 https://doi.org/10.56947/amcs.v35.873

3D NAVIER-STOKES EQUATION WITH MULTIVALUED FRICTION

SIMON ZONGO[1] AND BILA ADOLPHE KYELEM[2,*]

ABSTRACT. We prove the existence of weak solutions and their continuous dependence on the data for a three-dimensional incompressible Navier-Stokes system subject to multivalued friction boundary conditions, within a Hilbert space framework. Under suitable assumptions, a fixed point argument yields the existence of a solution to the associated perturbed stationary problem. Using nonlinear semigroup theory, we then establish the existence of a mild solution to the non-stationary problem and prove a contraction principle. Finally, by combining the Crandall-Liggett implicit discretization scheme with a priori estimates for the approximating mild solutions, we obtain weak solutions that depend continuously on the initial data.

Keywords. Navier-Stokes equations, multivalued friction, mild solution, weak solution, nonlinear semigroups.
2020 Mathematics Subject Classification. Primary 35Q30, 76D05; Secondary 47H20, 35D30, 47H04.

1. INTRODUCTION

In this paper, we study the existence of weak solutions and their continuous dependence on the initial data for the following three-dimensional Navier-Stokes system with multivalued friction boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, & \text{in } \Omega\times(0,T),\\ \operatorname{div}(u) = 0, & \text{in } \Omega\times(0,T),\\ u = 0, & \text{on } \Gamma_D\times(0,T),\\ u_n = 0, \quad \sigma_\tau \in h(\cdot, u_\tau), & \text{on } \Gamma_C\times(0,T),\\ u(0,x) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with sufficiently smooth boundary, and $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_C}$ is a disjoint partition. The number $T > 0$ is a fixed final time, $u : \Omega\times[0,T] \to \mathbb{R}^3$ is the velocity field, $p : \Omega\times[0,T] \to \mathbb{R}$ is the fluid pressure, and ν is the kinematic viscosity coefficient. The initial datum is $u_0 \in L^2(\Omega)^3$,

Date: Received: Jun 3, 2026; Revised: Jun 30, 2026; Accepted: Jul 8, 2026.
* Corresponding author

and $f \in L^2(0, T; L^2(\Omega)^3)$ is the external force. Moreover, $u_n = u \cdot n$ denotes the normal component of u on $\partial\Omega$, whereas $u_\tau = u - (u \cdot n)n$ denotes its tangential component. Finally, $h : \Gamma_C \times \mathbb{R}^3 \to 2^{\mathbb{R}^3}$ is a multivalued operator modelling the friction boundary condition on Γ_C.

The Navier-Stokes equations describe the evolution of the velocity and pressure of a viscous fluid. They were first introduced by Navier in 1821 (see [20]) as an extension of equations formulated for the equilibrium and motion of elastic solids. Subsequent contributions were made by Cauchy in 1828, Poisson in 1829, and Saint-Venant in 1843. In 1845, in connection with his work on pendulum oscillations, Stokes derived the equations in the form now commonly associated with his name. The Navier-Stokes equations may be viewed as viscous extensions of the Euler equations.

The Navier-Stokes equations are a cornerstone of fluid mechanics and are essential for describing and understanding numerous three-dimensional physical and industrial phenomena. They are used, for example, to model atmospheric dynamics, ocean currents, water flow through pipes, and airflow around wings. In both their full and simplified forms, these equations also arise in the design of trains, aircraft, and auto-mobiles, in the study of blood flow, in power-plant engineering, and in pollution analysis. Consequently, they have attracted considerable attention in the literature; see, for instance, [4, 5, 6, 8, 11, 12, 17, 18, 26, 27].

Major topics in the theory of the Navier-Stokes equations include the existence and uniqueness of solutions, global resolvability, smoothness and regularity, and the stability of solutions.

Although the existence of weak solutions and several regularity results for the non-stationary Navier-Stokes equations have been established in classical works such as [6, 10, 15, 23, 28, 30], important questions concerning smoothness and stability remain unresolved.

Subsequent studies have addressed non-homogeneous boundary conditions and friction laws proportional to the velocity, establishing existence and stability under strong regularity assumptions. Further results on non-homogeneous and friction boundary conditions may be found in [9, 12, 22, 24, 25, 27, 29]. Closely related multivalued boundary value problems, in which the boundary condition is expressed through a maximal monotone graph, have also been investigated for other classes of nonlinear elliptic and parabolic equations; see, e.g., [21]. By contrast, the three-dimensional case involving multivalued friction on only part of the boundary, together with stability and regularity properties of weak solutions, remains largely open.

The aim of this work is to prove the existence of weak solutions to problem (1.1) and their continuous dependence on the initial data by using techniques from nonlinear semigroup theory; see [1, 3, 7]. Continuous dependence on the initial data is a fundamental component of well-posedness.

To apply nonlinear semigroup theory and to obtain weak solutions of (1.1) that depend continuously on the initial data, we begin with the following perturbed stationary problem:

$$\begin{cases} \beta(u) - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, & \text{in } \Omega, \\ \operatorname{div}(u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D, \\ u_n = 0, \quad \sigma_\tau \in h(\cdot, u_\tau), & \text{on } \Gamma_C, \end{cases} \tag{1.2}$$

where the perturbation operator $\beta : \mathbb{R}^3 \to \mathbb{R}^3$ is a monotone graph that will later be taken to be the identity. A solution of the stationary problem (1.2) will be used to establish the existence of a mild solution to the non-stationary problem (1.1).

The rest of the paper is organized as follows. Section 2 introduces the functional framework, the auxiliary results, and the assumptions used to study the non-stationary problem (1.1) through the stationary problem (1.2). Section 3 is devoted to the existence of weak solutions to the stationary problem (1.2). Section 4 establishes the existence of mild solutions to (1.1) by means of nonlinear semigroup theory. In the final section, we prove the existence of weak solutions and their continuous dependence on the data.

2. Preliminaries

In this section, we introduce the functional framework and recall the auxiliary results used in the analysis. We also state the assumptions imposed throughout the paper.

Motivated by the physical setting, we work in the following Hilbert spaces:

$$\mathcal{H} = \{v \in L^2(\Omega)^3 : \operatorname{div} v = 0,\ v|_{\Gamma_D} = 0,\ v\cdot n|_{\Gamma_C} = 0\}$$

and

$$\mathcal{V} = \{v \in H^1(\Omega)^3 : \operatorname{div} v = 0,\ v|_{\Gamma_D} = 0,\ v\cdot n|_{\Gamma_C} = 0\},$$

with continuous and dense embeddings $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$, where $\mathcal{V}'$ denotes the dual of $\mathcal{V}$. We equip $\mathcal{H}$ with the $L^2(\Omega)^3$ norm and $\mathcal{V}$ with the $H^1(\Omega)^3$ norm. We also set

$$L^2_0(\Omega) = \left\{p \in L^2(\Omega) : \int_\Omega p\,dx = 0\right\}.$$

Following [13], let $b : \mathbb{R} \to \mathbb{R}$ be a continuous, nondecreasing function of bounded variation such that $b(0) = 0$, and define

$$B(z) = \int_0^z s\,db(s). \tag{2.1}$$

Functions of this type will be used extensively in the final section. We therefore introduce a common notation and record some of their basic properties. For any monotone function $m : \mathbb{R} \to \mathbb{R}$ with $m(0) = 0$ and any $w \in \mathbb{R}$, define $B_{m,w}$ by

$$B_{m,w}(z) = \int_w^z m(s-w)\,db(s).$$

Observe that, in (2.1), $B = B_{I_d,0}$, where $I_d : \mathbb{R} \to \mathbb{R}$ is the identity function. Moreover,

$$B_{m,w}(\hat{z}) - B_{m,w}(z) \geq m(z-w)(b(\hat{z}) - b(z)) \quad \text{for all } z, \hat{z} \in \mathbb{R}. \tag{2.2}$$

To establish the existence of weak solutions of the stationary problem (1.2), we need the two following theorems, the first of which states Kakutani-Fan-Glicksberg's fixed point theorem and the second claims Lusin's theorem.

Theorem 2.1. *Let $S \subset X$ be a nonempty, compact, convex subset of a Hausdorff locally convex topological vector space X. Let $\varphi : S \to 2^S$ be a multivalued map with closed graph and nonempty convex values. Then, the set of fixed points*

$$\Big\{x \in S : x \in \varphi(x)\Big\}$$

is nonempty and compact.

Theorem 2.2. *Let $O \subset \mathbb{R}^d$ be compact and let $f : O \to \mathbb{C}$. Then, the following assertions are equivalent:*

(1) *f is measurable;*
(2) *For every $\varepsilon > 0$, there exists a closed set $F \subset O$ with $|O \setminus F| < \varepsilon$ such that $f_{|F}$ is continuous; that is,*

$$\forall x_k, x \in F, \quad x_k \to x \in F \implies f(x_k) \to f(x).$$

We now define the notions of solutions used in this paper. The space $\mathcal{V}$ consists of the test functions compatible with the boundary conditions. We also set

$$E = \{\phi \in L^2(0,T;\mathcal{V}) \cap L^\infty(Q)^3 |\ \phi_t \in \mathcal{V}' + L^2(Q)^3\}$$

where $Q = (0,T) \times \Omega$. We then introduce the following definitions.

