Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 35 (2026) 1-10 https://doi.org/10.56947/amcs.v35.860

SOME RESULTS ON SYMMETRIZED AH-CONVEXITY

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Abstract. The Hermite-Hadamard inequality is applied to several classes of functions in many fields of analysis. This inequality has become a powerful mathematical tool for researchers. The idea of convexity has undergone numerous generalizations and extensions in recent years. This study presents the concept of a symmetrized AH-convex function and proves Hermite-Hadamard type inequalities via symmetrized AH-convex function and symmetrized $h-AH$-convex function.

Keywords. Symmetrized convexity, Hermite Hadamard inequality, AH-convexity.
2020 Mathematics Subject Classification. Primary: 60G05, 60G07,60K99. Secondary: 26D15, 26A51, 60G99.

1. Introduction and preliminaries

The following inequalities, referred to as the Hermite-Hadamard inequality,

$$\mathcal{Q}\left(\frac{\rho+\delta}{2}\right) \leq \frac{1}{\delta-\rho}\int_{\rho}^{\delta} \mathcal{Q}\left(\varkappa\right) d\varkappa \leq \frac{\mathcal{Q}(\rho)+\mathcal{Q}(\delta)}{2}$$

where $\rho, \delta \in \mathbb{R}, \rho \neq \delta$ and $\mathcal{Q}$ be a convex function.
This inequality has been a vital inequality in convex analysis. We refer the reader to [13] for further details.
The Hermite-Hadamard inequality is widely used in nonlinear analysis.
It is hard to collect all generalizations and applications of convex sets and functions. However, we refer the reader to [1, 2, 4, 7, 8, 9, 12, 14, 15, 16, 17, 19, 20] and [21, 22, 23, 24, 25, 27].

Definition 1.1 ([10, 28])**.** Let $p \in \mathbb{R}\backslash\{0\}$ and $\mathcal{I} \subset (0, \infty)$ be a real interval. A function $\mathcal{Q} : \mathcal{I} \to \mathbb{R}$ is said to be a p-convex function, if

$$\mathcal{Q}\left(\left[tu^p + (1-t)v^p\right]^{\frac{1}{p}}\right) \leq t\mathcal{Q}(u) + (1-t)\mathcal{Q}(v) \tag{1.1}$$

holds for all $u, v \in \mathcal{I}$ and $t \in [0, 1]$.

If the inequality in (1.1) is reversed, then $\mathcal{Q}$ is said to be p-concave.
It is clear from Definition 1.1 that $p-$convexity reduces to ordinary convexity for

Date: Received: May 22, 2026; Revised: Jun 19, 2026; Accepted: Jun 26, 2026.
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$p=1$ of functions defined on $\mathcal{I} \subset (0,\infty)$.
Given that the condition (1.1) can be expressed as

$$(\mathcal{Q}\circ f)\left([tu^p+(1-t)v^p]^{\frac{1}{p}}\right) \leq t(\mathcal{Q}\circ f)(u^p)+(1-t)(\mathcal{Q}\circ f)(v^p)\,, f(x)=x^{1/p}$$

Consequently, $\mathcal{Q}:\mathcal{I}\subseteq(0,\infty)\to\mathbb{R}$ is p-convex on $\mathcal{I}$ if and only if $\mathcal{Q}\circ f$ is convex on $f^{-1}(\mathcal{I})=\{s^p, s\in\mathcal{I}\}$.

Definition 1.2 ([11, 18])**.** A function $\mathcal{Q}:\mathcal{I}\subseteq(0,\infty)\to\mathbb{R}$ is said to be *GA*-convex function if

$$\mathcal{Q}\left(u^t v^{(1-t)}\right) \leq t\mathcal{Q}(u)+(1-t)\mathcal{Q}(v)$$

holds for any $u,v\in[\rho,\delta]$ and $t\in[0,1]$.

