

ON COMMON FIXED POINT THEOREMS USING $(E.A)$ -PROPERTY AND SATISFYING Φ -MAPS IN $\mathcal{S}$ -METRIC SPACES

GURUCHARAN SINGH SALUJA

ABSTRACT. The purpose of this paper is to prove some common fixed point theorems for the pair of self-mappings satisfying $(E.A)$ -property and some Φ -type contractive condition in the setting of $\mathcal{S}$ -metric spaces. We also illustrate some examples to support the results. The results of findings in this paper generalize, extend, unify and enrich some previous works in the existing literature.

1. INTRODUCTION

In 1922, S. Banach ([6]) established a result, which is well-known as Banach Fixed Point Theorem (or Banach Contraction Principle, or in short BCP) to establish the existence of solutions for nonlinear operator equations and integral equations.

In the literature, there are plenty of extensions of the famous Banach contraction principle, which states that every self mapping $\mathcal{Q}$ defined on a complete metric space $(\mathcal{Q}, \rho)$ satisfying

$$\rho(\mathcal{Q}(x), \mathcal{Q}(y)) \leq \alpha \rho(x, y), \quad (1.1)$$

for all $x, y \in \mathcal{Q}$, where $\alpha \in (0, 1)$, has a unique fixed point and for every $q_0 \in \mathcal{Q}$ a sequence $\{\mathcal{Q}^n q_0\}_{n \geq 1}$ is convergent to the fixed point. Inequality (1.1) also implies the continuity of $\mathcal{Q}$. Some of the extensions weaken right side of inequality in the condition (1.1) by replacing α with a mapping, see e.g., Boyd and Wong [7] and Rakotch [19]. In other results, the underlying space is more general, see e.g., [3, 4, 5, 8, 9, 12, 15, 22, 24, 25, 26, 27].

In 2012, Sedghi et al. [20] introduced the notion of an $\mathcal{S}$ -metric as a generalization of a metric and give some of their properties. Following is the definition of generalized metric spaces or $\mathcal{S}$ -metric spaces (see, [20]).

Definition 1.1. ([20]) Let X be a nonempty set and let $\mathcal{S}: X^3 \rightarrow [0, \infty)$ be a function satisfying the following conditions for all $u, v, z, t \in X$:

(S1) $\mathcal{S}(u, v, z) = 0$ if and only if $u = v = z$;

Date: Received: Jul 1, 2022; Accepted: Jul 27, 2022.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.

Key words and phrases. Common fixed point, $(E.A)$ property, Φ -type contractive condition, $\mathcal{S}$ -metric space.

(S2) $\mathcal{S}(u, v, z) \leq \mathcal{S}(u, u, t) + \mathcal{S}(v, v, t) + \mathcal{S}(z, z, t)$.

Then the function $\mathcal{S}$ is called an $\mathcal{S}$ -metric on X and the pair $(X, \mathcal{S})$ is called an $\mathcal{S}$ -metric space.

Some examples of $\mathcal{S}$ -metric spaces are as follows.

Example 1.2. ([20])

(1) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X , then $\mathcal{S}(u, v, z) = \|v + z - 2u\| + \|v - z\|$ is an $\mathcal{S}$ -metric on X .

(2) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X , then $\mathcal{S}(u, v, z) = \|u - z\| + \|v - z\|$ is an $\mathcal{S}$ -metric on X .

Example 1.3. ([21]) Let $X = \mathbb{R}$ be the real line. Then $\mathcal{S}(u, v, z) = |u - z| + |v - z|$ for all $u, v, z \in \mathbb{R}$ is an $\mathcal{S}$ -metric on X . This $\mathcal{S}$ -metric on X is called the usual $\mathcal{S}$ -metric on X .

Example 1.4. ([16]) Let X be a non-empty set and d be an ordinary metric on X . Then $\mathcal{S}(u, v, z) = d(u, z) + d(v, z)$ for all $u, v, z \in \mathbb{R}$ is an $\mathcal{S}$ -metric on X .

Example 1.5. ([23]) Let X be a non-empty set and d_1, d_2 be two ordinary metrics on X . Then $\mathcal{S}(u, v, z) = d_1(u, z) + d_2(v, z)$ for all $u, v, z \in X$ is an $\mathcal{S}$ -metric on X .

Recently, Sedghi et al. [23] (*Mathematical Sciences (2018)12:137-143*) have proved some common fixed point theorems for four mappings satisfying generalized contractive condition in the setting of $\mathcal{S}$ -metric spaces and give examples to validate the results. The results presented in this paper extend and improve several results in the literature.

In the present paper, we prove some unique common fixed point theorems using (E.A)-property and satisfying some Φ -type contractive condition in the setting of $\mathcal{S}$ -metric spaces and give some corollaries of the main result. We also illustrate some examples to support the results. Our results generalize, extend and enrich several existing results in the literature.

2. BASIC PROPERTIES AND LEMMAS

We need the following definitions and lemmas in the sequel.

Definition 2.1. Let $(X, \mathcal{S})$ be an $\mathcal{S}$ -metric space. For $r > 0$ and $x \in X$ we define the open ball $\mathcal{B}_{\mathcal{S}}(u, r)$ and closed ball $\mathcal{B}_{\mathcal{S}}[u, r]$ with center u and radius r as follows, respectively:

$$\mathcal{B}_{\mathcal{S}}(u, r) = \{v \in X : \mathcal{S}(v, v, u) < r\},$$

$$\mathcal{B}_{\mathcal{S}}[u, r] = \{v \in X : \mathcal{S}(v, v, u) \leq r\}.$$

Example 2.2. ([21]) Let $X = \mathbb{R}$. Denote $\mathcal{S}(u, v, z) = |v + z - 2u| + |v - z|$ for all $u, v, z \in \mathbb{R}$. Then

$$\begin{aligned} \mathcal{B}_{\mathcal{S}}(1, 2) &= \{v \in \mathbb{R} : \mathcal{S}(v, v, 1) < 2\} = \{v \in \mathbb{R} : |v - 1| < 1\} \\ &= \{v \in \mathbb{R} : 0 < v < 2\} = (0, 2), \end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_S[2, 4] &= \{v \in \mathbb{R} : \mathcal{S}(v, v, 2) \leq 4\} = \{v \in \mathbb{R} : |v - 2| \leq 2\} \\ &= \{v \in \mathbb{R} : 0 \leq v \leq 4\} = [0, 4].\end{aligned}$$

Definition 2.3. ([20], [21]) Let $(X, \mathcal{S})$ be an $\mathcal{S}$ -metric space and $A \subset X$.

(1) The subset A is said to be an open subset of X , if for every $u \in A$ there exists $r > 0$ such that $\mathcal{B}_S(u, r) \subset A$.

(2) A sequence $\{u_n\}$ in X converges to $u \in X$ if $\mathcal{S}(u_n, u_n, u) \rightarrow 0$ as $n \rightarrow \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\mathcal{S}(u_n, u_n, u) < \varepsilon$. We denote this by $\lim_{n \rightarrow \infty} u_n = u$ or $u_n \rightarrow u$ as $n \rightarrow \infty$.

(3) A sequence $\{u_n\}$ in X is called a Cauchy sequence if $\mathcal{S}(u_n, u_n, u_m) \rightarrow 0$ as $n, m \rightarrow \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $\mathcal{S}(u_n, u_n, u_m) < \varepsilon$.

(4) The $\mathcal{S}$ -metric space $(X, \mathcal{S})$ is called complete if every Cauchy sequence in X is convergent in X .

(5) Let τ be the set of all $A \subset X$ with $u \in A$ and there exists $r > 0$ such that $\mathcal{B}_S(u, r) \subset A$. Then τ is a topology on X (induced by the $\mathcal{S}$ -metric space).

(6) A nonempty subset A of X is $\mathcal{S}$ -closed if closure of A is equal to A .

Definition 2.4. ([20]) Let $(X, \mathcal{S})$ be an $\mathcal{S}$ -metric space. A mapping $\mathcal{T}: X \rightarrow X$ is said to be a contraction if there exists a constant $0 \leq \mathcal{L} < 1$ such that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}z) \leq \mathcal{L} \mathcal{S}(u, v, z) \quad (2.1)$$

for all $u, v, z \in X$. If the $\mathcal{S}$ -metric space $(X, \mathcal{S})$ is complete then the mapping defined as above has a unique fixed point.

Definition 2.5. ([20]) Let $(X, \mathcal{S})$ and $(Y, \mathcal{S}')$ be two $\mathcal{S}$ -metric spaces. A function $g: X \rightarrow Y$ is said to be continuous at a point $u_0 \in X$ if for every sequence $\{u_n\}$ in X with $\mathcal{S}(u_n, u_n, u_0) \rightarrow 0$, $\mathcal{S}'(g(u_n), g(u_n), g(u_0)) \rightarrow 0$ as $n \rightarrow \infty$. We say that g is continuous on X if g is continuous at every point $u_0 \in X$.

