

COLLINEAR CONFIGURATION IN THE CIRCULAR RESTRICTED FOUR-BODY PROBLEM WITH A VARIABLE MASS

ABDULLAH A. ANSARI^{1, 2, *}

ABSTRACT. This paper investigates the motion properties of the fourth infinitesimal variable mass body under the gravitational forces of the three collinear spherical primaries. We determine the equations of motion and quasi-Jacobi integral for this configuration. The locations of parking points, regions of motion and Poincaré surfaces of section are investigated. Finally, the stability states for the parking points are examined.

1. INTRODUCTION

The restricted four-body problem is the extension of the restricted three-body problem in the celestial mechanics. In the restricted four-body problem, three bodies are taken as primaries and fourth body is known as infinitesimal body for which researchers are interested to studied the motion properties. These primaries can be placed either at the vertices of the triangle (known as triangular configuration) or at the same straight line (known as collinear configuration). Till now many scientists have studied the restricted four body problem with triangular configuration. Where they have taken many perturbations such as shapes of the primaries (spherical, point, triaxial, oblate, heterogeneous etc.), solar radiation pressure etc. [9], [17], [15], [13], [18], [12], [8], [19], [23], [7], [1], [20, 21].

The effect of variation of mass is studied by many researchers [22], [5], [4], [24], [3, 6].

This work is organized in seven sections as: The history of literature is presented in section 1. The equations of motion and quasi-Jacob integral are determined in section 2 while section 3 presents the location of parking points. Sections 4 illustrate the regions of motion and section 5 explained the Poincaré surfaces of section. Section 6 examined the stability states for the parking points and finally paper ends with the conclusion in section 7.

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* Corresponding author.

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2. EQUATIONS OF MOTION

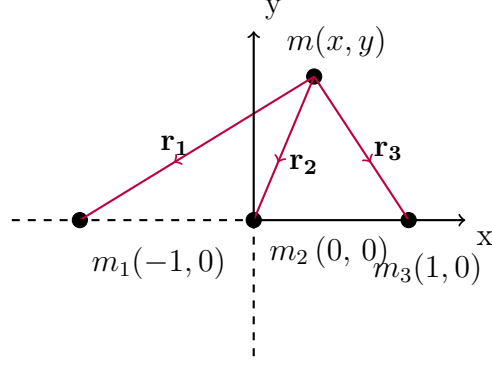


FIGURE 1. Collinear configuration in the circular restricted four-body problem

Assuming with four bodies, out of these, three spherical primary bodies are placed at x-axis and fourth infinitesimal body is moving in xy-plane under the gravitational forces of the three primary bodies while fourth body is not affecting them.

Before doing the investigations regarding the equations of motion, we would like to determine the magnitude of the mean motion n . For this we suppose that the distances between m_1 , m_2 and m_2 , m_3 are d_1 and d_2 respectively, and also m_1 as well as m_3 are moving circular around their common center of mass which is taken as origin where m_2 is placed. Now we can write the expression for the mean motion as:

$$n^2 = \frac{G(m_1 + m_3)}{(d_1 + d_2)^3} + \frac{G m_2}{(d_1 + d_2)} \left(\frac{1}{d_1^2} + \frac{1}{d_2^2} \right). \quad (2.1)$$

After fixing the units as the mass of the single primary, the distance between two primaries and the gravitational constant G are separately taken as unity, we get the value of n as $\frac{\sqrt{5}}{2}$.