The notion of symmetrized convex functions was presented by Dragomir [5], who also studied some of its characteristics. This notion was introduced as follows: Consider the function $\mathcal{Q}:[\rho,\delta]\to\mathbb{R}$ and its symmetrical form on $[\rho,\delta]$ denoted by $\breve{\mathcal{Q}}_{[\rho,\delta]}(\varkappa)$, or $\breve{\mathcal{Q}}(\varkappa)$ when $[\rho,\delta]$ is implicit is defined as follows:

$$\breve{\mathcal{Q}}(\varkappa)=\frac{1}{2}[\mathcal{Q}(\varkappa)+\mathcal{Q}(\rho+\delta-\varkappa)], \varkappa\in[\rho,\delta].$$

The anti symmetrical transform of $\mathcal{Q}$ on the interval $[a,b]$ is denoted by $\hat{\mathcal{Q}}_{[a,b]}$ or simply $\hat{\mathcal{Q}}$ and is defined by

$$\hat{\mathcal{Q}}(\varkappa)=\frac{1}{2}[\mathcal{Q}(\varkappa)-(\rho+\delta-\varkappa)],\ \varkappa\in[\rho,\delta].$$

The function $\mathcal{Q}:[\rho,\delta]\to\mathbb{R}$ is said to be symmetrized convex (concave) on $[\rho,\delta]$ if it's symmetrical form $\breve{\mathcal{Q}}$ is convex (concave) on $[\rho,\delta]$. It is clear that $\breve{\mathcal{Q}}+\hat{\mathcal{Q}}=\mathcal{Q}$. The function $\breve{\mathcal{Q}}$ is convex if $\mathcal{Q}$ is convex on $[\rho,\delta]$, but the opposite isn't true.

Theorem 1.3 ([5])**.** *Assume that $\mathcal{Q}:[\rho,\delta]\to\mathbb{R}$ is symmetrized convex. The following inequalities hold:*
i)

$$\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)\leq\frac{1}{\delta-\rho}\int_\rho^\delta \mathcal{Q}(\kappa)d\kappa\leq\frac{\mathcal{Q}(\rho)+\mathcal{Q}(\delta)}{2}.$$

ii) For all $\kappa\in[\rho,\delta]$:

$$\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)\leq\breve{\mathcal{Q}}(\kappa)\leq\frac{\mathcal{Q}(\rho)+\mathcal{Q}(\delta)}{2}.$$

iii) If $\Psi:[\rho,\delta]\to[0,\infty)$ is integrable on $[\rho,\delta]$, then

$$\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)\int_\rho^\delta\Psi(\kappa)d\kappa\leq\int_\rho^\delta\Psi(\kappa)\breve{\mathcal{Q}}(\kappa)d\kappa\leq\frac{\mathcal{Q}(\rho)+\mathcal{Q}(\delta)}{2}\int_\rho^\delta\Psi(\kappa)d\kappa.$$

For a function $\mathcal{Q}:[\rho,\delta]\subseteq(0,\infty)\to\mathbb{R}$, we consider the symmetrical transform of $\mathcal{Q}$ on the interval $[\rho,\delta]$, denoted by $\bar{\mathcal{Q}}$, defined by $\bar{\mathcal{Q}}(\kappa)=\frac{1}{2}\left[\mathcal{Q}(\kappa)+\mathcal{Q}\left(\frac{\rho\delta}{\kappa}\right)\right]$. A function $\mathcal{Q}:[\rho,\delta]\subseteq(0,\infty)\to\mathbb{R}$ is said to be symmetrized *GA*-convex (concave) on $[\rho,\delta]$ if $\bar{\mathcal{Q}}$ convex (concave) on $[\rho,\delta]$.

Theorem 1.4 ([11]). *Assume that $\Psi : [\rho, \delta] \subseteq (0, \infty) \to \mathbb{R}$ is GA-convex and $\mathcal{Q} : [\rho, \delta] \subseteq (0, \infty) \to \mathbb{R}$ is symmetrized GA-convex and integrable on $[\rho, \delta]$. Then*

$$\begin{aligned}
&\frac{1}{\ln \delta/\rho} \frac{\mathcal{Q}(\rho) + \mathcal{Q}(\delta)}{2} \int_\rho^\delta \frac{\Psi(\kappa)}{\kappa} d\kappa - \frac{\mathcal{Q}(\rho) + \mathcal{Q}(\delta)}{2} \frac{\Psi(\rho) + \Psi(\delta)}{2} \\
&\qquad + \frac{1}{\ln \delta/\rho} \frac{\Psi(\rho) + \Psi(\delta)}{2} \int_\rho^\delta \frac{\mathcal{Q}(\kappa)}{\kappa} d\kappa \\
&\leq \frac{1}{\ln \delta/\rho} \int_\rho^\delta \frac{\bar{\mathcal{Q}}(\kappa)\Psi(\kappa)}{\kappa} d\kappa,
\end{aligned}$$