Definition 2.6. Let X be a non-empty set and let $\mathcal{R}, f: X \rightarrow X$ be two self mappings of X . Then a point $u \in X$ is called a

- (i) fixed point of operator $\mathcal{R}$ if $\mathcal{R}(u) = u$;
- (ii) common fixed point of $\mathcal{R}$ and f if $\mathcal{R}(u) = f(u) = u$.

Definition 2.7. ([2]) Let $\mathcal{A}$ and $\mathcal{B}$ be single valued self-mappings on a set X . If $z = \mathcal{A}u = \mathcal{B}u$ for some $u \in X$, then u is called a coincidence point of $\mathcal{A}$ and $\mathcal{B}$, and z is called a point of coincidence of $\mathcal{A}$ and $\mathcal{B}$.

Definition 2.8. ([13]) Let $\mathcal{A}$ and $\mathcal{B}$ be single valued self-mappings on a set X . Mappings $\mathcal{A}$ and $\mathcal{B}$ are said to be commuting if $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u$ for all $u \in X$.

Definition 2.9. ([14]) Let $\mathcal{A}$ and $\mathcal{B}$ be single valued self-mappings on a set X . Mappings $\mathcal{A}$ and $\mathcal{B}$ are said to be weakly compatible if they commute at their coincidence points, i.e., if $\mathcal{A}u = \mathcal{B}u$ for some $u \in X$ implies $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u$.

Definition 2.10. ([1]) Let $(X, \mathcal{S})$ be an $\mathcal{S}$ -metric space and let $\mathcal{A}, \mathcal{Q}: X \rightarrow X$ be two self mappings of X . The pair $(\mathcal{A}, \mathcal{Q})$ is said to satisfy $(E.A)$ -property if there exists a sequence $\{y_n\}$ in X such that $\lim_{n \rightarrow \infty} \mathcal{A}y_n = \lim_{n \rightarrow \infty} \mathcal{Q}y_n = r$ for some $r \in X$.

Example 2.11. Let $X = [0, 1]$ and let $\mathcal{A}, \mathcal{Q}: X \rightarrow X$ be defined by $\mathcal{A}(u) = \frac{u}{2}$ and $\mathcal{Q}(u) = \frac{u}{4}$. Define the function $\mathcal{S}: X^3 \rightarrow [0, \infty)$ by

$$\mathcal{S}(u, v, z) = \begin{cases} 0, & \text{if } u = v = z, \\ \max\{u, v, z\}, & \text{if otherwise,} \end{cases}$$

for all $u, v, z \in X$, then $\mathcal{S}$ is an $\mathcal{S}$ -metric on X . Now, we show that the pair $(\mathcal{A}, \mathcal{Q})$ satisfies the (E.A) property. For this, consider the sequence $\{y_n\} = \{\frac{1}{2n+1}\}_{n \geq 1}$. Clearly $\{y_n\}$ is in X and note that $\mathcal{A}y_n = \frac{y_n}{2} = \frac{1}{2(2n+1)}$ and $\mathcal{Q}y_n = \frac{y_n}{4} = \frac{1}{4(2n+1)}$ for all $n \in \mathbb{N}$. This implies that

$$\begin{aligned} \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, 0) &= \mathcal{S}\left(\frac{1}{2(2n+1)}, \frac{1}{2(2n+1)}, 0\right) = \max\left\{\frac{1}{2(2n+1)}, \frac{1}{2(2n+1)}, 0\right\} \\ &= \frac{1}{2(2n+1)} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This shows that $\mathcal{A}y_n \rightarrow 0$ as $n \rightarrow \infty$.

Also note that

$$\begin{aligned} \mathcal{S}(\mathcal{Q}y_n, \mathcal{Q}y_n, 0) &= \mathcal{S}\left(\frac{1}{4(2n+1)}, \frac{1}{4(2n+1)}, 0\right) = \max\left\{\frac{1}{4(2n+1)}, \frac{1}{4(2n+1)}, 0\right\} \\ &= \frac{1}{4(2n+1)} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This shows that $\mathcal{Q}y_n \rightarrow 0$ as $n \rightarrow \infty$. Thus there exists a sequence $\{y_n\}$ in X such that $\mathcal{A}y_n \rightarrow 0$ and $\mathcal{Q}y_n \rightarrow 0$ as $n \rightarrow \infty$. Hence the pair $(\mathcal{A}, \mathcal{Q})$ satisfies (E.A) property.

Lemma 2.12. ([20], Lemma 2.5) *Let $(X, \mathcal{S})$ be an $\mathcal{S}$ -metric space. Then, we have $\mathcal{S}(u, u, v) = \mathcal{S}(v, v, u)$ for all $u, v \in X$.*

Lemma 2.13. ([20], Lemma 2.12) *Let $(X, \mathcal{S})$ be an $\mathcal{S}$ -metric space. If $u_n \rightarrow u$ and $v_n \rightarrow v$ as $n \rightarrow \infty$ then $\mathcal{S}(u_n, u_n, v_n) \rightarrow \mathcal{S}(u, u, v)$ as $n \rightarrow \infty$.*

Lemma 2.14. ([10], Lemma 8) *Let $(X, \mathcal{S})$ be an $\mathcal{S}$ -metric space and A is a nonempty subset of X . Then A is said to be $\mathcal{S}$ -closed if and only if for any sequence $\{u_n\}$ in A such that $u_n \rightarrow u$ as $n \rightarrow \infty$, then $u \in A$.*

Lemma 2.15. ([20]) *Let $(X, \mathcal{S})$ be an $\mathcal{S}$ -metric space. If $r > 0$ and $u \in X$, then the ball $\mathcal{B}_{\mathcal{S}}(u, r)$ is a subset of X .*

Lemma 2.16. ([21]) *The limit of $\{u_n\}$ in $\mathcal{S}$ -metric space $(X, \mathcal{S})$ is unique.*

Lemma 2.17. ([20]) *Let $(X, \mathcal{S})$ be an $\mathcal{S}$ -metric space. Then the convergent sequence $\{u_n\}$ in X is Cauchy.*

In the following lemma we see the relationship between a metric and $\mathcal{S}$ -metric.

Lemma 2.18. ([11]) *Let (X, d) be a metric space. Then the following properties are satisfied:*

- (1) $\mathcal{S}_d(u, v, z) = d(u, z) + d(v, z)$ for all $u, v, z \in X$ is an $\mathcal{S}$ -metric on X .
- (2) $u_n \rightarrow u$ in (X, d) if and only if $u_n \rightarrow u$ in $(X, \mathcal{S}_d)$.
- (3) $\{u_n\}$ is Cauchy in (X, d) if and only if $\{u_n\}$ is Cauchy in $(X, \mathcal{S}_d)$.
- (4) (X, d) is complete if and only if $(X, \mathcal{S}_d)$ is complete.

We call the function $\mathcal{S}_d$ defined in Lemma 2.18 (1) as the $\mathcal{S}$ -metric generated by the metric d . It can be found an example of an $\mathcal{S}$ -metric which is not generated by any metric in [11, 18].

Proposition 2.19. ([2]) *Let $\mathcal{A}$ and $\mathcal{B}$ be weakly compatible self mappings on a set X . If $\mathcal{A}$ and $\mathcal{B}$ have a unique point of coincidence $z = \mathcal{A}u = \mathcal{B}u$, then z is the unique common fixed point of $\mathcal{A}$ and $\mathcal{B}$.*

In 1977, Matkowski [17] (*Proc. Amer. Math. Soc.* 62 (1977), 344-348) introduced the Φ -maps as the following: let Φ be the set of all functions ϕ such that $\phi: [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function satisfying $\lim_{n \rightarrow \infty} \phi^n(t) = 0$ for all $t \in (0, \infty)$. If $\phi \in \Phi$, then ϕ is called a Φ -map. Furthermore, if ϕ is a Φ -map, then

- (ϕ_1) $\phi(t) < t$ for all $t \in (0, \infty)$;
- (ϕ_2) $\phi(0) = 0$.

3. COMMON FIXED POINT THEOREMS

In this section, we prove some unique common fixed point theorems for the pair of self-mappings satisfying (E.A) property and some Φ -type contractive conditions in the setting of $\mathcal{S}$ -metric spaces.