Definition 2.3. Let $\Omega \subset \mathbb{R}^3$ be a bounded open domain with Lipschitz boundary, and let $T > 0$. A pair (u,p) is called a weak solution to problem (1.1) if

$$u \in L^2(0,T;\mathcal{V}) \cap L^\infty(Q)^3, \quad p \in L^2(0,T;L^2_0(\Omega))$$

and if u satisfies, for every $\varphi \in E \cap L^\infty(Q)$,

$$\begin{aligned} &-\int_0^T \int_\Omega u \cdot \varphi_t \, dx\, dt + \int_\Omega u_0 \cdot \varphi(0)\, dx + \nu \int_0^T \int_\Omega \nabla u : \nabla \varphi \, dx\, dt \\ &\quad + \int_0^T \int_\Omega (u\cdot\nabla)u \cdot \varphi \, dx\, dt + \int_0^T \int_{\Gamma_C} \xi \cdot \varphi_\tau \, dS\, dt = \int_0^T \int_\Omega f \cdot \varphi \, dx\, dt \end{aligned} \tag{2.3}$$

where $\xi(x,t) \in h(x, u_\tau(x,t))$ for *a.e.*, $(x,t) \in \Gamma_C \times (0,T)$.

Definition 2.4. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary, and let $T > 0$. A pair (u,p) is called a weak solution to problem (1.2) if

$$u \in \mathcal{V}, \quad p \in L^2_0(\Omega)$$

and if, for every $v \in \mathcal{V}$,

$$\int_\Omega \beta(u)\cdot v\, dx + \nu \int_\Omega \nabla u \nabla v\, dx + \int_\Omega (u\cdot\nabla)u\cdot v\, dx + \int_{\Gamma_C} \xi\cdot v_\tau\, dS = \int_\Omega f\cdot v\, dx, \tag{2.4}$$

where $\xi(x) \in h(x, u_\tau(x))$ for *a.e.*, $x \in \Gamma_C$.

The existence results are established under the following assumptions on the friction multifunction h.

- (h_1) for every $s \in \mathbb{R}^3$, the multifunction $x \mapsto h(x,s)$ admits a measurable selection; that is, there exists a measurable function $g_s : \Gamma_C \to \mathbb{R}^3$ such that $g_s(x) \in h(x,s)$ for all $x \in \Gamma_C$;
- (h_2) the set $h(x,s)$ is nonempty, compact, and convex in $\mathbb{R}^3$ for every $s \in \mathbb{R}^3$ and for $a.e.$, $x \in \Gamma_C$;
- (h_3) the graph of $s \mapsto h(x,s)$ is closed in $\mathbb{R}^3$ for $a.e.$, $x \in \Gamma_C$;
- (h_4) for every $s \in \mathbb{R}^3$, for $a.e.$, $x \in \Gamma_C$, and for every $\xi \in h(x,s)$, there exist $c_1 \in L^2(\Gamma_C)$ and $c_2 \geq 0$ such that $|\xi| \leq c_1(x) + c_2|s|$;
- (h_5) the constant c_2 is chosen so that $c_2\|\gamma\|^2 < c_3$, where $\gamma : \mathcal{V} \to W$ is the trace operator on $\mathcal{V}$, with norm $\|\gamma\| = \|\gamma\|_{\mathcal{L}(\mathcal{V},W)}$, $W = (L^2(\Gamma_C))^3$, and c_3 is the positive constant
$$c_3 = \min\{\nu, 1\}.$$

As an example of a friction multifunction, consider the Coulomb law
$$h(x,s) = \begin{cases} \{\sigma \frac{s}{|s|} : |\sigma| \leq \mu|p_n(x)|\}, & s \neq 0, \\ \{\xi \in \mathbb{R}^3 : |\xi| \leq \mu|p_n(x)|\}, & s = 0, \end{cases}$$
where μ is the friction coefficient and $p_n(x)$ is the normal pressure. This friction law satisfies assumptions (h_1)-(h_5).
We now set $W := (L^2(\Gamma_C))^3$ and define the associated global multivalued operator $\mathsf{H} : W \to 2^W$ by
$$\mathsf{H}(u) := \{\xi \in W : \xi(x) \in h(x, u(x)) \text{ for } a.e., \, x \in \Gamma_C\}.$$
We also recall the following lemma, which will be used below.

Lemma 2.5 ([19]). *Assume that the multifunction h satisfies (h_1)-(h_5). Then, H has the following properties:*
(i) $\mathsf{H}(u)$ is nonempty, convex, and closed for every $u \in W$;
(ii) the graph of H is closed in $(W, strong) \times (W, weak)$;
(iii) $\|\xi\|_W \leq \|c_1\|_{L^2(\Gamma_C)} + c_2\|\gamma\|_{\mathcal{L}(\mathcal{V},W)}\|u\|_{\mathcal{V}}$ for every $\xi \in \mathsf{H}(u)$.

We next define the operators associated with the terms in equation (2.4). The linear operator $A : \mathcal{V} \to \mathcal{V}'$ is defined by
$$\langle Au, v\rangle := \int_\Omega \nabla u \nabla v \, dx, \quad \forall u, v \in \mathcal{V}.$$
The bilinear operator $\mathsf{B} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}'$ is defined by
$$\langle \mathsf{B}(u,w), v\rangle := \int_\Omega (u \cdot \nabla)w \cdot v \, dx, \quad \forall u, v, w \in \mathcal{V}.$$
The elements F and $\beta(u)$ of $\mathcal{V}'$ are defined by
$$\langle F, v\rangle := \int_\Omega f \cdot v \, dx \quad \text{and} \quad \langle \beta(u), v\rangle := \int_\Omega \beta(u) \cdot v \, dx.$$
We thus obtain the following weak formulation of problem (1.2):
$$\langle \beta(u) + \nu Au + \mathsf{B}(u,u), v\rangle + (\xi, v_\tau)_W = \langle F, v\rangle, \quad \forall v \in \mathcal{V}, \quad \xi \in \mathsf{H}(u_\tau), \tag{2.5}$$

where

$$(\xi, v_\tau)_W := \int_{\Gamma_C} \xi \cdot v_\tau \, dS.$$

We recall from [19] the following three lemmas, which will be used to establish the existence of weak solutions.

Lemma 2.6 ([19]). *The operator A is linear, continuous, and coercive:*

$$\langle Au, u\rangle = \|\nabla u\|^2_{L^2(\Omega)^{3\times 3}}, \quad \forall u \in \mathcal{V}.$$

Lemma 2.7 ([19]). *For every $u, v, w \in \mathcal{V}$,*

$$\langle \mathsf{B}(u, w), v\rangle \leq c_{\mathsf{B}} \|\nabla u\|_{L^2(\Omega)^{3\times 3}} \|\nabla w\|_{L^2(\Omega)^{3\times 3}} \|\nabla v\|_{L^2(\Omega)^{3\times 3}}$$

and

$$\langle \mathsf{B}(u, v), v\rangle = 0, \quad \langle \mathsf{B}(u, w), v\rangle = -\langle \mathsf{B}(u, v), w\rangle.$$

Lemma 2.8 ([19]). *The operator B is sequentially weakly continuous; that is, if $u_k \rightharpoonup u$ and $w_k \rightharpoonup w$ weakly in $\mathcal{V}$, then*

$$\langle \mathsf{B}(w_k, u_k), v\rangle \to \langle \mathsf{B}(w, u), v\rangle, \quad \forall v \in \mathcal{V}.$$

Hence, for every $f \in L^2(\Omega)^3$, a function $u \in \mathcal{V}$ is called a weak solution to the perturbed stationary Navier-Stokes problem with friction (1.2) if

$$\langle \beta(u) + \nu Au + \mathsf{B}(u, u), v\rangle + (\xi, v_\tau)_W = \langle F, v\rangle, \quad \forall v \in \mathcal{V}, \quad \xi \in \mathsf{H}(u_\tau). \tag{2.6}$$

The existence of weak solutions to problem (1.2) is studied in the next section.

3. Existence of weak solutions to the perturbed stationary problem

Throughout this section, we assume that (h_1)-(h_5) hold. We begin by defining the following subset $S \subset W \times \mathcal{V}$ in which a solution will be sought:

$$S = \left\{ (\xi, w) \in W \times \mathcal{V} \;\middle|\; \begin{cases} \|\xi\|_W \leq \frac{c_3\|c_1\|_{L^2(\Gamma_C)} + c_2\|\gamma\|\|F\|_{\mathcal{V}'}}{c_3 - c_2\|\gamma\|^2} \\ \\ \|w\| \leq \frac{\|\gamma\|\|c_1\|_{L^2(\Gamma_C)} + \|F\|_{\mathcal{V}'}}{c_3 - c_2\|\gamma\|^2} \end{cases} \right\}. \tag{3.1}$$

To prove the existence of weak solutions to (1.2), we establish several auxiliary lemmas that provide the decomposition used throughout this section.
Consider the following auxiliary problem.
For fixed $(\xi, w) \in W \times \mathcal{V}$, find $u \in \mathcal{V}$ such that, for every $v \in \mathcal{V}$,

$$\langle \beta(u) + \nu Au + \mathsf{B}(w, u), v\rangle + (\xi, v_\tau)_W = \langle F, v\rangle. \tag{3.2}$$

Lemma 3.1. *Assume that $\beta = I$, where I is the identity operator. Then, for every $(\xi, w) \in W \times \mathcal{V}$, problem (3.2) admits a unique solution $u \in \mathcal{V}$.*

Proof. Define $\mathsf{M} : \mathcal{V} \to \mathcal{V}'$ by

$$\mathsf{M}(u) := \beta(u) + \nu Au + \mathsf{B}(w, u).$$

The mapping M is linear. Moreover, since $\beta : \mathbb{R}^3 \to \mathbb{R}^3$ is monotone, Lemma 2.6 and the identity $\langle \mathsf{B}(w,u),u\rangle = 0$ give

$$\begin{aligned}\langle \mathsf{M}(u),u\rangle = \langle \beta(u),u\rangle + \nu\langle Au,u\rangle + \langle \mathsf{B}(w,u),u\rangle &\geq \nu\|\nabla u\|_{L^2(\Omega)^{3\times 3}} + \|u\|^2_{L^2(\Omega)^3}\\ &\geq c_3\|u\|^2_{H^1(\Omega)^3} = c_3\|u\|^2_{\mathcal{V}}.\end{aligned}$$