For a function $Q : [\rho, \delta] \subseteq (0, \infty) \to \mathbb{R}$, consider its symmetrical form denoted by $P_{(\mathcal{Q};p)}$ on the interval $[\rho, \delta]$, defined by

$$P_{(\mathcal{Q};p)}(\eta) = \frac{1}{2} \left[Q(\eta) + \mathcal{Q} \left([\rho^p + \delta^p - \eta^p]^{1/p} \right) \right].$$

A function $\mathcal{Q} : [\rho, \delta] \subseteq (0, \infty) \to \mathbb{R}$ is called symmetrized p-convex (concave) on $[\rho, \delta]$ if $P_{(\mathcal{Q};p)}$ is $p-$convex (concave) on $[\rho, \delta]$.

Theorem 1.5 ([10]). *Let $p \in \mathbb{R} \backslash \{0\}$ and $\mathcal{Q} : [\rho, \delta] \subseteq (0, \infty) \to \mathbb{R}$ be symmetrized p-convex on $[\rho, \delta]$. Then*

$$\begin{aligned}
&\frac{1}{2} \left[\mathcal{Q} \left(\left[\rho^p + \delta^p - \frac{\eta^p + \varrho^p}{2} \right]^{1/p} \right) + \mathcal{Q} \left(\left[\frac{\eta^p + \varrho^p}{2} \right]^{1/p} \right) \right] \\
\leq &\frac{p}{2(\varrho^p - \eta^p)} \left[\int_\eta^\varrho \frac{\mathcal{Q}(\ell)}{\ell^{1-p}} d\ell + \int_{[\rho^P + \delta^p - \varrho^p]^{1/p}}^{[\rho^p + \delta^p - \eta^p]^{1/p}} \frac{\mathcal{Q}(\ell)}{\ell^{1-p}} d\ell \right] \\
\leq &\frac{1}{4} \left[\mathcal{Q} \left([\rho^p + \delta^p - \eta^p]^{1/p} \right) + \mathcal{Q} \left([\rho^p + \delta^p - \varrho^p]^{1/p} \right) + \mathcal{Q}(\eta) + \mathcal{Q}(\varrho) \right],
\end{aligned}$$

hold for any $\eta, \varrho \in [\rho, \delta]$ with $\eta \neq \varrho$.

Motivated by the above results, in this paper we give a new class of symmetrized convex functions, called symmetrized AH-convexity and we will investigate some related results.

In this study, the symmetrized AH-convex functions and related Hermite–Hadamard inequalities could be utilized in numerical integration error bounding, optimization algorithms, information theory, and other areas of computer science. They could support the analysis of algorithm convergence. In digital signal processing, these inequalities could serve as a theoretical foundation for signal denoising, thereby improving noise reduction. In information theory, these inequalities could help in the analysis of entropy-related concepts. Also, the quadrature formulas (like the Boole, Milne, and Simpson's rules) serve as a pivotal tool for approximating definite integrals and by using this new class of functions, several integral inequalities could be derived to calculate the error of certain quadrature rules.

Recall that the mapping $\mathcal{Q}:[\rho,\delta]\to\mathbb{R}\backslash\{0\}$ is called AH-convex [6] if

$$\mathcal{Q}(mu+(1-m)v)\leq\frac{1}{\frac{m}{\mathcal{Q}(u)}+\frac{1-m}{\mathcal{Q}(v)}},$$

holds for any $u,v\in[\rho,\delta]$ and $m\in[0,1]$.
Also, it is said that the function $\psi:[\rho,\delta]\to\mathbb{R}$ is symmetric on $[\rho,\delta]$ if for all $\varkappa\in[\rho,\delta]$, we have $\psi(\rho+\delta-\varkappa)=\psi(\varkappa)$.