Theorem 3.1. *Let $(X, \mathcal{S})$ be a $\mathcal{S}$ -metric space and let $\mathcal{A}, \mathcal{B}, \mathcal{R}, \mathcal{T}: X \rightarrow X$ be four self-mappings of X satisfying the following conditions:*

(i)

$$\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) \leq \phi \left(\max \left\{ \mathcal{S}(\mathcal{R}u, \mathcal{R}u, \mathcal{T}v), \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u), \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v), \right. \right. \\ \left. \left. \frac{1}{2}[\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{T}v) + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{R}u)] \right\} \right) \quad (3.1)$$

for all $u, v \in X$, where $\phi \in \Phi$;

- (ii) the pairs $(\mathcal{A}, \mathcal{R})$ and $(\mathcal{B}, \mathcal{T})$ are weakly compatible;
- (iii) one of the pairs $(\mathcal{A}, \mathcal{R})$ or $(\mathcal{B}, \mathcal{T})$ satisfy (E.A) property;
- (iv) $\mathcal{A}(X) \subseteq \mathcal{T}(X)$ and $\mathcal{B}(X) \subseteq \mathcal{R}(X)$.

If the range of one of the mappings $\mathcal{R}(X)$ or $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\mathcal{A}, \mathcal{B}, \mathcal{R}$ and $\mathcal{T}$ have a unique common fixed point in X .

Proof. First, we suppose that the pair $(\mathcal{A}, \mathcal{R})$ satisfy (E.A) property. Then by definition 2.10, there exists a sequence $\{y_n\}$ in X such that $\lim_{n \rightarrow \infty} \mathcal{A}y_n = \lim_{n \rightarrow \infty} \mathcal{R}y_n = \alpha$ for some $\alpha \in X$. Further, since $\mathcal{A}(X) \subseteq \mathcal{T}(X)$, there exists a sequence $\{z_n\}$ in X such that $\lim_{n \rightarrow \infty} \mathcal{A}y_n = \lim_{n \rightarrow \infty} \mathcal{T}z_n$. Hence $\lim_{n \rightarrow \infty} \mathcal{T}z_n = \alpha$. We claim that $\lim_{n \rightarrow \infty} \mathcal{B}z_n = \alpha$. If not, then putting $u = y_n$ and $v = z_n$ in inequality (3.1) and using Lemma 2.12, property of ϕ and (S1), we have

$$\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)$$

$$\begin{aligned}
&\leq \phi\left(\max\left\{\mathcal{S}(\mathcal{R}y_n, \mathcal{R}y_n, \mathcal{T}z_n), \mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n), \mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{T}z_n), \right.\right. \\
&\quad \left.\left.\frac{1}{2}[\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{T}z_n) + \mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{R}y_n)]\right\}\right) \\
&= \phi\left(\max\left\{\mathcal{S}(\mathcal{R}y_n, \mathcal{R}y_n, \mathcal{A}y_n), \mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n), \mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n), \right.\right. \\
&\quad \left.\left.\frac{1}{2}[\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{A}y_n) + \mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n)]\right\}\right) \\
&= \phi\left(\max\left\{0, \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n), \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n), \frac{1}{2}[\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)]\right\}\right) \\
&\leq \phi\left(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)\right) \\
&< \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n), \tag{3.2}
\end{aligned}$$

which is a contradiction. Hence, we have $\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n) = 0$, that is, $\mathcal{A}y_n = \mathcal{B}z_n$. Letting $n \rightarrow \infty$, we get $\lim_{n \rightarrow \infty} \mathcal{A}y_n = \lim_{n \rightarrow \infty} \mathcal{B}z_n = \alpha$.

Now, first we assume that $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\alpha = \mathcal{T}\beta$ for some $\beta \in X$. Subsequently, we have

$$\lim_{n \rightarrow \infty} \mathcal{B}z_n = \lim_{n \rightarrow \infty} \mathcal{A}y_n = \lim_{n \rightarrow \infty} \mathcal{R}y_n = \lim_{n \rightarrow \infty} \mathcal{T}z_n = \mathcal{T}\beta = \alpha.$$

We claim that $\mathcal{B}\beta = \mathcal{T}\beta$. For this, putting $u = y_n$ and $v = \beta$ in inequality (3.1) and using Lemma 2.12, we have

$$\begin{aligned}
\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}\beta) &\leq \phi\left(\max\left\{\mathcal{S}(\mathcal{R}y_n, \mathcal{R}y_n, \mathcal{T}\beta), \mathcal{S}(\mathcal{B}\beta, \mathcal{B}\beta, \mathcal{A}y_n), \mathcal{S}(\mathcal{B}\beta, \mathcal{B}\beta, \mathcal{T}\beta), \right.\right. \\
&\quad \left.\left.\frac{1}{2}[\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{T}\beta) + \mathcal{S}(\mathcal{B}\beta, \mathcal{B}\beta, \mathcal{R}y_n)]\right\}\right). \tag{3.3}
\end{aligned}$$

Letting $n \rightarrow \infty$ in (3.3) and using Lemma 2.12, property of ϕ and (S1), we get

$$\begin{aligned}
\mathcal{S}(\mathcal{T}\beta, \mathcal{T}\beta, \mathcal{B}\beta) &\leq \phi\left(\max\left\{\mathcal{S}(\mathcal{T}\beta, \mathcal{T}\beta, \mathcal{T}\beta), \mathcal{S}(\mathcal{B}\beta, \mathcal{B}\beta, \mathcal{T}\beta), \mathcal{S}(\mathcal{B}\beta, \mathcal{B}\beta, \mathcal{T}\beta), \right.\right. \\
&\quad \left.\left.\frac{1}{2}[\mathcal{S}(\mathcal{T}\beta, \mathcal{T}\beta, \mathcal{T}\beta) + \mathcal{S}(\mathcal{B}\beta, \mathcal{B}\beta, \mathcal{T}\beta)]\right\}\right) \\
&= \phi\left(\max\left\{0, \mathcal{S}(\mathcal{T}\beta, \mathcal{T}\beta, \mathcal{B}\beta), \mathcal{S}(\mathcal{T}\beta, \mathcal{T}\beta, \mathcal{B}\beta), \frac{1}{2}\mathcal{S}(\mathcal{T}\beta, \mathcal{T}\beta, \mathcal{B}\beta)\right\}\right) \\
&= \phi\left(\mathcal{S}(\mathcal{T}\beta, \mathcal{T}\beta, \mathcal{B}\beta)\right) \\
&< \mathcal{S}(\mathcal{T}\beta, \mathcal{T}\beta, \mathcal{B}\beta), \tag{3.4}
\end{aligned}$$

which is a contradiction. Hence, we have $\mathcal{S}(\mathcal{T}\beta, \mathcal{T}\beta, \mathcal{B}\beta) = 0$, that is, $\mathcal{T}\beta = \mathcal{B}\beta = \alpha$. Hence β is a coincidence point of the mappings $\mathcal{B}$ and $\mathcal{T}$, that is, the pair $(\mathcal{B}, \mathcal{T})$. Now, the weak compatibility of the pair $(\mathcal{B}, \mathcal{T})$ implies that $\mathcal{B}\mathcal{T}\beta = \mathcal{T}\mathcal{B}\beta$ or $\mathcal{B}\alpha = \mathcal{T}\alpha$.

On the other hand, since $\mathcal{B}(X) \subseteq \mathcal{R}(X)$, there exists $\mu \in X$ such that $\mathcal{B}\beta = \mathcal{R}\mu$. Thus $\mathcal{T}\beta = \mathcal{B}\beta = \mathcal{R}\mu = \alpha$. Let us show that μ is a coincidence point of the pair $(\mathcal{A}, \mathcal{R})$, that is, $\mathcal{A}\mu = \mathcal{R}\mu = \alpha$. If not, then putting $u = \mu$ and $v = \beta$ in

inequality (3.1) and using Lemma 2.12, property of ϕ and (S1), we get

$$\begin{aligned}
\mathcal{S}(\mathcal{A}\mu, \mathcal{A}\mu, \mathcal{B}\beta) &\leq \phi\left(\max\left\{\mathcal{S}(\mathcal{R}\mu, \mathcal{R}\mu, \mathcal{T}\beta), \mathcal{S}(\mathcal{B}\beta, \mathcal{B}\beta, \mathcal{A}\mu), \mathcal{S}(\mathcal{B}\beta, \mathcal{B}\beta, \mathcal{T}\beta), \right.\right. \\
&\quad \left.\left.\frac{1}{2}[\mathcal{S}(\mathcal{A}\mu, \mathcal{A}\mu, \mathcal{T}\beta) + \mathcal{S}(\mathcal{B}\beta, \mathcal{B}\beta, \mathcal{R}\mu)]\right\}\right) \\
&= \phi\left(\max\left\{\mathcal{S}(\alpha, \alpha, \alpha), \mathcal{S}(\mathcal{A}\mu, \mathcal{A}\mu, \mathcal{B}\beta), \mathcal{S}(\alpha, \alpha, \alpha), \right.\right. \\
&\quad \left.\left.\frac{1}{2}[\mathcal{S}(\alpha, \alpha, \alpha) + \mathcal{S}(\mathcal{A}\mu, \mathcal{A}\mu, \mathcal{B}\beta)]\right\}\right) \\
&= \phi\left(\max\left\{0, \mathcal{S}(\mathcal{A}\mu, \mathcal{A}\mu, \mathcal{B}\beta), 0, \frac{1}{2}\mathcal{S}(\mathcal{A}\mu, \mathcal{A}\mu, \mathcal{B}\beta)\right\}\right) \\
&= \phi\left(\mathcal{S}(\mathcal{A}\mu, \mathcal{A}\mu, \mathcal{B}\beta)\right) \\
&< \mathcal{S}(\mathcal{A}\mu, \mathcal{A}\mu, \mathcal{B}\beta),
\end{aligned} \tag{3.5}$$