Furthermore, for all $u, v \in \mathcal{V}$,

$$\begin{aligned}&\langle \mathsf{M}(u),v\rangle = \langle \beta(u),v\rangle + \nu\langle Au,v\rangle + \langle \mathsf{B}(w,u),v\rangle \leq \|u\|_{L^2(\Omega)^3}\|v\|_{L^2(\Omega)^3}\\ +\ & \nu\|\nabla u\|_{L^2(\Omega)^{3\times 3}}\|v\|_{L^2(\Omega)^3} + c_{\mathsf{B}}\|\nabla w\|_{L^2(\Omega)^{3\times 3}}\|\nabla u\|_{L^2(\Omega)^{3\times 3}}\|\nabla v\|_{L^2(\Omega)^{3\times 3}}\\ \leq\ & C_w\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}\end{aligned}$$

where $C_w > 0$ depends on $\|\nabla w\|_{L^2(\Omega)^{3\times 3}}$.
Define $\ell \in \mathcal{V}'$ by

$$\ell(v) = \langle F, v\rangle - (\xi, v_\tau)_W.$$

Then,

$$\begin{aligned}|\ell(v)| &\leq (\|F\|_{\mathcal{V}'} + \|\xi\|_W\|\gamma\|)\,\|v\|_{L^2(\Omega)^3}\\ &\leq (\|F\|_{\mathcal{V}'} + \|\xi\|_W\|\gamma\|)\|v\|_{\mathcal{V}}.\end{aligned}$$

By the Lax-Milgram theorem (see [16]), problem (3.2) admits a unique solution $u \in \mathcal{V}$. □

We now define the mapping

$$\Lambda_1 : W \times \mathcal{V} \to \mathcal{V}, \quad (\xi, w) \mapsto u,$$

where u is the unique solution to (3.2). The following estimate holds.

Lemma 3.2. *Assume that $\beta = I$. Then, for every $(\xi, w) \in W \times \mathcal{V}$, the solution $u = \Lambda_1(\xi, w)$ satisfies*

$$\|\Lambda_1(\xi,w)\|_{\mathcal{V}} \leq \frac{\|\gamma\|}{c_3}\,\|\xi\|_W + \frac{\|F\|_{\mathcal{V}'}}{c_3}.$$

Proof. We have

$$\langle \beta(\Lambda_1(\xi,w)) + \nu A\Lambda_1(\xi,w) + \mathsf{B}(w,\Lambda_1(\xi,w)), v\rangle + (\xi, v_\tau)_W = \langle F, v\rangle,\ \forall v \in \mathcal{V}. \quad (3.3)$$

Taking $v = u = \Lambda_1(\xi,w)$ in (3.3), we obtain

$$\langle \mathsf{M}(u),u\rangle = \langle F,u\rangle - (\xi,u_\tau)_W \geq c_3\|u\|^2_{\mathcal{V}}.$$

Since

$$\langle F,u\rangle - (\xi,u_\tau)_W \leq (\|F\|_{\mathcal{V}'} + \|\xi\|_W\|\gamma\|)\|v\|_{\mathcal{V}}$$

we obtain

$$c_3\|u\|^2_{\mathcal{V}} \leq (\|F\|_{\mathcal{V}'} + \|\xi\|_W\|\gamma\|)\|u\|_{\mathcal{V}}. \quad (3.4)$$

This yields the desired estimate. □

Lemma 3.3. *Assume that $\beta = I$. Let $\xi_k \rightharpoonup \xi$ weakly in W, let $w_k \rightharpoonup w$ weakly in $\mathcal{V}$, and assume that $\Lambda_1(\xi_k, w_k) \rightharpoonup \zeta$ weakly in $\mathcal{V}$. Then,*

$$\zeta = \Lambda_1(\xi, w),$$

that is, the graph of Λ_1 is sequentially weakly closed in $W \times \mathcal{V} \times \mathcal{V}$.

Proof. By the definition of Λ_1, for every $v \in \mathcal{V}$ we have

$$\langle \beta(\Lambda_1(\xi_k, w_k)) + \nu A\Lambda_1(\xi_k, w_k) + \mathsf{B}(w_k, \Lambda_1(\xi_k, w_k)), v\rangle + (\xi_k, v_\tau)_W = \langle F, v\rangle. \quad (3.5)$$

Passing to the limit and using the linearity and continuity of β and A, we obtain

$$\beta(\Lambda_1(\xi_k, w_k)) \rightharpoonup \beta(\zeta), \text{ in } \mathcal{V}'$$

and

$$A\Lambda_1(\xi_k, w_k) \rightharpoonup A\zeta, \text{ in } \mathcal{V}'.$$

Moreover, B is sequentially weakly continuous by Lemma 2.8. Consequently,

$$\mathsf{B}(w_k, \Lambda_1(\xi_k, w_k)) \rightharpoonup \mathsf{B}(w, \zeta), \text{ in } \mathcal{V}'.$$

Furthermore, since $\xi_k \rightharpoonup \xi$ in W,

$$(\xi_k, v_\tau)_W \to (\xi, v_\tau)_W.$$

Hence, for every $v \in \mathcal{V}$,

$$\langle \beta(\zeta) + \nu A\zeta + \mathsf{B}(w, \zeta), v\rangle + (\xi, v_\tau)_W = \langle F, v\rangle.$$

The uniqueness of the solution to problem (3.2) implies that $\zeta = \Lambda_1(\xi, w)$. $\square$

We now introduce the multivalued mapping

$$\Lambda : W \times \mathcal{V} \longrightarrow 2^{W\times\mathcal{V}}, \quad (\xi, w) \mapsto \Lambda(\xi, w) = \mathsf{H}(w_\tau) \times \{\Lambda_1(\xi, w)\},$$

where $\mathsf{H}(w_\tau) \subset W$ is associated with the multivalued boundary condition.
To apply the Kakutani-Fan-Glicksberg's fixed point theorem, we must verify that $\Lambda(\xi, w)$ is nonempty, convex, and compact in $W \times \mathcal{V}$ for every (ξ, w), and that the graph of Λ is sequentially weakly closed in $W \times \mathcal{V}$.
Each fixed point $(\xi, w) \in W \times \mathcal{V}$ satisfying $(\xi, w) \in \Lambda(\xi, w)$ also satisfies $\xi \in \mathsf{H}(w_\tau)$ and $w = \Lambda_1(\xi, w)$. Hence,

$$\langle \beta(w) + \nu Aw + \mathsf{B}(w, w), v\rangle + (\xi, v_\tau)_W = \langle F, v\rangle, \quad \forall v \in \mathcal{V} \quad (3.6)$$

which coincides with the variational formulation in (2.4). Therefore, a fixed point of Λ yields a weak solution (ξ, w).

Lemma 3.4. *Assume that* (h_1)-(h_5) *hold and* $\beta = I$*. Then, the following assertions hold:*
(a) $\Lambda(S) \subset S$*;*
(b) If $x \in \Lambda(x)$*, then* $x \in S$*.*

Proof. Let $(\eta, u) \in \Lambda(\xi, w)$ with $(\xi, w) \in S$. Then, $\eta \in \mathsf{H}(w_\tau)$ and $u = \Lambda_1(\xi, w)$. Moreover, by Lemma 3.2 and property (iii) of Lemma 2.5,

$$\|u\|_{\mathcal{V}} \le \frac{\|\gamma\|}{c_3}\|\xi\|_W + \frac{\|F\|_{\mathcal{V}'}}{c_3} \quad \text{and} \quad \|\eta\|_W \le \|c_1\|_{L^2(\Gamma_C)} + c_2\|w\|_W.$$

Since $(\xi, w) \in S$, the defining bounds for $\|\xi\|_W$ and $\|w\|_{\mathcal{V}}$ yield

$$\|u\| \le \frac{\|\gamma\|\|c_1\| + \|F\|_{\mathcal{V}'}}{c_3 - c_2\|\gamma\|^2} \quad \text{and} \quad \|\eta\|_W \le \frac{c_3\|c_1\| + c_2\|\gamma\|\|F\|_{\mathcal{V}'}}{c_3 - c_2\|\gamma\|^2}.$$

Consequently, $\Lambda(S) \subset S$, which proves (a).

If $(\eta, u) \in \Lambda(\eta, u)$, repeated application of the preceding estimates shows that $(\eta, u) \in S$. This proves (b). ☐

Lemma 3.5. *Assume that (h_1)-(h_5) hold and $\beta = I$. Then, the graph of Λ is sequentially weakly closed in $(W \times \mathcal{V}) \times (W \times \mathcal{V})$.*

Proof. Let $\xi_k \rightharpoonup \xi$ in W, $w_k \rightharpoonup w$ in $\mathcal{V}$, and $(\eta_k, u_k) \in \Lambda(\xi_k, w_k)$ with $\eta_k \rightharpoonup \eta$ and $u_k \rightharpoonup u$. By the definition of Λ, $\eta_k \in \mathsf{H}(w_{k\tau})$ and $u_k = \Lambda_1(\xi_k, w_k)$. Lemma 3.3 therefore gives $u = \Lambda_1(\xi, w)$. Moreover, the compactness of the trace operator γ yields $w_{k\tau} \to w_\tau$ strongly in W. By property (ii) of Lemma 2.5, the graph of H is sequentially closed; hence $\eta \in \mathsf{H}(w_\tau)$. Consequently,

$$(\eta, u) \in \Lambda(\xi, w)$$

which completes the proof that the graph of Λ is sequentially weakly closed in $(W \times \mathcal{V}) \times (W \times \mathcal{V})$. ☐

We next recall the following lemma, which is an important tool in proving the existence result for problem (1.2).