2. On symmetrized AH-convexity

Consider the real interval $[\rho,\delta]$ with $\rho<\delta$ and the function $\mathcal{Q}:[\rho,\delta]\to\mathbb{R}\backslash\{0\}$. We take the symmetrical form of the function $\mathcal{Q}$ denoted by $\tilde{\mathcal{Q}}$ defined in the following way:

$$\tilde{\mathcal{Q}}(\varkappa)=\frac{2}{\frac{1}{\mathcal{Q}(\varkappa)}+\frac{1}{\mathcal{Q}(\rho+\delta-\varkappa)}}\; for\; all\; \varkappa\in[\rho,\delta],$$

and the anti-symmetrical form, denoted by $\overset{\circ}{\mathcal{Q}}$, is given by

$$\overset{\circ}{\mathcal{Q}}(\varkappa)=\frac{1}{2}\left(\frac{\mathcal{Q}(\varkappa)}{\mathcal{Q}(\rho+\delta-\varkappa)}+1\right)\; for\; all\; \varkappa\in[\rho,\delta].$$

Definition 2.1. A function $\mathcal{Q}:[\rho,\delta]\to\mathbb{R}\backslash\{0\}$ is said to be symmetrized AH-convex (AH-concave) if $\tilde{\mathcal{Q}}$ is AH-convex (concave).

It is obvious that: $\mathcal{Q}=\tilde{\mathcal{Q}}\times\overset{\circ}{\mathcal{Q}}$.

Lemma 2.2. *If $\mathcal{Q}:[\rho,\delta]\to(0,\infty)$ is AH-convex function then its symmetrical form $\tilde{\mathcal{Q}}$ is also AH-convex .*

Proof. For any $\varkappa,s\in[\rho,\delta]$ and $m\in(0,1)$ we have

$$\begin{aligned}\tilde{\mathcal{Q}}(m\varkappa+(1-m)s)&=\frac{2}{\frac{1}{\mathcal{Q}(m\varkappa+(1-m)s)}+\frac{1}{\mathcal{Q}(\rho+\delta-(m\varkappa+(1-m)s))}}\\&\leq\frac{2}{\frac{m}{\mathcal{Q}(\varkappa)}+\frac{1-m}{\mathcal{Q}(s)}+\frac{m}{\mathcal{Q}(\rho+\delta-\varkappa)}+\frac{1-m}{\mathcal{Q}(\rho+\delta-s)}}\\&=\frac{2}{m\left(\frac{1}{\mathcal{Q}(\varkappa)}+\frac{1}{\mathcal{Q}(\rho+\delta-\varkappa)}\right)+(1-m)\left(\frac{1}{\mathcal{Q}(s)}+\frac{1}{\mathcal{Q}(\rho+\delta-s)}\right)}\\&=\frac{1}{\frac{m}{\tilde{\mathcal{Q}}(\varkappa)}+\frac{1-m}{\tilde{\mathcal{Q}}(s)}}.\end{aligned}$$

□

Theorem 2.3. *Suppose that $\mathcal{Q}:[\rho,\delta]\to\mathbb{R}\backslash\{0\}$ is symmetrized AH-convex. Then*

$$\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)\leq\tilde{\mathcal{Q}}(\varkappa)\leq\frac{2}{\frac{1}{\mathcal{Q}(\rho)}+\frac{1}{\mathcal{Q}(\delta)}},\tag{2.1}$$

for any $\varkappa\in[\rho,\delta]$.

Proof. Let $\varkappa \in [\rho, \delta]$, by using the $AH-$convexity of $\tilde{\mathcal{Q}}$ on $[\rho, \delta]$ we get

$$\begin{aligned}\mathcal{Q}\left(\frac{\rho+\delta}{2}\right) &= \tilde{\mathcal{Q}}\left(\frac{\rho+\delta}{2}\right) \\ &= \tilde{\mathcal{Q}}\left(\frac{\rho+\delta-\varkappa+\varkappa}{2}\right) \\ &\leq \frac{2}{\frac{1}{\tilde{\mathcal{Q}}(\rho+\delta-\varkappa)}+\frac{1}{\tilde{\mathcal{Q}}(\varkappa)}} \\ &= \tilde{\mathcal{Q}}(\varkappa),\end{aligned}$$