Following the procedure given by [2] and [14], the equations of motion of fourth body of variable mass in the non-dimensional units can be performed as:

$$\begin{aligned} \frac{\dot{m}}{m}(\dot{x} - n y) + \ddot{x} - 2 n \dot{y} &= \frac{\partial U}{\partial x}, \\ \frac{\dot{m}}{m}(\dot{y} + n x) + \ddot{y} + 2 n \dot{x} &= \frac{\partial U}{\partial y}, \end{aligned} \quad (2.2)$$

where,

$$\begin{aligned}
U &= \frac{n^2}{2}(x^2 + y^2) + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}, \\
r_1^2 &= (x+1)^2 + y^2, \\
r_2^2 &= x^2 + y^2, \\
r_3^2 &= (x-1)^2 + y^2.
\end{aligned} \tag{2.3}$$

Due to variable mass of the fourth body, we will use Jean's law [11] and Meshcherskii space-time transformations [16]. These are as follow:

$$m = m_0 e^{-\alpha_1 t}, \quad (x, y) = \alpha_2^{-1/2}(\xi, \eta), \tag{2.4}$$

where α_1 is variation constant and $\alpha_2 = \frac{m}{m_0}$, where m_0 is the starting mass of the fourth body.

Utilizing Eqs. (2.2) and (2.4), we obtain

$$\begin{aligned}
\ddot{\xi} - 2n\dot{\eta} &= \frac{\partial V}{\partial \xi}, \\
\ddot{\eta} + 2n\dot{\xi} &= \frac{\partial V}{\partial \eta},
\end{aligned} \tag{2.5}$$

where,

$$\begin{aligned}
V &= \left(\frac{n^2}{2} + \frac{\alpha_1^2}{8} \right) (\xi^2 + \eta^2) + \alpha_2^{3/2} \left\{ \frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} \right\}, \\
\rho_1^2 &= (\xi + \alpha_2^{1/2})^2 + \eta^2, \\
\rho_2^2 &= \xi^2 + \eta^2, \\
\rho_3^2 &= (\xi - \alpha_2^{1/2})^2 + \eta^2.
\end{aligned} \tag{2.6}$$

The Quasi-Jacobi integral will be admitted from the Eq. (2.5) as

$$v^2 = 2V - C - 2 \int_{t_0}^t \left(\frac{\partial V}{\partial t} \right) dt, \tag{2.7}$$

where $v = \sqrt{\dot{\xi}^2 + \dot{\eta}^2}$ is the velocity of the infinitesimal body and C is the quasi-Jacobi energy constant. If we suppose $\alpha_1 = 0$ and $\alpha_2 = 1$, then the above variable mass system will reduce to the constant mass system.

3. PARKING POINTS

Parking points can be obtained by solving the right hand side of Eq. (2.5) numerically as:

$$\begin{aligned}\frac{\partial V}{\partial \xi} &= 0, \\ \frac{\partial V}{\partial \eta} &= 0.\end{aligned}\tag{3.1}$$