Lemma 3.6 ([19])**.** *Assume that (h_1)-(h_5) hold and $\beta = I$. Then, the graph of $\Lambda|_S$ is weakly closed in $W \times \mathcal{V}$.*

We now establish the existence of a weak solution to problem (1.2).

Theorem 3.7. *Assume that (h_1)-(h_5) hold and $\beta = I$. Then problem (1.2) admits a solution $(\xi, u) \in S$. Moreover, any solution necessarily belongs to S.*

Proof. By Lemmas 3.4 and 3.6, the mapping $\Lambda|_S$ satisfies the hypotheses of the Kakutani-Fan-Glicksberg fixed point theorem: S is nonempty, convex, and weakly compact; $\Lambda|_S$ has nonempty convex values; and the graph of $\Lambda|_S$ is closed.
Thus, there exists $(\xi, w) \in S$ such that $(\xi, w) \in \Lambda(\xi, w)$, and this fixed point yields a solution to (2.4). Therefore, there exists $u \in \mathcal{V}$ satisfying (2.4).
Furthermore, define the functional

$$G := \beta(u) - \nu\Delta u + (u \cdot \nabla)u - f \in \mathcal{V}'.$$

Then, for every test function $\varphi \in \mathcal{V}$ satisfying $\operatorname{div} \varphi = 0$,

$$\langle G, \varphi \rangle_{\mathcal{V}', \mathcal{V}} = 0,$$

which means that G annihilates all divergence-free test functions.
The de Rham theorem, equivalently the Helmholtz-Leray decomposition, guarantees the existence of a pressure distribution

$$p \in L^2(\Omega)/\mathbb{R} \quad \text{such that} \quad G = \nabla p \quad \text{in } \mathcal{V}'.$$

Consequently, the weak formulation of the system can be written as

$$\beta(u) - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{in } \mathcal{V}',$$

where the pressure p is uniquely determined up to an additive constant. Equivalently, problem (1.2) admits a solution. ☐

We now turn to the existence of mild solutions to (1.1) using nonlinear semigroup theory.

4. Existence of mild solutions to the non-stationary problem

In this section, for each $f \in L^2(\Omega)^3$, let us denote $\mathcal{S}_{\text{sol}}(f)$ the subset of S consisting of the pairs solutions (ξ, u) satisfying (2.4) with respect to f. Since problem (1.2) admits at least one solution satisfying (2.4), we have $\mathcal{S}_{\text{sol}}(f) \neq \emptyset$. Furthermore, $\mathcal{S}_{\text{sol}}(f)$ is compact subset of S. The major challenge for some physical applications and perspective, is to find one solution for each $f \in L^2(\Omega)^3$. One of the physical treatment is to consider the unique solution u_f on the compact set $\mathcal{S}_{\text{sol}}(f)$ which produces the largest mechanical dissipation of

$$\varepsilon(u) = \|u\|^2_{L^2(\Omega)^3} + \nu\|\nabla u\|^2_{L^2(\Omega)^{3\times 3}} + \int_{\Gamma_C} \xi \cdot u_\tau dS. \tag{4.1}$$

In our paper, to obtain the uniqueness of weak solution to the stationary perturbed Navier-Stokes equation (1.2), we will work under a low Reynolds number condition. Outside the scope of mathematics, using a low Reynolds number is essential for engineering micro-technologies, understanding biological systems and ensuring computational accuracy.
There are in fact two types of Reynolds number used in mathematical analysis of the Navier-Stokes equation and their relationship plays an important role such as uniqueness and stability theory: the one is named the classical Reynolds number denoted by $\mathcal{R}_e$, which is defined from the problem data as follows

$$\mathcal{R}_e = \frac{UL}{\nu}$$

where U is a characteristic velocity, L is a characteristic length while ν is a kinematic viscosity; the other designated by $\mathcal{R}_e(u)$ is dependent of each solution u to Navier-Stokes equation and is defined such as

$$\mathcal{R}_e(u) = \frac{C\|\nabla u\|}{\nu}$$

where C depends only on the domain Ω through the Sobolev's constant. Observe that the classical Reynolds number is convenient because it is known before solving the problem while the solution-dependent Reynolds number is more precise because it measures the actual balance between convection and viscosity in the computed flow.
Finally, we mention that there is the relationship between the classical Reynolds number $\mathcal{R}_e$ and the Reynolds number $\mathcal{R}_e(u)$ with respect to the solution u, which relationship is expressed for all solution u by

$$\mathcal{R}_e(u) \leq K\mathcal{R}_e$$

with K is positive constant.
In the rest of the following work, to simplify we can take $\mathcal{R}_e(u) \leq \mathcal{R}_e$ and we use the following assumption:

$$\mathcal{R}_e < 1. \tag{4.2}$$

Immediately, we obtain this important remark which plays an important role in the rest of our work.

Remark 4.1. Under hypothesis (h_1)-(h_5) and $\beta = I$, the use of (4.2) gives a uniqueness of the solution to the stationary Navier-Stokes equation (1.1).

For the next steps, let us now proceed under the condition (4.2). We mention that a classical construction of mild solutions consists in approximating the non-stationary problem (1.1), for $\varepsilon > 0$, by the following implicit time-discretization scheme:

$$\begin{cases} \dfrac{\beta(u_i^\varepsilon) - \beta(u_{i-1}^\varepsilon)}{t_i^\varepsilon - t_{i-1}^\varepsilon} = \nu\Delta u_i^\varepsilon - (u_i^\varepsilon \cdot \nabla)u_i^\varepsilon - \nabla p_i^\varepsilon + f_i^\varepsilon, \quad \text{for } i = 1, \dots, n, \\ u_i^\varepsilon \in \mathcal{V} \cap L^\infty(\Omega). \end{cases} \tag{4.3}$$

where $n \in \mathbb{N}$, $0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_n^\varepsilon$, $p_i^\varepsilon \in L_0^2(\Omega) \cap L^\infty(\Omega)^3$, and $f_i^\varepsilon \in L^\infty(\Omega)^3$ for $i = 1, \dots, n$, such that

$$\sum_{i=1}^{n} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \|f(t) - f_i^\varepsilon\|_{L^2(\Omega)^3} dt, \quad \max_{i=1,\dots,n} (t_i^\varepsilon - t_{i-1}^\varepsilon), \quad T - t_n^\varepsilon, \quad \|\beta_0 - \beta_0^\varepsilon\|_{L^2(\Omega)^3}$$

all converge to 0 as $\varepsilon \to 0$.
The functions u_ε and p_ε are defined piecewise constantly by

$$p_\varepsilon = p_i^\varepsilon,\ u_\varepsilon = u_i^\varepsilon \text{ on } \ (t_{i-1}^\varepsilon, t_i^\varepsilon),\ i = 1, \dots, n, \quad u_\varepsilon(0) = u_0^\varepsilon,\ \beta(u_\varepsilon)(0) = \beta_0^\varepsilon.$$

Following [13, 14], this construction uses nonlinear semigroup theory to define the associated operator and establish the existence of mild solutions.

Definition 4.2. A mild solution to (1.1) is a function $u \in C([0, T]; L^2(\Omega)^3)$ that is the uniform limit of the piecewise-constant approximations u_ε.

The main result of this section is the following theorem.

Theorem 4.3. *Assume that (h_1)-(h_5) hold and $\beta = I$. Moreover, suppose that (4.2) is satisfied. Then, for every $(u_0, f) \in L^2(\Omega)^3 \times L^2(Q)^3$, there exists a unique mild solution u to (1.1). Furthermore, the following contraction principle holds: for every $0 \le t \le T$, if u and $\hat{u}$ are mild solutions to (1.1) corresponding, respectively, to the data (u_0, f) and $(\hat{u}_0, \hat{f})$ in $L^2(\Omega)^3 \times L^2(Q)^3$, then*

$$\|u(t) - \hat{u}(t)\|_{L^2(\Omega)^3} \le \|u_0 - \hat{u}_0\|_{L^2(\Omega)^3} + \int_0^t \|f(s) - \hat{f}(s)\|_{L^2(\Omega)^3} ds.$$

By nonlinear semigroup theory, Theorem 4.3 follows from the following proposition.

Proposition 4.4. *Assume that (h_1)–(h_5) hold and $\beta = I$, and let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Assume, in addition, that (4.2) is satisfied. Then, there exists an operator*

$$A := \Big\{ (\beta(u),\ f - \beta(u)) \in L^2(\Omega)^3 \times L^2(\Omega)^3 :\ u \ \textit{is the solution of } (1.2) \Big\}.$$

such that:

(i) A is T-accretive, and even completely accretive, in $L^2(\Omega)^3$;
(ii) $R(I + \sigma A) = L^2(\Omega)^3$ for every $\sigma > 0$;
(iii) $\overline{D(A)}^{L^2(\Omega)^3} = \{w \in L^2(\Omega)^3 : w(x) \in R(\beta) \text{ for a.e. } x \in \Omega\}$.