which is a contradiction. Hence, we have $\mathcal{S}(\mathcal{A}\mu, \mathcal{A}\mu, \mathcal{B}\beta) = 0$, that is, $\mathcal{S}(\mathcal{A}\mu, \mathcal{A}\mu, \mathcal{R}\mu) = 0$ and hence $\mathcal{A}\mu = \mathcal{R}\mu = \alpha$. Thus μ is a coincidence point of the mappings $\mathcal{A}$ and $\mathcal{R}$. Further, the weak compatibility of the pair $(\mathcal{A}, \mathcal{R})$ implies that $\mathcal{A}\mathcal{R}\mu = \mathcal{R}\mathcal{A}\mu$ or $\mathcal{A}\alpha = \mathcal{R}\alpha$. Hence α is a common coincidence point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ and $\mathcal{T}$.

Now to show that α is a common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ and $\mathcal{T}$. For this, we put $u = \mu$ and $v = \alpha$ in (3.1) and using Lemma 2.12, (S1) and the property of ϕ , we get

$$\begin{aligned}
\mathcal{S}(\alpha, \alpha, \mathcal{B}\alpha) &= \mathcal{S}(\mathcal{A}\mu, \mathcal{A}\mu, \mathcal{B}\alpha) \\
&\leq \phi\left(\max\left\{\mathcal{S}(\mathcal{R}\mu, \mathcal{R}\mu, \mathcal{T}\alpha), \mathcal{S}(\mathcal{B}\alpha, \mathcal{B}\alpha, \mathcal{A}\mu), \mathcal{S}(\mathcal{B}\alpha, \mathcal{B}\alpha, \mathcal{T}\alpha), \right.\right. \\
&\quad \left.\left.\frac{1}{2}[\mathcal{S}(\mathcal{A}\mu, \mathcal{A}\mu, \mathcal{T}\alpha) + \mathcal{S}(\mathcal{B}\alpha, \mathcal{B}\alpha, \mathcal{R}\mu)]\right\}\right) \\
&= \phi\left(\max\left\{\mathcal{S}(\alpha, \alpha, \mathcal{B}\alpha), \mathcal{S}(\alpha, \alpha, \mathcal{B}\alpha), \mathcal{S}(\mathcal{B}\alpha, \mathcal{B}\alpha, \mathcal{B}\alpha), \right.\right. \\
&\quad \left.\left.\frac{1}{2}[\mathcal{S}(\alpha, \alpha, \mathcal{B}\alpha) + \mathcal{S}(\alpha, \alpha, \mathcal{B}\alpha)]\right\}\right) \\
&= \phi\left(\max\left\{\mathcal{S}(\alpha, \alpha, \mathcal{B}\alpha), \mathcal{S}(\alpha, \alpha, \mathcal{B}\alpha), 0, \mathcal{S}(\alpha, \alpha, \mathcal{B}\alpha)\right\}\right) \\
&= \phi\left(\mathcal{S}(\alpha, \alpha, \mathcal{B}\alpha)\right) \\
&< \mathcal{S}(\alpha, \alpha, \mathcal{B}\alpha),
\end{aligned} \tag{3.6}$$

which is a contradiction. Hence, we have $\mathcal{S}(\alpha, \alpha, \mathcal{B}\alpha) = 0$, that is, $\alpha = \mathcal{B}\alpha$. Consequently, $\mathcal{A}\alpha = \mathcal{B}\alpha = \mathcal{R}\alpha = \mathcal{T}\alpha = \alpha$. This shows that α is a common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ and $\mathcal{T}$.

Similar argument arises if we assume that $\mathcal{R}(X)$ is a complete subspace of $(X, \mathcal{S})$.

Similarly, the property (E.A) of the pair $(\mathcal{B}, \mathcal{T})$ will give the similar result.

Now, we show the uniqueness of the common fixed point. For this, let us assume that δ be another common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ and $\mathcal{T}$ with $\delta \neq \alpha$. From inequality (3.1) and using Lemma 2.12, the property of ϕ and (S1) for $u = \delta$

and $v = \alpha$, we have

$$\begin{aligned}
\mathcal{S}(\delta, \delta, \alpha) &= \mathcal{S}(\mathcal{A}\delta, \mathcal{A}\delta, \mathcal{B}\alpha) \\
&\leq \phi\left(\max\left\{\mathcal{S}(\mathcal{R}\delta, \mathcal{R}\delta, \mathcal{T}\alpha), \mathcal{S}(\mathcal{B}\alpha, \mathcal{B}\alpha, \mathcal{A}\delta), \mathcal{S}(\mathcal{B}\alpha, \mathcal{B}\alpha, \mathcal{T}\alpha), \right.\right. \\
&\quad \left.\left.\frac{1}{2}[\mathcal{S}(\mathcal{A}\delta, \mathcal{A}\delta, \mathcal{T}\alpha) + \mathcal{S}(\mathcal{B}\alpha, \mathcal{B}\alpha, \mathcal{R}\delta)]\right\}\right) \\
&= \phi\left(\max\left\{\mathcal{S}(\delta, \delta, \alpha), \mathcal{S}(\alpha, \alpha, \delta), \mathcal{S}(\alpha, \alpha, \alpha), \frac{1}{2}[\mathcal{S}(\delta, \delta, \alpha) + \mathcal{S}(\alpha, \alpha, \delta)]\right\}\right) \\
&= \phi\left(\max\left\{\mathcal{S}(\delta, \delta, \alpha), \mathcal{S}(\delta, \delta, \alpha), 0, \frac{1}{2}[\mathcal{S}(\delta, \delta, \alpha) + \mathcal{S}(\delta, \delta, \alpha)]\right\}\right) \\
&= \phi\left(\max\left\{\mathcal{S}(\delta, \delta, \alpha), \mathcal{S}(\delta, \delta, \alpha), 0, \mathcal{S}(\delta, \delta, \alpha)\right\}\right) \\
&= \phi\left(\mathcal{S}(\delta, \delta, \alpha)\right) \\
&< \mathcal{S}(\delta, \delta, \alpha),
\end{aligned} \tag{3.7}$$

which is a contradiction. Hence, we conclude that $\mathcal{S}(\delta, \delta, \alpha) = 0$, that is, $\alpha = \delta$. This shows that the common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ and $\mathcal{T}$ is unique. This completes the proof. $\square$

Corollary 3.2. *Let $(X, \mathcal{S})$ be a $\mathcal{S}$ -metric space and let $\mathcal{A}, \mathcal{R}: X \rightarrow X$ be two self-mappings of X satisfying the following conditions:*

(i)

$$\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{A}v) \leq k \max\left\{\mathcal{S}(\mathcal{R}u, \mathcal{R}u, \mathcal{R}v), \mathcal{S}(\mathcal{A}v, \mathcal{A}v, \mathcal{A}u), \mathcal{S}(\mathcal{A}v, \mathcal{A}v, \mathcal{R}v), \right. \\
\left. \frac{1}{2}[\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{R}v) + \mathcal{S}(\mathcal{A}v, \mathcal{A}v, \mathcal{R}u)]\right\},$$

for all $u, v \in X$, where $0 < k < 1$;

(ii) the pair $(\mathcal{A}, \mathcal{R})$ is weakly compatible;

(iii) the pair $(\mathcal{A}, \mathcal{R})$ satisfies (E.A) property;

(iv) $\mathcal{A}(X) \subseteq \mathcal{R}(X)$.

If the range of the mapping $\mathcal{R}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\mathcal{A}$ and $\mathcal{R}$ have a unique common fixed point in X .