then, we obtain the first part of (2.1).
Also, for any $\varkappa \in [\rho, \delta],$ there exist a number $m \in [0,1]$ such that $\varkappa = m\rho + (1-m)\delta$, then

$$\begin{aligned}\tilde{\mathcal{Q}}(\varkappa) &= \tilde{\mathcal{Q}}\left(\rho\frac{\varkappa-\delta}{\rho-\delta}+\delta\frac{\rho-\varkappa}{\rho-\delta}\right) \\ &\leq \frac{1}{\frac{\varkappa-\delta}{\rho-\delta}\frac{1}{\tilde{\mathcal{Q}}(\rho)}+\frac{\rho-\varkappa}{\rho-\delta}\frac{1}{\tilde{\mathcal{Q}}(\delta)}}.\end{aligned}$$

By using the fact that,

$$\begin{aligned}\tilde{\mathcal{Q}}(\rho) &= \tilde{\mathcal{Q}}(\delta) \\ &= \frac{2}{\frac{1}{\mathcal{Q}(\rho)}+\frac{1}{\mathcal{Q}(\delta)}},\end{aligned}$$

we obtain the second part of (2.1). □

Remark 2.4. If $\mathcal{Q} : [\rho, \delta] \to \mathbb{R}\backslash\{0\}$ is symmetrized AH-convex function, then we have the bounds

$$\inf_{\varkappa\in[\rho,\delta]} \tilde{\mathcal{Q}}(\varkappa) = \mathcal{Q}\left(\frac{\rho+\delta}{2}\right)$$

and

$$\sup_{\varkappa\in[\rho,\delta]} \tilde{\mathcal{Q}}(\varkappa) = \frac{2}{\frac{1}{\mathcal{Q}(\rho)}+\frac{1}{\mathcal{Q}(\delta)}}.$$

Corollary 2.5. *Let $\psi : [\rho, \delta] \to (0, \infty)$ be a symmetric and integrable function on $[\rho, \delta]$.*
If $\mathcal{Q} : [\rho, \delta] \to (0, \infty)$ is symmetrized AH-convex, then

$$\frac{1}{2}\left(\frac{1}{\mathcal{Q}(\rho)}+\frac{1}{\mathcal{Q}(\delta)}\right)\int_\rho^\delta \psi(\varkappa)d\varkappa \leq \int_\rho^\delta \frac{\psi(\varkappa)}{\mathcal{Q}(\varkappa)}d\varkappa \leq \frac{1}{\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)}\int_\rho^\delta \psi(\varkappa)d\varkappa.$$

Proof. Let $\varkappa \in [\rho, \delta]$. Since $\mathcal{Q}$ is symmetrized AH-convex, by using (2.1)

$$\mathcal{Q}\left(\frac{\rho+\delta}{2}\right) \leq \tilde{\mathcal{Q}}(\varkappa) \leq \frac{2}{\frac{1}{\mathcal{Q}(\rho)}+\frac{1}{\mathcal{Q}(\delta)}},$$

then,

$$\frac{1}{2}\left[\frac{1}{\mathcal{Q}(\rho)}+\frac{1}{\mathcal{Q}(\delta)}\right]\int_\rho^\delta \psi(\varkappa)d\varkappa \leq \int_\rho^\delta \frac{\psi(\varkappa)}{\tilde{\mathcal{Q}}(\varkappa)}d\varkappa \leq \frac{1}{\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)}\int_\rho^\delta \psi(\varkappa)d\varkappa.$$

Using the facts that

$$\int_\rho^\delta \frac{\psi(\varkappa)}{\tilde{\mathcal{Q}}(\varkappa)}d\varkappa = \frac{1}{2}\left(\int_\rho^\delta \frac{\psi(\varkappa)}{\mathcal{Q}(\varkappa)}d\varkappa + \int_\rho^\delta \frac{\psi(\varkappa)}{\mathcal{Q}(\rho+\delta-\varkappa)}d\varkappa\right),$$

and

$$\int_\rho^\delta \frac{\psi(\varkappa)}{\mathcal{Q}(\rho+\delta-\varkappa)}d\varkappa = \int_\rho^\delta \frac{\psi(\rho+\delta-u)}{\mathcal{Q}(u)}du = \int_\rho^\delta \frac{\psi(u)}{\mathcal{Q}(u)}du,$$