Proof. (i) Let $u \in \mathcal{V}$ be a solution to (1.2). Then, for every $v \in \mathcal{V}$,

$$\int_\Omega \beta(u)\cdot v\,dx+\nu\int_\Omega \nabla u\nabla v\,dx+\int_\Omega (u\cdot\nabla)u\cdot v\,dx+\int_{\Gamma_C}\xi\cdot v_\tau\,ds=\int_\Omega f\cdot v\,dx, \quad (4.4)$$

where $\xi \in h(x,u_\tau)$. Let $u,\hat{u}\in\mathcal{V}$ be two solutions associated with f and $\hat{f}$, respectively, and set $w=u-\hat{u}$. Taking $v=w$ as the test function and subtracting the two variational formulations, we obtain

$$\begin{aligned}&\left\langle\beta(u)-\beta(\hat{u}),\,w\right\rangle_{L^2(\Omega)^3}+\nu\left(\nabla w,\nabla w\right)_{L^2(\Omega)^{3\times3}}\\&\quad+\left((u\cdot\nabla)u-(\hat{u}\cdot\nabla)\hat{u},\;w\right)_{L^2(\Omega)^3}+\left\langle\xi-\hat{\xi},\;w_\tau\right\rangle_{L^2(\Gamma_C)^3}\\&=\int_\Omega(f-\hat{f})\cdot\mathrm{sign}(w)\,|w|\,dx,\end{aligned} \quad (4.5)$$

where $\xi\in h(x,u_\tau)$ and $\hat{\xi}\in h(x,\hat{u}_\tau)$, and

$$\begin{aligned}(f-\hat{f})\cdot\mathrm{sign}(w)|w| &= f_1\,\mathrm{sign}(u_1-\hat{u}_1)|u_1-\hat{u}_1|+f_2\,\mathrm{sign}(u_2-\hat{u}_2)|u_2-\hat{u}_2|\\&+ f_3\,\mathrm{sign}(u_3-\hat{u}_3)|u_3-\hat{u}_3|.\end{aligned}$$

Since β is the identity operator,

$$\left\langle\beta(u)-\beta(\hat{u}),\,w\right\rangle_{L^2(\Omega)^3}=\|w\|^2_{L^2(\Omega)^3}.$$

Moreover,

$$\nu\left(\nabla w,\nabla w\right)_{L^2(\Omega)^{3\times3}}=\nu\|\nabla w\|^2_{L^2(\Omega)^{3\times3}}.$$

The difference of the convective terms can be written as

$$(u\cdot\nabla)u-(\hat{u}\cdot\nabla)\hat{u}=(w\cdot\nabla)u+(\hat{u}\cdot\nabla)w.$$

Thus,

$$\int_\Omega[(u\cdot\nabla)u-(\hat{u}\cdot\nabla)\hat{u}]\cdot w dx=\int_\Omega(w\cdot\nabla)u\cdot w\,dx+\int_\Omega(\hat{u}\cdot\nabla)w\cdot w\,dx.$$

Since the flow is incompressible and $\hat{u}\cdot n=0$ on $\partial\Omega$,

$$\int_\Omega(\hat{u}\cdot\nabla)w\cdot w\,dx=0.$$

In $3D$ Navier-Stokes case, it is known that

$$\Big|\int_\Omega(w\cdot\nabla)u\cdot w\,dx\Big|\le C\|\nabla u\|_{L^2(\Omega)^3}\|\nabla w\|^2_{L^2(\Omega)^{3\times3}}.$$

Consequently, since $\displaystyle\int_{\Gamma_C}(\xi-\hat{\xi})\cdot(u_\tau-\hat{u}_\tau)dS\ge0$, one obtains

$$\|w\|^2_{L^2(\Omega)^3}+(\nu-C\|\nabla u\|_{L^2(\Omega)^3})\|\nabla w\|^2_{L^2(\Omega)^{3\times3}}\le\int_\Omega(f-\hat{f})\cdot\mathrm{sign}(w)\,|w|\,dx.$$

Hence, taking into account that $\mathcal{R}_e(u)\le\mathcal{R}_e<1$

$$0\le\|w\|^2_{L^2(\Omega)^3}+\nu(1-\mathcal{R}_e(u))\|\nabla w\|^2_{L^2(\Omega)^{3\times3}}\le\int_\Omega(f-\hat{f})\cdot\mathrm{sign}(w)\,|w|\,dx$$

and A is completely accretive.
(ii) We prove that $I+\sigma A$ is surjective.
Let $g \in L^2(\Omega)^3$. By the existence result for the stationary problem (1.2), there exists $u \in D(A)$ satisfying

$$\beta(u)+\sigma A(\beta(u))=g.$$

Consequently, the range of $I+\sigma A$ is the whole space $L^2(\Omega)^3$; that is,

$$R(I+\sigma A)=L^2(\Omega)^3,$$

which proves the surjectivity of $I+\sigma A$.
(iii) Finally, we establish that the domain $D(A)$ is dense in $L^2(\Omega)^3$; equivalently,

$$\forall f \in L^2(\Omega)^3,\ \exists (f_n) \subset D(A) \quad \text{such that} \quad f_n \to f \text{ in } L^2(\Omega)^3.$$

Let $f \in L^2(\Omega)^3$ and $\delta>0$, and set $\beta(u_\delta)=(I+\delta A)^{-1} f$. Then

$$(\beta(u_\delta),\, \frac{1}{\delta}(f-\beta(u_\delta))) \in A.$$

Hence, for every $v \in \mathcal{V}$,

$$\int_\Omega \beta(u_\delta)\cdot v\, dx+\delta\Big[\nu\int_\Omega \nabla u_\delta \nabla v\, dx+\int_\Omega (u_\delta\cdot\nabla)u_\delta\cdot v\, dx+\int_{\Gamma_C}\xi_\delta\cdot v_\tau\, dS\Big]=\int_\Omega f\cdot v\, dx,$$

with $\xi_\delta \in \mathsf{H}(u_{\delta,\tau})$. Since β is the identity, we have $\beta(u_\delta)=u_\delta$. Taking $v=\beta(u_\delta)$ and using $\int_\Omega (u_\delta\cdot\nabla)u_\delta\cdot u_\delta\, dx=0$ to obtain

$$\int_\Omega |\beta(u_\delta)|^2\, dx+\delta\Big[\nu\int_\Omega |\nabla u_\delta|^2\, dx+\int_{\Gamma_C}\xi_\delta\cdot u_{\delta,\tau}\, dS\Big]=\int_\Omega f\cdot u_\delta\, dx.$$

Since $\int_{\Gamma_C}\xi_\delta\cdot u_{\delta,\tau}\, dS \geq 0$ and $u_\delta=\beta(u_\delta)$,

$$\begin{aligned}\int_\Omega |\beta(u_\delta)|^2\, dx &\leq \int_\Omega f\cdot u_\delta\, dx\\ &\leq \frac{1}{2}\int_\Omega |f|^2 dx+\frac{1}{2}\int_\Omega |u_\delta|^2\, dx\\ &\leq \frac{1}{2}\int_\Omega |f|^2 dx+\frac{1}{2}\int_\Omega |\beta(u_\delta)|^2\, dx.\end{aligned}$$

Consequently,

$$\int_\Omega |\beta(u_\delta)|^2\, dx \leq \int_\Omega |f|^2 dx.$$

Now, for every $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned}\Big|\delta\Big[\nu\int_\Omega \nabla u_\delta\nabla\phi\, dx &+ \int_\Omega (u_\delta\cdot\nabla)u_\delta\cdot\phi\, dx+\int_{\Gamma_C}\xi_\delta\cdot\phi_\tau\, dS\Big]\Big|\\ &= \Big|\int_\Omega f\cdot\phi\, dx-\int_\Omega \beta(u_\delta)\cdot\phi\, dx\Big|\\ &\leq \Big(\|f\|_{L^2(\Omega)^3}\|\phi\|_{L^2(\Omega)^3}+\|\beta(u_\delta)\|_{L^2(\Omega)^3}\|\phi\|_{L^2(\Omega)^3}\Big)\\ &\leq 2\|f\|_{L^2(\Omega)^3}\|\phi\|_{L^2(\Omega)^3}.\end{aligned}$$

Therefore,

$$\delta\Big[\nu\int_\Omega \nabla u_\delta \nabla\phi\,dx + \int_\Omega (u_\delta\cdot\nabla)u_\delta\cdot\phi\,dx + \int_{\Gamma_C}\xi_\delta\cdot\phi_\tau\,dS\Big] \longrightarrow 0 \quad \text{as } \delta\to 0.$$

The sequence $(\beta(u_\delta))$ is bounded in $L^2(\Omega)^3$. Returning to the resolvent equation and letting $\delta\to 0$, we obtain

$$\beta(u_\delta)\to f \quad \text{in } (C_0^\infty(\Omega))'.$$

Since $(\beta(u_\delta))_{\delta>0}$ is uniformly bounded, we may extract a subsequence such that

$$\beta(u_\delta)\to f \quad \text{in } L^2(\Omega)^3 \quad \text{and almost everywhere in } \Omega.$$

Since $\|\beta(u_\delta)\|_{L^2(\Omega)^3}\le \|f\|_{L^2(\Omega)^3}$, the dominated convergence theorem yields

$$\beta(u_\delta)\to f \quad \text{in } L^2(\Omega)^3.$$

$$\overline{D(A)}^{\|\cdot\|_{L^2(\Omega)^3}} = \{w\in L^2(\Omega)^3 : w(x)\in R(\beta) \text{ a.e. in } \Omega\}.$$

□

By Proposition 4.4, the nonlinear operator A is m-accretive in $L^2(\Omega)^3$. Moreover, since $\beta(u)=u$, the general theory of nonlinear semigroups (see [3]) implies that the abstract evolution problem corresponding to (1.1) admits a unique mild solution $u\in C([0,T];L^2(\Omega)^3)$ for every initial datum $u_0\in\overline{D(A)}^{L^2(\Omega)^3}$ and every right-hand side $f\in L^2(0,T;L^2(\Omega)^3)$.