Proof. Putting $\phi(t) = kt$, $t \geq 0$, $0 \leq k < 1$, $\mathcal{A} = \mathcal{B}$ and $\mathcal{R} = \mathcal{T}$ in inequality (3.1). Then all conditions of Theorem 3.1 are satisfied and hence the result follows. $\square$

Example 3.3. Let $X = [0, 2]$ and $\mathcal{S}(u, v, z) = \max\{|u - v|, |v - z|, |z - u|\}$ for all $u, v, z \in X$ and $\phi \in \Phi$. Denote $\mathcal{A}, \mathcal{R}: X \rightarrow X$ by

$$\mathcal{A}(u) = 1 \quad \text{and} \quad \mathcal{R}(u) = 2 - u.$$

We obtain that $\mathcal{A}$ and $\mathcal{R}$ satisfy the inequality of Corollary 3.2. Indeed, we have

$$\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{A}v) = 0,$$

and

$$\phi\left(\max\left\{\mathcal{S}(\mathcal{R}u, \mathcal{R}u, \mathcal{R}v), \mathcal{S}(\mathcal{A}v, \mathcal{A}v, \mathcal{A}u), \mathcal{S}(\mathcal{A}v, \mathcal{A}v, \mathcal{R}v), \right.\right. \\
\left.\left.\frac{1}{2}[\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{R}v) + \mathcal{S}(\mathcal{A}v, \mathcal{A}v, \mathcal{R}u)]\right\}\right)$$

$$= \phi \left(\max \left\{ |u - v|, 0, |1 - v|, \frac{1}{2} [|1 - v| + |1 - u|] \right\} \right).$$

That is,

$$\begin{aligned} & \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{A}v) \\ & \leq \phi \left(\max \left\{ \mathcal{S}(\mathcal{R}u, \mathcal{R}u, \mathcal{R}v), \mathcal{S}(\mathcal{A}v, \mathcal{A}v, \mathcal{A}u), \mathcal{S}(\mathcal{A}v, \mathcal{A}v, \mathcal{R}v), \right. \right. \\ & \quad \left. \left. \frac{1}{2} [\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{R}v) + \mathcal{S}(\mathcal{A}v, \mathcal{A}v, \mathcal{R}u)] \right\} \right). \end{aligned}$$

It is obvious that the range of $\mathcal{R}$ contains the range of $\mathcal{A}$ and $\mathcal{R}(X)$ is a complete subspace of $(X, \mathcal{S})$. Also, $\mathcal{A}$ and $\mathcal{R}$ are weakly compatible maps. Now, we show that the pair $(\mathcal{A}, \mathcal{R})$ satisfies $(E.A)$ property. For this, consider the sequence $\{r_n\} = \{1 + \frac{1}{2n+1}\}_{n \geq 1}$. Clearly the sequence $\{r_n\}$ is in X and note that $\mathcal{A}r_n = 1$ and $\mathcal{R}r_n = 2 - r_n = 2 - \{1 + \frac{1}{2n+1}\}$ for all $n \in \mathbb{N}$. This will implies that

$$\mathcal{S}(\mathcal{A}r_n, \mathcal{A}r_n, 1) = \mathcal{S}(1, 1, 1) = \max\{1, 1, 1\} = 1 \text{ as } n \rightarrow \infty.$$

This shows that $\mathcal{A}r_n \rightarrow 1$ as $n \rightarrow \infty$.

Also note that

$$\begin{aligned} \mathcal{S}(\mathcal{R}r_n, \mathcal{R}r_n, 1) &= \mathcal{S}\left(2 - \{1 + \frac{1}{2n+1}\}, 2 - \{1 + \frac{1}{2n+1}\}, 1\right) \\ &= \max \left\{ 2 - \{1 + \frac{1}{2n+1}\}, 2 - \{1 + \frac{1}{2n+1}\}, 1 \right\} \\ &= 2 - \{1 + \frac{1}{2n+1}\} \rightarrow 1 \text{ as } n \rightarrow \infty. \end{aligned}$$

This shows that $\mathcal{R}r_n \rightarrow 1$ as $n \rightarrow \infty$.

Thus there exists a sequence $\{r_n\}$ in X such that $\mathcal{A}r_n \rightarrow 1$ and $\mathcal{R}r_n \rightarrow 1$ as $n \rightarrow \infty$. Hence the pair $(\mathcal{A}, \mathcal{R})$ satisfies $(E.A)$ property.

Hence all the assumptions in Corollary 3.2 are satisfied. Consequently $\mathcal{A}$ and $\mathcal{R}$ have a unique common fixed point, say, $u = 1$ in X .

Theorem 3.4. *Let $(X, \mathcal{S})$ be a $\mathcal{S}$ -metric space and let $\mathcal{A}, \mathcal{B}, \mathcal{R}, \mathcal{T}: X \rightarrow X$ be four self-mappings of X satisfying the following conditions:*

(i)

$$\begin{aligned} \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) &\leq \max \left\{ \phi(\mathcal{S}(\mathcal{R}u, \mathcal{R}u, \mathcal{T}v)), \phi(\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u)), \right. \\ &\quad \phi(\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)), \\ &\quad \phi\left(\frac{1}{2}[\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{T}v) + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{R}u)]\right), \\ &\quad \left. \phi\left(\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u) \frac{[1 + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)]}{[1 + \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v)]}\right) \right\}, \end{aligned} \tag{3.8}$$

for all $u, v \in X$, where $\phi \in \Phi$;

- (ii) the pairs $(\mathcal{A}, \mathcal{R})$ and $(\mathcal{B}, \mathcal{T})$ are weakly compatible;
- (iii) one of the pairs $(\mathcal{A}, \mathcal{R})$ or $(\mathcal{B}, \mathcal{T})$ satisfy $(E.A)$ property;
- (iv) $\mathcal{A}(X) \subseteq \mathcal{T}(X)$ and $\mathcal{B}(X) \subseteq \mathcal{R}(X)$.

If the range of one of the mappings $\mathcal{R}(X)$ or $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\mathcal{A}, \mathcal{B}, \mathcal{R}$ and $\mathcal{T}$ have a unique common fixed point in X .

Proof. First, we suppose that the pair $(\mathcal{A}, \mathcal{R})$ satisfy (E.A) property. Then by definition 2.10, there exists a sequence $\{y_n\}$ in X such that $\lim_{n \rightarrow \infty} \mathcal{A}y_n = \lim_{n \rightarrow \infty} \mathcal{R}y_n = \alpha$ for some $\alpha \in X$. Further, since $\mathcal{A}(X) \subseteq \mathcal{T}(X)$, there exists a sequence $\{z_n\}$ in X such that $\lim_{n \rightarrow \infty} \mathcal{A}y_n = \lim_{n \rightarrow \infty} \mathcal{T}z_n$. Hence $\lim_{n \rightarrow \infty} \mathcal{T}z_n = \alpha$. We claim that $\lim_{n \rightarrow \infty} \mathcal{B}z_n = \alpha$. If not, then putting $u = y_n$ and $v = z_n$ in inequality (3.8) and using Lemma 2.12, (S1) and the property of ϕ , we have

$$\begin{aligned}
\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n) &\leq \max \left\{ \phi(\mathcal{S}(\mathcal{R}y_n, \mathcal{R}y_n, \mathcal{T}z_n)), \phi(\mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n)), \right. \\
&\quad \phi(\mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{T}z_n)), \\
&\quad \left. \phi\left(\frac{1}{2}[\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{T}z_n) + \mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{R}y_n)]\right), \right. \\
&\quad \left. \phi\left(\mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n) \frac{[1 + \mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{T}z_n)]}{[1 + \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)]}\right) \right\} \\
&= \max \left\{ \phi(\mathcal{S}(\mathcal{R}y_n, \mathcal{R}y_n, \mathcal{R}y_n)), \phi(\mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n)), \right. \\
&\quad \phi(\mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n)), \\
&\quad \left. \phi\left(\frac{1}{2}[\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{A}y_n) + \mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n)]\right), \right. \\
&\quad \left. \phi\left(\mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n) \frac{[1 + \mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n)]}{[1 + \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)]}\right) \right\} \\
&= \max \left\{ \phi(0), \phi(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)), \phi(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)), \right. \\
&\quad \phi\left(\frac{1}{2}[0 + \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)]\right), \\
&\quad \left. \phi\left(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n) \frac{[1 + \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)]}{[1 + \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)]}\right) \right\} \\
&= \max \left\{ 0, \phi(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)), \phi(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)), \right. \\
&\quad \left. \phi\left(\frac{1}{2}[\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)]\right), \phi\left(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)\right) \right\} \\
&= \phi(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)) \\
&< \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n), \tag{3.9}
\end{aligned}$$

which is a contradiction. Hence, we have $\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n) = 0$, that is, $\mathcal{A}y_n = \mathcal{B}z_n$. Letting $n \rightarrow \infty$, we get $\lim_{n \rightarrow \infty} \mathcal{A}y_n = \lim_{n \rightarrow \infty} \mathcal{B}z_n = \alpha$.