we get the required inequalities. □

Proposition 2.6. *Let $\mathcal{Q}:[\rho,\delta]\to(0,\infty)$ be continuous AH-convex. Then for all $\varkappa\in[\rho,\delta]$, the following inequality holds:*

$$\frac{1}{\delta-\rho}\int_\rho^\delta \tilde{\mathcal{Q}}^2(\varkappa)d\varkappa \leq 4\left(\frac{1}{\mathcal{Q}(\rho)}+\frac{1}{\mathcal{Q}(\delta)}\right)^{-2}. \tag{2.2}$$

Proof. Let $\varkappa\in[\rho,\delta]$, from AH-convexity of $\mathcal{Q}$ we obtain

$$\begin{aligned}\mathcal{Q}(\varkappa) &= \mathcal{Q}\left(\frac{\varkappa-\delta}{\rho-\delta}\rho+\frac{\rho-\varkappa}{\rho-\delta}\delta\right)\\ &\leq \left(\frac{\varkappa-\delta}{\mathcal{Q}(\rho)(\rho-\delta)}+\frac{\rho-\varkappa}{(\rho-\delta)\mathcal{Q}(\delta)}\right)^{-1},\end{aligned}$$

then,

$$\begin{aligned}\int_\rho^\delta \mathcal{Q}^2(\varkappa)d\varkappa &\leq \int_\rho^\delta \left(\frac{\varkappa-\delta}{\mathcal{Q}(\rho)(\rho-\delta)}+\frac{\rho-\varkappa}{(\rho-\delta)\mathcal{Q}(\delta)}\right)^{-2}d\varkappa\\ &= (\delta-\rho)\mathcal{Q}(\rho)\mathcal{Q}(\delta).\end{aligned}$$

Using the Lemma 2.2, we get

$$\int_\rho^\delta \tilde{\mathcal{Q}}^2(\varkappa)d\varkappa \leq (\delta-\rho)\tilde{\mathcal{Q}}(\rho)\tilde{\mathcal{Q}}(\delta).$$

Since $\tilde{\mathcal{Q}}(\rho)=\tilde{\mathcal{Q}}(\delta)=\frac{2}{\frac{1}{\mathcal{Q}(\rho)}+\frac{1}{\mathcal{Q}(\delta)}}$, we deduce the desired result (2.2). □

Proposition 2.7. *Let $\mathcal{Q}:[\rho,\delta]\to(0,\infty)$ be a convex function. Then the following inequality holds*

$$\tilde{\mathcal{Q}}\left(\sum_{i=1}^n \alpha_i\varkappa_i\right) \leq 2\left(\frac{\mathcal{Q}(\sum_{i=1}^n \alpha_i\varkappa_i)}{\mathcal{Q}(\rho+\delta-\sum_{i=1}^n \alpha_i\varkappa_i)}+1\right)^{-1}\sum_{i=1}^n \alpha_i\mathcal{Q}(\varkappa_i),$$

where $\alpha_i\in(0,1)$ *for all* $i\in\{1;2;...;n\}$ *with* $\sum_{i=1}^n \alpha_i=1$ *and* $\varkappa_1,...,\varkappa_n\in[\rho,\delta]$.

Proof. Since $\mathcal{Q}$ is convex then, it follows from Jensen inequality [3] for convex function that $\mathcal{Q}\left(\sum_{i=1}^{n}\alpha_i\varkappa_i\right)\leq\sum_{i=1}^{n}\alpha_i\mathcal{Q}\left(\varkappa_i\right)$ where $\varkappa_i\in[\rho,\delta]$, $\alpha_i\in(0,1)$ for all $i\in\{1;2;...;n\}$ and $\sum_{i=1}^{n}\alpha_i=1$.
Also, by using this equality: $\mathcal{Q}=\tilde{\mathcal{Q}}\times\overset{\circ}{\mathcal{Q}}$ we get the desired result. □

Consider the positive functions $h:(0,1)\subset\mathbb{R}\to\mathbb{R}\quad(h\neq 0)$ and $\Phi:\mathcal{J}\subset\mathbb{R}\to\mathbb{R}$, where $(0,1)\subset\mathcal{J}$. Φ is called h-convex , if for all $u,v\in\mathcal{J}$ and $t\in(0,1)$, we have

$$\Phi(tu+(1-t)v)\leq h(t)\Phi(u)+h(1-t)\Phi(v).$$

See [26] for more details.