Remark 4.5. The Crandall-Liggett's approach also shows that there exists $T>0$ such that the mild solution u belongs to $L^2([0,T];\mathcal{V})\cap L^\infty([0,T];\mathcal{H})$ for every initial datum $u_0\in\mathcal{H}$ and every $f\in L^2([0,T];\mathcal{V}')$.

5. Existence of weak solutions

To prove the existence of weak solutions, we use an energy estimate similar to that in Lemma 1.5 of [2].

Proposition 5.1. *Assume that (h_1)–(h_5) hold, $\beta=I$, and $u_0\in\mathcal{V}$. Let $\Omega\subset\mathbb{R}^N$ be a bounded domain, and assume that (4.2) holds. Let $\beta_0^\varepsilon\in L^\infty(\Omega)^3$, $f_\varepsilon\in L^\infty(Q)^3$, and let u_ε be the unique mild solution to problem (1.1). Then u_ε is a weak solution to*

$$\begin{cases} \beta(u_\varepsilon)_t-\nu\Delta u_\varepsilon+(u_\varepsilon\cdot\nabla)u_\varepsilon+\nabla p_\varepsilon=f_\varepsilon, & \text{in } \Omega\times(0,T),\\ \operatorname{div}(u_\varepsilon)=0, & \text{in } \Omega\times(0,T),\\ u_\varepsilon=0, & \text{on } \Gamma_D\times(0,T),\\ u_\varepsilon\cdot n=0,\quad \sigma_\tau\in h(u_{\varepsilon\tau}), & \text{on } \Gamma_C\times(0,T),\\ u_\varepsilon(0,x)=\beta_0^\varepsilon, & \text{in } \Omega. \end{cases} \tag{5.1}$$

Furthermore, for any two weak solutions u_ε and $\hat u_\varepsilon$ corresponding respectively to the data $(u_{\varepsilon,0},f_\varepsilon)$ and $(\hat u_{\varepsilon,0},\hat f_\varepsilon)$, we have

$$\|(u_\varepsilon-\hat u_\varepsilon)(t)\|_{L^2(\Omega)^3}\le\|u_{\varepsilon,0}-\hat u_{\varepsilon,0}\|_{L^2(\Omega)^3}+\int_0^t\|(f_\varepsilon-\hat f_\varepsilon)(s)\|_{L^2(\Omega)^3}\,ds,\quad \forall t\in[0,T].$$

Proof. For $i = 0, 1, \dots, n$, let u_i^ε denote the unique weak energy solution to

$$\varepsilon f_i^\varepsilon + \beta(u_{i-1}^\varepsilon) \in (I + \varepsilon A)(\beta(u_i^\varepsilon)).$$

Then, for every $\varphi \in \mathcal{V}$,

$$\begin{aligned} &\nu \int_\Omega \nabla u_i^\varepsilon \nabla \varphi \, dx + \int_\Omega (u_i^\varepsilon \cdot \nabla) u_i^\varepsilon \cdot \varphi \, dx \\ &+ \int_\Omega \frac{\big(\beta(u_i^\varepsilon) - \beta(u_{i-1}^\varepsilon)\big)}{\varepsilon} \varphi \, dx + \int_{\Gamma_C} \xi_i^\varepsilon \cdot \varphi_\tau \, dS = \int_\Omega f_i^\varepsilon \varphi \, dx, \end{aligned} \tag{5.2}$$

with $\xi_i^\varepsilon \in \mathsf{H}(u_{i,\tau}^\varepsilon)$. Taking $\varphi = u_i^\varepsilon$ in (5.2), integrating over $(t_{i-1}^\varepsilon, t_i^\varepsilon)$, and summing the resulting equalities for $i = 1, \dots, n$, we obtain

$$\begin{aligned} &\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Omega \frac{\big(\beta(u_i^\varepsilon) - \beta(u_{i-1}^\varepsilon)\big)}{\varepsilon} u_i^\varepsilon \, dxdt + \nu \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Omega |\nabla u_i^\varepsilon|^2 \, dxdt \\ &+ \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_{\Gamma_C} \xi_i^\varepsilon \cdot u_{i,\tau}^\varepsilon \, dSdt = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Omega f_i^\varepsilon u_i^\varepsilon \, dxdt. \end{aligned} \tag{5.3}$$

Assume now that $t_i^\varepsilon - t_{i-1}^\varepsilon = \varepsilon$. Then $f_\varepsilon(t) = f_i^\varepsilon$, $\beta(u_\varepsilon(t)) = \beta(u_i^\varepsilon)$, and $u_\varepsilon(t) = u_i^\varepsilon$ for $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon)$ and $i = 1, \dots, n$, while $\beta(u_\varepsilon(0)) = \beta_0^\varepsilon$ and $\xi^\varepsilon(t) = \xi_i^\varepsilon$.
Since $u = (u_1, u_2, u_3)$ and $\beta(u) = u$,

$$\begin{aligned} \frac{\big(\beta(u_i^\varepsilon) - \beta(u_{i-1}^\varepsilon)\big)}{\varepsilon} u_i^\varepsilon &= \frac{u_{i_1}^\varepsilon - u_{i-1_1}^\varepsilon}{\varepsilon} u_{i_1}^\varepsilon + \frac{u_{i_2}^\varepsilon - u_{i-1_2}^\varepsilon}{\varepsilon} u_{i_2}^\varepsilon + \frac{u_{i_3}^\varepsilon - u_{i-1_3}^\varepsilon}{\varepsilon} u_{i_3}^\varepsilon \\ &= \frac{b(u_{i_1}^\varepsilon) - b(u_{i-1_1}^\varepsilon)}{\varepsilon} u_{i_1}^\varepsilon + \frac{b(u_{i_2}^\varepsilon) - b(u_{i-1_2}^\varepsilon)}{\varepsilon} u_{i_2}^\varepsilon \\ &+ \frac{b(u_{i_3}^\varepsilon) - b(u_{i-1_3}^\varepsilon)}{\varepsilon} u_{i_3}^\varepsilon \end{aligned}$$

where $b : \mathbb{R} \to \mathbb{R}$ is defined by $b(x) = x$ and is continuous and nondecreasing, with $b(0) = 0$. Using (2.2), we obtain

$$\begin{aligned} &\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Omega \frac{\big(\beta(u_i^\varepsilon) - \beta(u_{i-1}^\varepsilon)\big)}{\varepsilon} u_i^\varepsilon \, dxdt = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Omega \frac{b(u_{i_1}^\varepsilon) - b(u_{i-1_1}^\varepsilon)}{\varepsilon} u_{i_1}^\varepsilon dxdt \\ + \;& \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Omega \frac{b(u_{i_2}^\varepsilon) - b(u_{i-1_2}^\varepsilon)}{\varepsilon} u_{i_2}^\varepsilon dxdt + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Omega \frac{b(u_{i_3}^\varepsilon) - b(u_{i-1_3}^\varepsilon)}{\varepsilon} u_{i_3}^\varepsilon dxdt \\ \geq \;& \int_\Omega \big(B(u_{\varepsilon_1}(T)) - B(u_{\varepsilon_1}(0))\big) dx + \int_\Omega \big(B(u_{\varepsilon_2}(T)) - B(u_{\varepsilon_2}(0))\big) dx \\ + \;& \int_\Omega \big(B(u_{\varepsilon_3}(T)) - B(u_{\varepsilon_3}(0))\big) dx. \end{aligned}$$

Therefore,

$$\begin{aligned}&\int_\Omega \big(B(u_{\varepsilon_1}(T)) - B(u_{\varepsilon_1}(0))\big)dx + \int_\Omega \big(B(u_{\varepsilon_2}(T)) - B(u_{\varepsilon_2}(0))\big)dx \\ &+ \int_\Omega \big(B(u_{\varepsilon_3}(T)) - B(u_{\varepsilon_3}(0))\big)dx + \nu \int_0^T\int_\Omega |\nabla u_\varepsilon(t)|^2 dxdt \\ &+ \int_0^T\int_{\Gamma_C} \xi_\varepsilon(t)\cdot u_{\varepsilon_\tau}(t)\, dSdt \le \int_0^T\int_\Omega f_\varepsilon(t)u_\varepsilon(t)dxdt.\end{aligned} \tag{5.4}$$

Moreover, $B(u_{\varepsilon_j}(T)) - B(u_{\varepsilon_j}(0)) \in L^\infty(\Omega)$ for $j = 1, 2, 3$, while $u_\varepsilon, f_\varepsilon \in L^\infty(Q)$ and $\xi_\varepsilon, u_{\varepsilon\tau} \in L^\infty([0,T]\times\Gamma_C)$. By assumption (h_4), (5.4) implies

$$\int_0^T\int_\Omega |\nabla u_\varepsilon(t)|^2 dxdt \le C \tag{5.5}$$

where C is independent of ε.
Therefore, (∇u_ε) is uniformly bounded in $(L^2(0,T;L^2(\Omega)^3))^3$, and (u_ε) is uniformly bounded in $L^2(0,T;\mathcal{V})$. Moreover, by assumption (h_4), (ξ_ε) is uniformly bounded in $L^2(\Sigma_C)^3$. Consequently, we may extract subsequences, not relabelled, such that

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^2(0,T;\mathcal{V}), \tag{5.6}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^2(Q)^{3\times 3}, \tag{5.7}$$

and

$$\xi_\varepsilon \rightharpoonup \xi \quad \text{in } L^2(\Sigma_C)^3. \tag{5.8}$$

We now prove that

$$(u_\varepsilon\cdot\nabla)u_\varepsilon \rightharpoonup (u\cdot\nabla)u \quad \text{in } L^2(Q)^3. \tag{5.9}$$