Now, first we assume that $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\alpha = \mathcal{T}\beta$ for some $\beta \in X$. Subsequently, we have

$$\lim_{n \rightarrow \infty} \mathcal{B}z_n = \lim_{n \rightarrow \infty} \mathcal{A}y_n = \lim_{n \rightarrow \infty} \mathcal{R}y_n = \lim_{n \rightarrow \infty} \mathcal{T}z_n = \mathcal{T}\beta = \alpha.$$

We claim that $\mathcal{B}\beta = \mathcal{T}\beta$, that is, β is a coincidence point of the mappings $\mathcal{B}$ and $\mathcal{T}$. Rest of the proof follows from Theorem 3.1. This completes the proof. $\square$

Theorem 3.5. *Let $(X, \mathcal{S})$ be a $\mathcal{S}$ -metric space and let $\mathcal{A}, \mathcal{B}, \mathcal{R}, \mathcal{T}: X \rightarrow X$ be four self-mappings of X satisfying the following conditions: (i)*

$$\begin{aligned}
 \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) \leq & k_1 \phi(\mathcal{S}(\mathcal{R}u, \mathcal{R}u, \mathcal{T}v)) + k_2 \phi(\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u)) \\
 & + k_3 \phi(\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)) \\
 & + k_4 \phi\left(\frac{1}{2}[\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{T}v) + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{R}u)]\right) \\
 & + k_5 \phi\left(\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u) \frac{[1 + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)]}{[1 + \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v)]}\right),
 \end{aligned} \tag{3.10}$$

for all $u, v \in X$, where $\phi \in \Phi$ and $k_1, k_2, k_3, k_4, k_5 > 0$ are nonnegative reals with $k_1 + k_2 + k_3 + k_4 + k_5 < 1$;

- (ii) the pairs $(\mathcal{A}, \mathcal{R})$ and $(\mathcal{B}, \mathcal{T})$ are weakly compatible;
- (iii) one of the pairs $(\mathcal{A}, \mathcal{R})$ or $(\mathcal{B}, \mathcal{T})$ satisfy (E.A) property;
- (iv) $\mathcal{A}(X) \subseteq \mathcal{T}(X)$ and $\mathcal{B}(X) \subseteq \mathcal{R}(X)$.

If the range of one of the mappings $\mathcal{R}(X)$ or $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\mathcal{A}, \mathcal{B}, \mathcal{R}$ and $\mathcal{T}$ have a unique common fixed point in X .

Proof. First, we suppose that the pair $(\mathcal{A}, \mathcal{R})$ satisfy (E.A) property. Then by definition 2.10, there exists a sequence $\{y_n\}$ in X such that $\lim_{n \rightarrow \infty} \mathcal{A}y_n = \lim_{n \rightarrow \infty} \mathcal{R}y_n = \alpha$ for some $\alpha \in X$. Further, since $\mathcal{A}(X) \subseteq \mathcal{T}(X)$, there exists a sequence $\{z_n\}$ in X such that $\lim_{n \rightarrow \infty} \mathcal{A}y_n = \lim_{n \rightarrow \infty} \mathcal{T}z_n$. Hence $\lim_{n \rightarrow \infty} \mathcal{T}z_n = \alpha$. We claim that $\lim_{n \rightarrow \infty} \mathcal{B}z_n = \alpha$. If not, then putting $u = y_n$ and $v = z_n$ in

inequality (3.10) and using Lemma 2.12, (S1) and the property of ϕ , we have

$$\begin{aligned}
\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n) &\leq k_1 \phi(\mathcal{S}(\mathcal{R}y_n, \mathcal{R}y_n, \mathcal{T}z_n)) + k_2 \phi(\mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n)) \\
&\quad + k_3 \phi(\mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{T}z_n)) \\
&\quad + k_4 \phi\left(\frac{1}{2}[\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{T}z_n) + \mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{R}y_n)]\right) \\
&\quad + k_5 \phi\left(\mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n) \frac{[1 + \mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{T}z_n)]}{[1 + \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)]}\right) \\
&= k_1 \phi(\mathcal{S}(\mathcal{R}y_n, \mathcal{R}y_n, \mathcal{R}y_n)) + k_2 \phi(\mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n)) \\
&\quad + k_3 \phi(\mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n)) \\
&\quad + k_4 \phi\left(\frac{1}{2}[\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{A}y_n) + \mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n)]\right) \\
&\quad + k_5 \phi\left(\mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n) \frac{[1 + \mathcal{S}(\mathcal{B}z_n, \mathcal{B}z_n, \mathcal{A}y_n)]}{[1 + \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)]}\right) \\
&= k_1 \phi(0) + k_2 \phi(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)) + k_3 \phi(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)) \\
&\quad + k_4 \phi\left(\frac{1}{2}[0 + \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)]\right) \\
&\quad + k_5 \phi\left(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n) \frac{[1 + \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)]}{[1 + \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)]}\right) \\
&= k_1 .0 + k_2 \phi(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)) + k_3 \phi(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)) \\
&\quad + k_4 \phi\left(\frac{1}{2}[\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)]\right) + k_5 \phi\left(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)\right) \Big\} \\
&\leq k_1 .0 + k_2 \phi(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)) + k_3 \phi(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)) \\
&\quad + k_4 \phi\left(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)\right) + k_5 \phi\left(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)\right) \Big\} \\
&= (k_2 + k_3 + k_4 + k_5) \phi(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)) \\
&\leq (k_1 + k_2 + k_3 + k_4 + k_5) \phi(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)) \\
&< \phi(\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n)) \text{ (since } k_1 + k_2 + k_3 + k_4 + k_5 < 1) \\
&< \mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n), \tag{3.11}
\end{aligned}$$

which is a contradiction. Hence, we have $\mathcal{S}(\mathcal{A}y_n, \mathcal{A}y_n, \mathcal{B}z_n) = 0$, that is, $\mathcal{A}y_n = \mathcal{B}z_n$. Letting $n \rightarrow \infty$, we get $\lim_{n \rightarrow \infty} \mathcal{A}y_n = \lim_{n \rightarrow \infty} \mathcal{B}z_n = \alpha$.

Now, we assume that $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\alpha = \mathcal{T}\beta$ for some $\beta \in X$. Subsequently, we have

$$\lim_{n \rightarrow \infty} \mathcal{B}z_n = \lim_{n \rightarrow \infty} \mathcal{A}y_n = \lim_{n \rightarrow \infty} \mathcal{R}y_n = \lim_{n \rightarrow \infty} \mathcal{T}z_n = \mathcal{T}\beta = \alpha.$$

We claim that $\mathcal{B}\beta = \mathcal{T}\beta$, that is, β is a coincidence point of the mappings $\mathcal{B}$ and $\mathcal{T}$. Rest of the proof follows from Theorem 3.1. This completes the proof. $\square$

Remark 3.6. (i) Completeness of the space X is relaxed in Theorem 3.1, Theorem 3.4 and Theorem 3.5.

(ii) Continuity of the mappings $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ and $\mathcal{T}$ is relaxed in Theorem 3.1, Theorem 3.4 and Theorem 3.5.

Corollary 3.7. *Let $(X, \mathcal{S})$ be a complete $\mathcal{S}$ -metric space and let $\mathcal{A}: X \rightarrow X$ be a mapping satisfies the following condition: there exist $r_1, r_2 \in [0, 1)$ with $r_1 + r_2 < 1$ and some $\phi \in \Phi$ such that*

$$\begin{aligned} \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{A}v) \leq & r_1 \phi\left(\frac{\mathcal{S}(v, v, \mathcal{A}v)[1 + \mathcal{S}(u, u, \mathcal{A}u)]}{[1 + \mathcal{S}(u, u, v)]}\right) \\ & + r_2 \max \left\{ \phi(\mathcal{S}(u, u, v)), \phi(\mathcal{S}(u, u, \mathcal{A}u)), \right. \\ & \left. \phi(\mathcal{S}(v, v, \mathcal{A}v)), \phi\left(\frac{1}{2}[\mathcal{S}(u, u, \mathcal{A}v) + \mathcal{S}(v, v, \mathcal{A}u)]\right) \right\}, \end{aligned}$$

for all $u, v \in X$. Then $\mathcal{A}$ has a unique fixed point in X .

Corollary 3.8. *Let $(X, \mathcal{S})$ be a complete $\mathcal{S}$ -metric space and let $\mathcal{A}: X \rightarrow X$ be a mapping satisfies the following condition: there exist $r_1, r_2 \in [0, 1)$ with $r_1 + r_2 < 1$ and some $\phi \in \Phi$ such that*

$$\begin{aligned} \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{A}v) \leq & r_1 \phi\left(\frac{\mathcal{S}(v, v, \mathcal{A}v)[1 + \mathcal{S}(u, u, \mathcal{A}u)]}{[1 + \mathcal{S}(u, u, v)]}\right) \\ & + r_2 \phi\left(\max \left\{ \mathcal{S}(u, u, v), \mathcal{S}(u, u, \mathcal{A}u), \mathcal{S}(v, v, \mathcal{A}v), \right. \right. \\ & \left. \left. \frac{1}{2}[\mathcal{S}(u, u, \mathcal{A}v) + \mathcal{S}(v, v, \mathcal{A}u)] \right\}\right), \end{aligned} \quad (3.12)$$

for all $u, v \in X$. Then $\mathcal{A}$ has a unique fixed point in X .