Definition 2.8. Consider the positive function $h:(0,1)\to\mathbb{R}$. A positive function $\mathcal{Q}:[\rho,\delta]\to\mathbb{R}\backslash\{0\}$ is called $h-AH$-convex if

$$\mathcal{Q}(m\varkappa+(1-m)y)\leq\frac{1}{\frac{h(m)}{\mathcal{Q}(\varkappa)}+\frac{h(1-m)}{\mathcal{Q}(y)}},$$

holds for any $\varkappa,y\in[\rho,\delta]$ and $m\in(0,1)$.

Theorem 2.9. *Let $h:(0,1)\to\mathbb{R}$ be a strictly positive function. If $\mathcal{Q}:[\rho,\delta]\to(0,\infty)$ is an $h-AH$-convex function then*

$$\begin{aligned}(\delta-\rho)h\left(\frac{1}{2}\right)\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)&\leq\int_{\rho}^{\delta}\frac{d\varkappa}{\frac{1}{\mathcal{Q}(\varkappa)}+\frac{1}{\mathcal{Q}(\rho+\delta-\varkappa)}}\\&\leq\frac{1}{\frac{1}{\mathcal{Q}(\rho)}+\frac{1}{\mathcal{Q}(\delta)}}\int_{\rho}^{\delta}\frac{d\varkappa}{h(\frac{\varkappa-\delta}{\rho-\delta})+h(\frac{\rho-\varkappa}{\rho-\delta})}.\end{aligned}\tag{2.3}$$

Proof. Similar to the proof of Lemma 2.2, we prove that if $\mathcal{Q}$ is $h-AH$-convex then $\tilde{\mathcal{Q}}$ is also $h-AH$-convex.
Let $\varkappa\in[\rho,\delta]$, since $\tilde{\mathcal{Q}}$ is $h-AH$-convex, then

$$\begin{aligned}\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)&=\tilde{\mathcal{Q}}\left(\frac{\rho+\delta-\varkappa+\varkappa}{2}\right)\\&\leq\frac{1}{h(\frac{1}{2})\left(\frac{1}{\tilde{\mathcal{Q}}(\rho+\delta-\varkappa)}+\frac{1}{\tilde{\mathcal{Q}}(\varkappa)}\right)}\\&=\frac{1}{2h(\frac{1}{2})}\tilde{\mathcal{Q}}(\varkappa),\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{Q}}(\varkappa)&=\tilde{\mathcal{Q}}\left(\rho\frac{\varkappa-\delta}{\rho-\delta}+\delta\frac{\rho-\varkappa}{\rho-\delta}\right)\\&\leq\frac{2}{(\frac{1}{\mathcal{Q}(\rho)}+\frac{1}{\mathcal{Q}(\delta)})\left(h(\frac{\varkappa-\delta}{\rho-\delta})+h(\frac{\rho-\varkappa}{\rho-\delta})\right)}.\end{aligned}$$

Consequently,

$$2\,(\delta-\rho)\,h\left(\frac{1}{2}\right)\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)\leq\int_{\rho}^{\delta}\tilde{\mathcal{Q}}(\varkappa)d\varkappa\leq\frac{2}{\frac{1}{\mathcal{Q}(\rho)}+\frac{1}{\mathcal{Q}(\delta)}}\int_{\rho}^{\delta}\frac{d\varkappa}{h(\frac{\varkappa-\delta}{\rho-\delta})+h(\frac{\rho-\varkappa}{\rho-\delta})}.$$

□

If we take $h(\varkappa)=\frac{1}{\varkappa}$ in Theorem 2.9, then the inequality (2.3) becomes

$$2(\delta-\rho)\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)\leq\int_{\rho}^{\delta}\frac{d\varkappa}{\frac{1}{\mathcal{Q}(\varkappa)}+\frac{1}{\mathcal{Q}(\rho+\delta-\varkappa)}}\leq\frac{1}{\frac{1}{\mathcal{Q}(\rho)}+\frac{1}{\mathcal{Q}(\delta)}}\int_{\rho}^{\delta}\frac{d\varkappa}{\frac{\rho-\delta}{\varkappa-\delta}+\frac{\rho-\delta}{\rho-\varkappa}}.$$