From (5.2), for every $\varphi \in \mathcal{V}$,

$$\begin{aligned}&\int_\Omega \frac{(\beta(u_\varepsilon(t)) - \beta(u_\varepsilon(t-\varepsilon)))}{\varepsilon}\varphi\, dx \\ &= -\nu\int_\Omega \nabla u_\varepsilon\nabla\varphi\, dx - \int_\Omega (u_\varepsilon\cdot\nabla)u_\varepsilon\cdot\varphi\, dx - \int_{\Gamma_C}\xi_\varepsilon(t)\cdot\varphi_\tau\, dS + \int_\Omega f_\varepsilon(t)\varphi\, dx.\end{aligned}$$

By Hölder's inequality and the Gagliardo–Nirenberg inequality

$$\|u\|_{L^4(\Omega)^3} \le C\|u\|^{1/4}_{L^2(\Omega)^3}\|\nabla u\|^{3/4}_{L^2(\Omega)^{3\times3}}, \quad \forall u \in \mathcal{V},$$

applied to u_ε and φ, together with Poincaré's inequality for φ, we obtain

$$\begin{aligned}\left|\int_\Omega (u_\varepsilon\cdot\nabla)u_\varepsilon\cdot\varphi\right| &\le \|u_\varepsilon\|_{L^4(\Omega)^3}\|\nabla u_\varepsilon\|_{L^2(\Omega)^{3\times3}}\|\varphi\|_{L^4(\Omega)^3} \\ &\le C\|u_\varepsilon\|^{1/4}_{L^2(\Omega)^3}\|\nabla u_\varepsilon\|^{7/4}_{L^2(\Omega)^{3\times3}}\|\varphi\|^{1/4}_{L^2(\Omega)^3}\|\nabla\varphi\|^{7/4}_{L^2(\Omega)^{3\times3}} \\ &\le C\|u_\varepsilon\|^{1/4}_{L^2(\Omega)^3}\|\nabla u_\varepsilon\|^{7/4}_{L^2(\Omega)^{3\times3}}\|\nabla\varphi\|_{L^2(\Omega)^{3\times3}}.\end{aligned}$$

Hence,

$$\begin{aligned}\|(u_\varepsilon\cdot\nabla)u_\varepsilon\|_{\mathcal{V}'} &\le C\|u_\varepsilon\|^{1/4}_{L^2(\Omega)^3}\|\nabla u_\varepsilon\|^{7/4}_{L^2(\Omega)^{3\times3}}\\ &\le C\|u_\varepsilon\|^{1/4}_{L^2(\Omega)^3}\|\nabla u_\varepsilon\|^{1/4}_{L^2(\Omega)^{3\times3}}\|\nabla u_\varepsilon\|^{3/2}_{L^2(\Omega)^{3\times3}}\\ &\le C\|u_\varepsilon\|^{\frac12}_{\mathcal{V}}\|\nabla u_\varepsilon\|^{\frac32}_{L^2(\Omega)^{3\times3}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Big\|\frac{\beta(u_\varepsilon(t))-\beta(u_\varepsilon(t-\varepsilon))}{\varepsilon}\Big\|_{\mathcal{V}'}\\ \le\ & C\big(\|\nabla u_\varepsilon\|_{L^2(\Omega)^{3\times3}}+\|(u_\varepsilon\cdot\nabla)u_\varepsilon\|_{\mathcal{V}'}+\|\xi_\varepsilon\|_{L^2(\Gamma_C)^3}+\|f_\varepsilon\|_{L^2(\Omega)^3}\big).\end{aligned}$$

Furthermore,

$$\int_0^T\|(u_\varepsilon\cdot\nabla)u_\varepsilon\|^{\frac43}_{\mathcal{V}'}\le C\|u_\varepsilon\|^{\frac23}_{L^\infty(0,T;\mathcal{H})}\int_0^T\|\nabla u_\varepsilon\|^2_{L^2(\Omega)^{3\times3}}dt<+\infty.$$

Consequently,

$$\frac{\partial u_\varepsilon}{\partial t}=\frac{\partial\beta(u_\varepsilon)}{\partial t}\in L^{\frac43}(0,T;\mathcal{V}').$$

Since (u_ε) is uniformly bounded in $L^2(0,T;\mathcal{V})$ and $\mathcal{V}\hookrightarrow\hookrightarrow L^2(\Omega)^3\hookrightarrow\mathcal{V}'$, the Aubin–Lions–Simon lemma implies that (u_ε) is relatively compact in $L^2(0,T;L^2(\Omega)^3)$. After extracting a subsequence, still denoted by (u_ε), we obtain

$$u_\varepsilon\to u\quad\text{strongly in } L^2(0,T;L^2(\Omega)^3)\text{ and a.e. in }\Omega\times(0,T).\tag{5.10}$$

Combining this result with (5.7), we obtain

$$(u_\varepsilon\cdot\nabla)u_\varepsilon\rightharpoonup(u\cdot\nabla)u.$$

Since $\beta(u_\varepsilon)$ converges weakly to $\beta(u)$ in $L^2(Q)$, the Aubin–Lions lemma yields

$$\beta(u_\varepsilon)\to\beta(u)\quad\text{strongly in }\ C([0,T];\mathcal{V}').$$

Combining this strong convergence with the uniform boundedness of $\beta(u_\varepsilon)$, we obtain

$$\beta(u_\varepsilon(t))\rightharpoonup\beta(u(t))\text{ in }L^2(\Omega)^3\quad\text{as }\varepsilon\to0.\tag{5.11}$$

In (5.2), take $\varphi\in W^{1,1}(0,T;\mathcal{V}\cap L^\infty(\Omega))\cap E$ with $\varphi(T)=0$ as a test function and integrate with respect to time to obtain

$$\begin{aligned}&\nu\int_0^T\int_\Omega\nabla u_\varepsilon(t)\cdot\nabla\varphi(t)\,dxdt\\ &+\int_0^T\int_\Omega\frac{\big(\beta(u_\varepsilon(t))-\beta(u_\varepsilon(t-\varepsilon))\big)}{\varepsilon}\varphi(t)\,dxdt\\ &+\int_0^T\int_\Omega(u_\varepsilon\cdot\nabla)u_\varepsilon\cdot\varphi\,dxdt+\int_0^T\int_{\Gamma_C}\xi_\varepsilon(t)\cdot\varphi_\tau(t)\,dSdt\\ &=\int_0^T\int_\Omega f_\varepsilon(t)\varphi(t)\,dxdt.\end{aligned}\tag{5.12}$$

First, set

$$\begin{aligned} I_\varepsilon &= \int_0^T \int_\Omega \frac{\beta(u_\varepsilon(t)) - \beta(u_\varepsilon(t-\varepsilon))}{\varepsilon} \varphi(t)\, dxdt \\ &= \int_0^T \int_\Omega \frac{\beta(u_\varepsilon(t))}{\varepsilon} \varphi(t)\, dxdt - \int_0^T \int_\Omega \frac{\beta(u_\varepsilon(t-\varepsilon))}{\varepsilon} \varphi(t)\, dxdt. \end{aligned}$$

Set $s = t - \varepsilon$. Then

$$\int_0^T \int_\Omega \frac{\beta(u_\varepsilon(t-\varepsilon))}{\varepsilon} \varphi(t)\, dxdt = \int_{-\varepsilon}^{T-\varepsilon} \int_\Omega \frac{\beta(u_\varepsilon(s))}{\varepsilon} \varphi(s+\varepsilon)\, dxds.$$

Splitting the integrals gives

$$\begin{aligned} I_\varepsilon &= \int_0^{T-\varepsilon} \int_\Omega \frac{\beta(u_\varepsilon(t))}{\varepsilon} \big(\varphi(t) - \varphi(t+\varepsilon)\big)\, dxdt + \int_{T-\varepsilon}^T \int_\Omega \frac{\beta(u_\varepsilon(t))}{\varepsilon} \varphi(t)\, dxdt \\ &- \int_{-\varepsilon}^0 \int_\Omega \frac{\beta(u_\varepsilon(s))}{\varepsilon} \varphi(s+\varepsilon)\, dxds. \end{aligned}$$

For $s \in [-\varepsilon, 0]$, we have $\beta(u_\varepsilon(s)) = \beta_0$. Hence

$$\int_{-\varepsilon}^0 \int_\Omega \frac{\beta(u_\varepsilon(s))}{\varepsilon} \varphi(s+\varepsilon)\, dxds = \int_0^\varepsilon \int_\Omega \frac{\beta_0}{\varepsilon} \varphi(t)\, dxdt.$$

Since $\varphi(T) = 0$,

$$\lim_{\varepsilon \to 0} I_\varepsilon = -\int_0^T \int_\Omega \beta(u) \varphi_t\, dxdt - \int_\Omega \beta_0 \varphi(0)\, dx.$$

Finally, since $\beta(u) = u$, passing to the limit in (5.12) gives, for every $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$,

$$\begin{aligned} &\nu \int_0^T \int_\Omega \nabla u \cdot \nabla \varphi\, dxdt - \int_0^T \int_\Omega u \varphi_t\, dxdt - \int_\Omega u_0 \varphi(0)\, dx \\ &+ \int_0^T \int_\Omega (u \cdot \nabla) u \cdot \varphi dxdt + \int_0^T \int_{\Gamma_C} \xi \cdot \varphi_\tau\, dSdt = \int_0^T \int_\Omega f \varphi\, dxdt. \end{aligned} \tag{5.13}$$

Therefore, u is a weak solution to problem (1.1).