Remark 3.9. It is clear that the conclusions of the Corollary 3.8 remain valid if in condition (3.12), the second term of the right-hand side is replaced by one of the following terms:

$$\begin{aligned} & r_2 \phi(\mathcal{S}(u, u, v)); \quad r_2 \phi\left(\frac{1}{2}[\mathcal{S}(u, u, \mathcal{A}v) + \mathcal{S}(v, v, \mathcal{A}u)]\right); \\ & r_2 \max \left\{ \phi(\mathcal{S}(u, u, \mathcal{A}u)), \phi(\mathcal{S}(v, v, \mathcal{A}v)) \right\}; \\ \text{or } & r_2 \max \left\{ \phi(\mathcal{S}(u, u, v)), \phi(\mathcal{S}(u, u, \mathcal{A}u)), \phi(\mathcal{S}(v, v, \mathcal{A}v)) \right\}. \end{aligned}$$

Example 3.10. Let $X = [0, 1]$. We define the function $\mathcal{S}: X^3 \rightarrow [0, \infty)$ by

$$\mathcal{S}(u, v, z) = \begin{cases} 0, & \text{if } u = v = z, \\ \max\{u, v, z\}, & \text{if otherwise,} \end{cases}$$

for all $u, v, z \in X$, then $\mathcal{S}$ is an $\mathcal{S}$ -metric on X . Define four self-maps $\mathcal{A}, \mathcal{B}, \mathcal{R}, \mathcal{T}: X \rightarrow X$ on X by $\mathcal{A}(u) = \frac{u}{4}$, $\mathcal{B}(u) = \frac{u}{4}$, $\mathcal{T}(u) = u$ and $\mathcal{R}(u) = \frac{u}{2}$ for all $u \in X$. Let $u, v \in X$. We also define $\phi: [0, \infty) \rightarrow [0, \infty)$ by $\phi(\alpha) = \frac{\alpha}{2}$ for all $\alpha \in [0, \infty)$. Clearly ϕ is continuous on $[0, \infty)$ satisfying $\phi(0) = 0$ and $0 < \phi(\alpha) < \alpha$ for all $\alpha > 0$. Now consider the following cases:

Case I. (1) Let $u < v$. Then we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) &= \mathcal{S}\left(\frac{u}{4}, \frac{u}{4}, \frac{v}{4}\right) = \max\left\{\frac{u}{4}, \frac{u}{4}, \frac{v}{4}\right\} = \frac{v}{4}, \\ \mathcal{S}(\mathcal{R}u, \mathcal{R}u, \mathcal{T}v) &= \mathcal{S}\left(\frac{u}{2}, \frac{u}{2}, v\right) = \max\left\{\frac{u}{2}, \frac{u}{2}, v\right\} = v, \end{aligned}$$

$$\begin{aligned}
\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u) &= \mathcal{S}\left(\frac{v}{4}, \frac{v}{4}, \frac{u}{4}\right) = \max\left\{\frac{v}{4}, \frac{v}{4}, \frac{u}{4}\right\} = \frac{v}{4}, \\
\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v) &= \mathcal{S}\left(\frac{v}{4}, \frac{v}{4}, v\right) = \max\left\{\frac{v}{4}, \frac{v}{4}, v\right\} = v, \\
\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{T}v) &= \mathcal{S}\left(\frac{u}{4}, \frac{u}{4}, v\right) = \max\left\{\frac{u}{4}, \frac{u}{4}, v\right\} = v, \\
\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{R}u) &= \mathcal{S}\left(\frac{v}{4}, \frac{v}{4}, \frac{u}{2}\right) = \max\left\{\frac{v}{4}, \frac{v}{4}, \frac{u}{2}\right\} = \frac{v}{4}.
\end{aligned}$$

Now using inequality (3.1) and the property of ϕ , we have

$$\begin{aligned}
\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) &= \frac{v}{4} \\
&\leq \phi\left(\max\left\{\mathcal{S}(\mathcal{R}u, \mathcal{R}u, \mathcal{T}v), \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u), \right.\right. \\
&\quad \left.\left.\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v), \frac{1}{2}[\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{T}v) + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{R}u)]\right\}\right) \\
&= \phi\left(\max\left\{v, \frac{v}{4}, v, \frac{1}{2}\left[v + \frac{v}{4}\right]\right\}\right) = \phi(v) \\
&< v,
\end{aligned}$$

that is,

$$\frac{v}{4} \leq \frac{v}{2},$$

or,

$$\frac{1}{4} \leq \frac{1}{2},$$

which is true.

Hence we conclude that

$$\begin{aligned}
\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) &\leq \phi\left(\max\left\{\mathcal{S}(\mathcal{R}u, \mathcal{R}u, \mathcal{T}v), \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u), \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v), \right.\right. \\
&\quad \left.\left.\frac{1}{2}[\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{T}v) + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{R}u)]\right\}\right),
\end{aligned}$$

for all $u, v \in X$, where $\phi \in \Phi$.

(2) Now using inequality (3.8) of Theorem 3.4, we have

$$\begin{aligned}
\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) &= \frac{v}{4} \\
&\leq \max\left\{\phi(\mathcal{S}(\mathcal{R}u, \mathcal{R}u, \mathcal{T}v)), \phi(\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u)), \right. \\
&\quad \phi(\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)), \\
&\quad \phi\left(\frac{1}{2}[\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{T}v) + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{R}u)]\right), \\
&\quad \left.\phi\left(\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u) \frac{[1 + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)]}{[1 + \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v)]}\right)\right\} \\
&= \max\left\{\phi(v), \phi\left(\frac{v}{4}\right), \phi(v), \phi\left(\frac{5v}{8}\right), \phi\left(\frac{v(1+v)}{(v+4)}\right)\right\} \\
&= \phi(v) = \frac{v}{2},
\end{aligned}$$

that is,

$$\frac{v}{4} \leq \frac{v}{2},$$

or,

$$\frac{1}{4} \leq \frac{1}{2},$$

which is true.

Hence we conclude that

$$\begin{aligned} \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) \leq \max \bigg\{ & \phi(\mathcal{S}(\mathcal{R}u, \mathcal{R}u, \mathcal{T}v)), \phi(\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u)), \\ & \phi(\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)), \\ & \phi\left(\frac{1}{2}[\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{T}v) + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{R}u)]\right), \\ & \phi\left(\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u) \frac{[1 + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)]}{[1 + \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v)]}\right) \bigg\}, \end{aligned}$$

for all $u, v \in X$, where $\phi \in \Phi$.

(3) Now using inequality (3.10) of Theorem 3.4, we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) &= \frac{v}{4} \\ &\leq k_1 \cdot \phi(v) + k_2 \cdot \phi\left(\frac{v}{4}\right) + k_3 \cdot \phi(v) \\ &\quad + k_4 \cdot \phi\left(\frac{5v}{8}\right) + k_5 \cdot \phi\left(\frac{v(1+v)}{(v+4)}\right) \\ &\leq k_1 \cdot \frac{v}{2} + k_2 \cdot \frac{v}{8} + k_3 \cdot \frac{v}{2} \\ &\quad + k_4 \cdot \frac{5v}{16} + k_5 \cdot \frac{v(1+v)}{2(v+4)}. \end{aligned}$$

Putting $u = 0$ and $v = 1$ in the above inequality, we obtain

$$\begin{aligned} \frac{1}{4} &\leq \frac{k_1}{2} + \frac{k_2}{8} + \frac{k_3}{2} \\ &\quad + \frac{5k_4}{16} + \frac{k_5}{5}. \end{aligned}$$

The above inequality is satisfies for (i) $k_1 = \frac{1}{2}$, $k_2 = \frac{1}{3}$ and $k_3 = k_4 = k_5 = 0$, (ii) $k_2 = \frac{1}{4}$, $k_3 = \frac{1}{2}$ and $k_1 = k_4 = k_5 = 0$, (iii) $k_3 = \frac{1}{3}$, $k_4 = \frac{1}{2}$ and $k_1 = k_2 = k_5 = 0$, (iv) $k_3 = \frac{1}{4}$, $k_4 = \frac{1}{3}$, $k_5 = \frac{1}{3}$ and $k_1 = k_2 = 0$ with $k_1 + k_2 + k_3 + k_4 + k_5 < 1$ etc., that is, it satisfies for $k_i \in [0, 1)$ for $i = 1, 2, 3, 4, 5$ with $k_1 + k_2 + k_3 + k_4 + k_5 < 1$.