Proposition 2.10. *Assume that $\mathcal{Q}:[\rho,\delta]\to\mathbb{R}\backslash\{0\}$ is a symmetrized AH-convex function and $\Psi:[\rho,\delta]\to(0,\infty)$ is AH-convex function. Then*

$$\int_{\rho}^{\delta}\tilde{\Psi}^2(\varkappa)\tilde{\mathcal{Q}}(\varkappa)d\varkappa\leq 8(\delta-\rho)\left(\frac{1}{\mathcal{Q}(\rho)}+\frac{1}{\mathcal{Q}(\delta)}\right)^{-1}\left(\frac{1}{\Psi(\rho)}+\frac{1}{\Psi(\delta)}\right)^{-2}.$$

Proof. It follows from (2.1) that

$$\int_{\rho}^{\delta}\tilde{\Psi}^2(\varkappa)\tilde{\mathcal{Q}}(\varkappa)d\varkappa\leq\frac{2}{\frac{1}{\mathcal{Q}(\rho)}+\frac{1}{\mathcal{Q}(\delta)}}\int_{\rho}^{\delta}\tilde{\Psi}^2(\varkappa)d\varkappa.$$

Then by using (2.2), we obtain the desired result. □

Theorem 2.11. *Let $\mathcal{Q}:[\rho,\delta]\to\mathbb{R}\backslash\{0\}$ be a symmetrized AH-convex function and $\Psi:[\rho,\delta]\to(0,\infty)$ be an AH-convex function. Then*

$$\begin{aligned}&\frac{\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)}{2\Psi(\rho)}+\frac{\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)}{2\Psi(\delta)}-\frac{\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)}{\delta-\rho}\int_{\rho}^{\delta}\frac{d\varkappa}{\Psi(\varkappa)}\\&\geq\frac{1}{\Psi(\rho)(\delta-\rho)^2}\int_{\rho}^{\delta}(\delta-\varkappa)\tilde{\mathcal{Q}}(\varkappa)d\varkappa+\frac{1}{\Psi(\delta)(\delta-\rho)^2}\int_{\rho}^{\delta}(\varkappa-\rho)\tilde{\mathcal{Q}}(\varkappa)d\varkappa\\&\qquad-\frac{1}{\delta-\rho}\int_{\rho}^{\delta}\frac{\tilde{\mathcal{Q}}(\varkappa)}{\Psi(\varkappa)}d\varkappa.\end{aligned}$$

Proof. Let $\alpha\in[0,1]$, by using the AH-convexity of Ψ and the inequality (2.1), we obtain

$$\left\{\frac{\alpha}{\Psi(\rho)}+\frac{1-\alpha}{\Psi(\delta)}-\frac{1}{\Psi(\alpha\rho+(1-\alpha)\delta)}\right\}\left\{\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)-\tilde{\mathcal{Q}}(\alpha\rho+(1-\alpha)\delta)\right\}\geq 0.$$

Integrating the above inequality over $\alpha\in[0,1]$, we obtain

$$\frac{\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)}{2\Psi(\rho)}+\frac{\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)}{2\Psi(\delta)}-\int_0^1\frac{\mathcal{Q}\left(\frac{\rho+\delta}{2}\right)}{\Psi(\alpha\rho+(1-\alpha)\delta)}d\alpha$$
$$\geq\int_0^1\frac{\alpha\tilde{\mathcal{Q}}(\alpha\rho+(1-\alpha)\delta)}{\Psi(\rho)}d\alpha+\int_0^1\frac{(1-\alpha)\tilde{\mathcal{Q}}(\alpha\rho+(1-\alpha)\delta)}{\Psi(\delta)}d\alpha$$
$$-\int_0^1\frac{\tilde{\mathcal{Q}}(\alpha\rho+(1-\alpha)\delta)}{\Psi(\alpha\rho+(1-\alpha)\delta)}d\alpha.$$

Hence, the desired result can be obtained. □

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