Returning to Proposition 4.4, recall that A is m-accretive and therefore generates a contraction semigroup $(S(t))_{t\geq 0}$. Since the mild solution to (1.1) associated with the data $(u_{\varepsilon,0}, f_\varepsilon)$ is also a weak solution corresponding to the same data, it satisfies

$$u_\varepsilon(t) = S(t) u_{\varepsilon,0} + \int_0^t S(t-s) f_\varepsilon(s)\, ds, \quad \forall t \in [0,T].$$

Thus, for any weak solutions u_ε and $\hat{u}_\varepsilon$ to (1.1) corresponding, respectively, to the data $(u_{\varepsilon,0}, f_\varepsilon)$ and $(\hat{u}_{\varepsilon,0}, \hat{f}_\varepsilon)$,

$$\|(u_\varepsilon - \hat{u}_\varepsilon)(t)\|_{L^2(\Omega)^3} \leq \|u_{\varepsilon,0} - \hat{u}_{\varepsilon,0}\|_{L^2(\Omega)^3} + \int_0^t \|(f_\varepsilon - \hat{f}_\varepsilon)(s)\|_{L^2(\Omega)^3}\, ds, \ \forall t \in [0,T].$$

Finally, the de Rham theorem yields the fluid pressure p corresponding to the velocity field u. □

Conclusion The results obtained for this Navier-Stokes system with multivalued friction can be interpreted in the context of computational fluid dynamics, particularly in numerical simulation schemes involving non-smooth boundary conditions. Moreover, outside the scope of mathematics, using a low Reynolds number is essential for engineering micro-technologies, understanding biological systems and ensuring computational accuracy.

References

1. Ahmed, B. M., Akinyele, A., & Omosowon, J. B. (2024). The non-homogeneous Cauchy problem of semigroup of linear operators. *Annals of Mathematics and Computer Science*, Vol. 23. https://doi.org/10.56947/amcs.v23.29
2. Alt, H. W., & Luckhaus, S. (1983). Quasi-linear elliptic-parabolic differential equations. *Mathematische Zeitschrift*, 183(3), 311-341. https://doi.org/10.1007/BF01176474
3. Bénilan, Ph., Crandall, M. G., & Pazy, A. (forthcoming). *Evolution equations governed by accretive operators*.
4. Caraballo, T., Real, J., & Márquez, A. M. (2010). Three-dimensional system of globally modified Navier-Stokes equations with delay. *International Journal of Bifurcation and Chaos*, 20(9), 2869-2883. https://doi.org/10.1142/S0218127410027438
5. Chepyzhov, V. V., Titi, E. S., & Vishik, M. I. (2007). On the convergence of solutions of the Leray-alpha model to the trajectory attractor of the 3D Navier-Stokes system. *Discrete and Continuous Dynamical Systems*, 17(3), 481-500. https://doi.org/10.3934/dcds.2007.17.481
6. Cheskidov, A., & Shvydkoy, R. (2010). The regularity of weak solutions of the 3D Navier-Stokes equations in $B^{-1}_{\infty,\infty}$. *Archive for Rational Mechanics and Analysis*, 195(1), 159-169. https://doi.org/10.1007/s00205-008-0198-9
7. Crandall, M. G., & Liggett, T. M. (1971). Generation of semigroups of nonlinear transformations on general Banach spaces. *American Journal of Mathematics*, 93(2), 265-298. https://doi.org/10.2307/2373376
8. Doering, Ch. R., & Gibbon, J. D. (1995). *Applied Analysis of the Navier-Stokes Equations*. Cambridge University Press. https://doi.org/10.1017/CBO9780511608803
9. Duvaut, G. (1980). Équilibre d'un solide élastique avec contact unilatéral et frottement de Coulomb. *Comptes Rendus de l'Académie des Sciences, Paris*, 290, 263-265.
10. Foiaş, C., & Temam, R. (1989). Gevrey class regularity for the solutions of the Navier-Stokes equations. *Journal of Functional Analysis*, 87(2), 359-369. https://doi.org/10.1016/0022-1236(89)90015-3
11. Girault, V., & Raviart, P. A. (1986). *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*. Springer-Verlag. https://doi.org/10.1007/978-3-642-61623-5
12. Ionescu, I. R., & Nguyen, Q. L. (2002). Dynamic contact problems with slip dependent friction in viscoelasticity. *International Journal of Applied Mathematics and Computer Science*, 12(1), 71-80. https://doi.org/10.2478/amcs-2002-0005
13. Kyelem, B. A., Ouedraogo, A., & Zongo, F. D. Y. (2019). Existence and uniqueness of entropy solutions to nonlinear parabolic problem with homogeneous Dirichlet boundary conditions involving variable exponent. *SeMA Journal*, 76(2), 153-180. https://doi.org/10.1007/s40324-018-0170-5
14. Kyelem, B. A., Ouedraogo, A., & Zongo, F. D. Y. (2022). Existence and uniqueness of renormalized solutions to nonlinear multivalued parabolic problem with homogeneous Dirichlet boundary conditions involving variable exponent. *Gulf Journal of Mathematics*, Vol. 12, Issue 1, 41–55. https://doi.org/10.56947/gjom.v12i1.778
15. Landes, R. (1986). A remark on the existence proof of Hopf's solution of the Navier-Stokes equation. *Archiv der Mathematik*, 47(4), 367-371. https://doi.org/10.1007/BF01213486

16. Lax, P. D., & Milgram, A. N. (1954). Parabolic equations. In *Contributions to the Theory of Partial Differential Equations* (Annals of Mathematics Studies, Vol. 33, pp. 167-190). Princeton University Press. https://doi.org/10.1515/9781400882182-013
17. Leray, J. (1934). Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Mathematica*, 63, 193-248. https://doi.org/10.1007/BF02547354
18. Leray, J. (1935). Etude de diverses équations intégrales non linéaires et applications au mouvement d'un liquide visqueux. *Journal de Mathématiques Pures et Appliquées*, 14, 331-418. https://doi.org/10.1016/S0021-7824(35)80002-9
19. Łukaszewicz, G., & Kalita, P. (2016). *Navier-Stokes Equations: An Introduction with Applications* (Advances in Mechanics and Mathematics, Vol. 34). Springer. https://doi.org/10.1007/978-3-319-27760-8
20. Navier, C. L. (1821). Sur les lois des mouvements des fluides, en ayant égard à l'adhésion des molécules. *Annales de Chimie et de Physique*, XIX, 244-260.
21. Ouaro, S., & Sawadogo, N. (2024). Nonlinear multivalued homogeneous Robin boundary $p(u)$-laplacian problem. *Annals of Mathematics and Computer Science*, Vol. 24, pp. 57-84. https://doi.org/10.56947/amcs.v24.308
22. Ouya, J., & Ouedraogo, A. (2024). A well-defined Godunov-type scheme in a Navier-Stokes model with a friction term. *Gulf Journal of Mathematics*, *16*(2), 213–231. https://doi.org/10.56947/gjom.v16i2.1882
23. Pata, V., & Miranville, A. (2012). On the regularity of solutions to the Navier-Stokes equations. *Communications on Pure and Applied Analysis*, 11(2), 747-761. https://doi.org/10.3934/cpaa.2012.11.747
24. Stokes, G. G. (1880). On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids (1845). In *Mathematical and Physical Papers by George Gabriel Stokes* (Vol. I, pp. 75-129). Cambridge University Press. https://doi.org/10.1017/CBO9780511702266.004
25. Stokes, G. G. (1901). On the effect of the internal friction of fluids on the motion of pendulums (1850). In *Mathematical and Physical Papers by George Gabriel Stokes* (Vol. III, pp. 1-154). Cambridge University Press. https://doi.org/10.1017/CBO9780511702266.005
26. Temam, R. (1984). *Navier-Stokes Equations: Theory and Numerical Analysis* (3rd ed.). North-Holland. https://doi.org/10.1016/s0168-2024(08)x6001-4
27. Temam, R., & Ziane, M. (1996). Navier-Stokes equations in three-dimensional thin domains with various boundary conditions. *Advances in Differential Equations*, 1(4), 499-546. https://doi.org/10.57262/ade/1366895811
28. Vishik, M. I., & Chepyzhov, V. V. (2002). Trajectory and global attractors of three-dimensional Navier-Stokes system. *Mathematical Notes*, 71(2), 177-193. https://doi.org/10.1023/A:1013948919494
29. Wang, X. (2000). Effect of tangential derivative in the boundary layer on time averaged energy dissipation rate. *Physica D*, 144(1-2), 142-153. https://doi.org/10.1016/S0167-2789(00)00074-6
30. Ziane, M. (1998). On the 2D-Navier-Stokes equations with the free boundary condition. *Applied Mathematics and Optimization*, 38(1), 1-19. https://doi.org/10.1007/s002459900079
31. Ouya, J., & Ouedraogo, A. (2024). A well-defined Godunov-type scheme in a Navier-Stokes model with a friction term. Gulf Journal of Mathematics, 16(2), 213-231. https://doi.org/10.56947/gjom.v16i2.1882
32. Khaider, H., Ouidirne, F., Azanzal, A., & Allalou, C. (2024). Well-posedness for the stochastic incompressible Hall-magnetohydrodynamic system. Gulf Journal of Mathematics, 17(1), 191-204. https://doi.org/10.56947/gjom.v17i1.2080

[1] Département de Mathématiques et Informatique, Laboratoire de Mathématiques, Informatique et Applications (L@MIA), Université Norbert ZONGO, BP 376 Koudougou, Burkina Faso.
Email address: simonzongo14arti@gmail.com

[2] Département de Mathématiques et Informatique, Unité de Formation et de Recherche en Sciences et Technologies, Université Lédéa Bernard Ouédraogo, Burkina Faso.
Email address: kyeleadoc@yahoo.fr