Case II. Note that $\mathcal{A}(X) = [0, \frac{1}{4}]$, $\mathcal{B}(X) = [0, \frac{1}{4}]$, $\mathcal{T}(X) = [0, 1] = X$ and $\mathcal{R}(X) = [0, \frac{1}{2}]$. This will imply that $\mathcal{A}(X) \subset \mathcal{R}(X)$ and $\mathcal{B}(X) \subset \mathcal{T}(X)$.

Case III. Now we show that the pairs $(\mathcal{A}, \mathcal{R})$ and $(\mathcal{B}, \mathcal{T})$ are weakly compatible. For this, suppose that $\mathcal{T}u = \mathcal{B}u$ for $u \in X$. Then $u = \frac{u}{4}$. It follows that $u = 0$. Now, we consider $\mathcal{T}\mathcal{B}(u) = \mathcal{T}(\mathcal{B}u) = \mathcal{T}(0) = 0$ and $\mathcal{B}\mathcal{T}(u) = \mathcal{B}(\mathcal{T}u) = \mathcal{B}(0) = 0$. Thus, the pair $(\mathcal{B}, \mathcal{T})$ is weakly compatible. Now, let $\mathcal{A}u = \mathcal{R}u$ for $u \in X$. This implies that $\frac{u}{4} = \frac{u}{2}$ and hence $u = 0$. Now,

we consider $\mathcal{AR}(u) = \mathcal{A}(\mathcal{R}u) = \mathcal{A}(0) = 0$ and $\mathcal{RA}(u) = \mathcal{R}(\mathcal{A}u) = \mathcal{R}(0) = 0$. It follows that the pair $(\mathcal{A}, \mathcal{R})$ is also weakly compatible.

Case IV. Now we show that the pairs $(\mathcal{B}, \mathcal{T})$ satisfies (E.A) property. For this, consider the sequence $\{v_n\} = \{\frac{1}{2n+1}\}_{n \geq 1}$. Clearly the sequence $\{v_n\}$ is in X and note that $\mathcal{T}v_n = v_n = \frac{1}{2n+1}$ and $\mathcal{B}v_n = \frac{v_n}{4} = \frac{1}{4(2n+1)}$ for all $n \in \mathbb{N}$. This will imply that

$$\begin{aligned} \mathcal{S}(\mathcal{T}v_n, \mathcal{T}v_n, 0) &= \mathcal{S}\left(\frac{1}{2n+1}, \frac{1}{2n+1}, 0\right) = \max\left\{\frac{1}{2n+1}, \frac{1}{2n+1}, 0\right\} \\ &= \frac{1}{2n+1} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This shows that $\mathcal{T}v_n \rightarrow 0$ as $n \rightarrow \infty$.

Also note that

$$\begin{aligned} \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, 0) &= \mathcal{S}\left(\frac{1}{4(2n+1)}, \frac{1}{4(2n+1)}, 0\right) = \max\left\{\frac{1}{4(2n+1)}, \frac{1}{4(2n+1)}, 0\right\} \\ &= \frac{1}{4(2n+1)} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This shows that $\mathcal{B}v_n \rightarrow 0$ as $n \rightarrow \infty$.

Thus there exists a sequence $\{v_n\}$ in X such that $\mathcal{B}v_n \rightarrow 0$ and $\mathcal{T}v_n \rightarrow 0$ as $n \rightarrow \infty$. Hence the pair $(\mathcal{B}, \mathcal{T})$ satisfies (E.A) property.

Similarly, we can show that the pair $(\mathcal{A}, \mathcal{R})$ also satisfies (E.A) property.

Case V. As $\mathcal{A}(X) = [0, \frac{1}{4}]$, then $\mathcal{A}(X)$ is a complete subspace of X .

Thus all the conditions of Theorem 3.1, Theorem 3.4 and Theorem 3.5 are satisfied and hence the mappings $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ and $\mathcal{T}$ have a unique common fixed point, namely $u = 0 \in X$.

4. CONCLUSION

In this paper, we prove some unique common fixed point theorems in the setting of S -metric spaces for two pairs of weakly compatible mappings satisfying (E.A) property and give some corollaries of the main results. We also illustrate some examples to support the results. Our results extend, generalize and improve several results from the existing literature.

REFERENCES

- [1] M. Aamri and D. El. Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl. **270** (2002), 181-188
- [2] M. Abbas and B. E. Rhoades, *Common fixed point results for non-commuting mappings without continuity generalized metric spaces*, Appl. Math. Computation **215** (2009), 262-269.
- [3] T. Abdeljawad, E. Karapinar and K. Tas, *Existence and uniqueness of a commofixed point on partial metric spaces*, Appl. Math. Lett. **24** (2011), 1900-1904.
- [4] I. Arandjelović, Z. Kadelburg and S. Radenović, *Boyd-Wong type common fixed point results in cone metric spaces*, Appl. Math. Comput. **217** (2011), 7167-7171.
- [5] I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal. Gos. Ped. Inst. Unianowsk **30** (1989), 26-37.
- [6] S. Banach, *Sur les operation dans les ensembles abstraits et leur application aux equation integrals*, Fund. Math. **3** (1922), 133-181.

- [7] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458-464.
- [8] B. C. Dhage, *Generalized metric spaces mappings with fixed point*, Bull. Calcutta Math. Soc. **84** (1992), 329-336.
- [9] S. Gähler, *2-metrische Räume und ihrer topoloische struktur*, Math. Nachr. **26** (1963), 115-148.
- [10] A. Gupta, *Cyclic contraction on S-metric space*, Int. J. Anal. Appl. **3(2)** (2013), 119-130.
- [11] N. T. Hieu, N. T. Ly and N. V. Dung, *A generalization of Ciric quasi-contractions for maps on S-metric spaces*, Thai J. Math. **13(2)** (2015), 369-380.
- [12] L. -G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332(2)** (2007), 1468-1476.
- [13] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci. **9** (1986), 771-779.
- [14] G. Jungck, *Common fixed points for noncontinuous, nonself maps on nonnumetric spaces*, Far East J. Math. Sci. **4(2)** (1996), 195-215.
- [15] Z. Kadelburg, S. Radenović and V. Rakočević, *Remarks on Quasi-contraction on a cone metric space*, Appl. Math. Lett. **22** (2009), 1674-1679.
- [16] J. K. Kim, S. Sedghi, A. Gholidahneh and M. M. Rezaee, *Fixed point theorems in S-metric spaces*, East Asian Math. J. **32(5)** (2016), 677-684.
- [17] J. Matkowski, *Fixed point theorems for mapping with a contractive iterate at a point*, Proc. Amer. Math. Soc. **62** (1977), 344-348.
- [18] N. Y. Özgür and N. Tas, *Some new contractive mappings on S-metric spaces and their relationships with the mapping (S25)*, Math. Sci. **11(7)** (2017), 7-16.
- [19] E. Rakotch, *A note on contractive mappings*, Proc. Amer. Math. Soc. **13** (1969), 459-465.
- [20] S. Sedghi, N. Shobe and A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesnik **64(3)** (2012), 258-266.
- [21] S. Sedghi and N. V. Dung, *Fixed point theorems on S-metric spaces*, Mat. Vesnik **66(1)** (2014), 113-124.
- [22] S. Sedghi, M. M. Rezaee, T. Dosenovic and S. Radenovic, *Common fixed point theorems for contractive mappings satisfying Φ -maps in S-metric spaces*, Acta Univ. Sapientiae Math. **8(2)** (2016), 298-311.
- [23] S. Sedghi, N. Shobkolaei, M. Shahraki and T. Dosenovic, *Common fixed point of four maps in S-metric space*, Math. Sci. **12** (2018), 137-143.
- [24] K. Anthony Singh and M. R. Singh, *Some fixed point theorems of cone S_b -metric space*, J. Indian Acad. Math. **40(2)** (2018), 255-272.
- [25] N. Souayah and N. Mlaiki, *A fixed point theorem in S_b -metric spaces*, J. Math. Computer Sci. **16** (2016), 131-139.
- [26] E. Tarafdar, *An approach to fixed point theorems on uniform spaces*, Trans. Amer. Math. Soc. **191** (1974), 209-225.
- [27] N. Tas and N. Yilmaz Ozgur, *New generalized fixed point results on S_b -metric spaces*, arxiv:1703.01868v2 [math.gn] 17 apr.

H.N. 3/1005, GEETA NAGAR, RAIPUR, RAIPUR - 492001 (C.G.), INDIA.

Email address: saluja1963@gmail.com