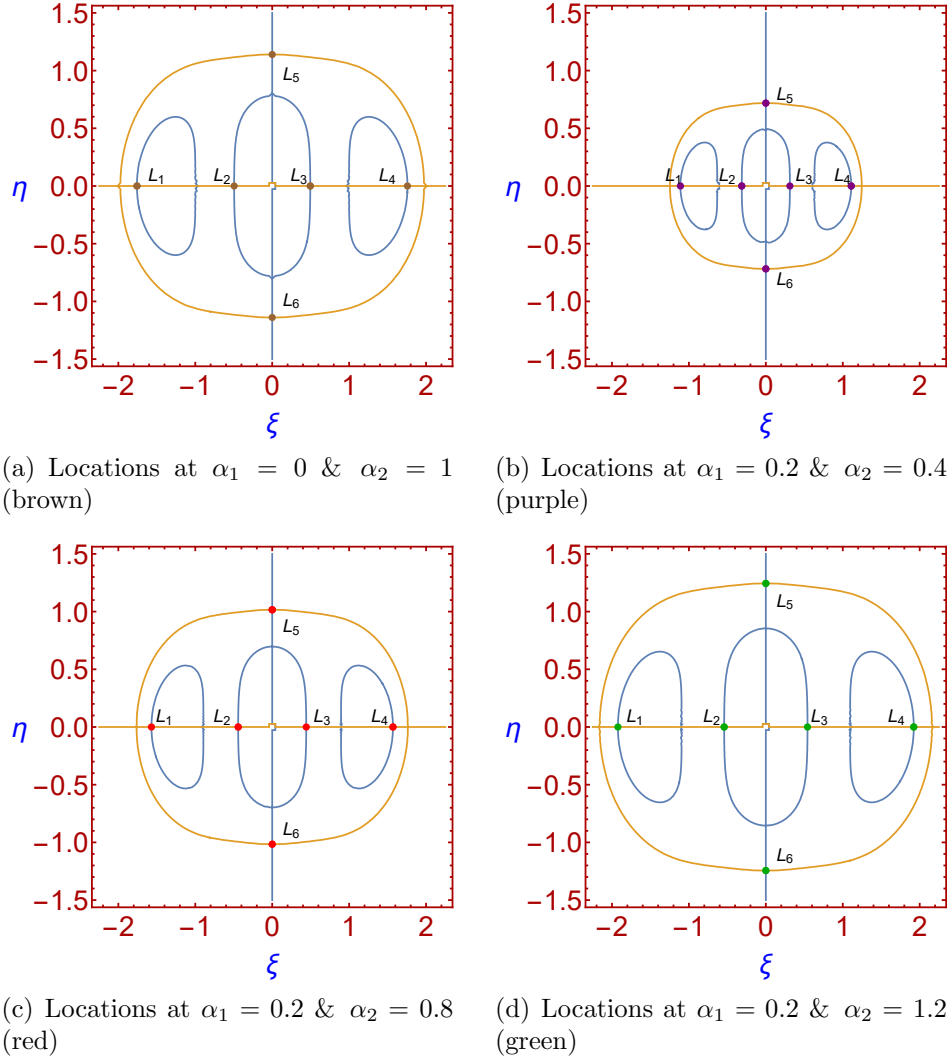
After solving Eq. (3.1) numerically, we get six parking points out of which four are collinear parking points $L_{1,2,3,4}$ and two are non-collinear parking points $L_{5,6}$. These are presented in figure (2). We observed from this figure that out of four collinear parking points, two $L_{1,2}$ and $L_{3,4}$ are symmetrical about η -axis respectively as well as $L_{5,6}$ are symmetrical about ξ -axis. We have plotted these for various values of variation parameters use ($\alpha_1 = 0$ & $\alpha_2 = 1$ (brown), $\alpha_1 = 0.2$ & $\alpha_2 = 0.4$ (purple), 0.8 (red), 1.2 (green)) and given in the sub-figures (2(a)), (2(b)), (2(c)), (2(d)). We observed from here that as increase the value of the parameters locations of the parking points are moving away. We also observed that when the primaries are placed at the vertices of an equilateral triangle, there are eleven parking points exist [8] while when we have considered the three primaries on the same straight line then we got six parking points.

4. ALLOWED REGIONS OF MOTION

Following the procedure given by [2], we have illustrated the allowed regions of motion of the infinitesimal boy for various values of the variation parameters used and given in figures (3) (at $\alpha_2 = 1$) and (4) (at $\alpha_2 = 0.4$). For this first we have evaluated the value of Jacobi constant (C) corresponding to each parking points and then with the help of these values we have plotted the allowed regions of motion which are shown in the figures as white parts while shaded parts are prohibited regions. In both the figures, sub-figures (a), (b) and (c) are representing the regions corresponding to the parking points $L_{1,4}$, $L_{2,3}$ and $L_{5,6}$ respectively. The sub-figure (a) reveals that infinitesimal body can move every where except near the parking points $L_{5,6}$ and the parking points $L_{1,4}$ act as gateway for the regions near the parking points $L_{2,3}$. The sub-figure (b) reveals that infinitesimal body can move every where except near the parking points $L_{1,4,5,6}$ and the parking points $L_{2,3}$ act as gateway for the regions around the parking points $L_{2,3}$. The sub-figure (c) reveals that infinitesimal body can move every where and there is no prohibited regions.

5. POINCARÉ SURFACES OF SECTION

We have illustrated the surfaces of section with the help of Eqs. (2.5) and (2.7). First of all, we have to change the system of equations into phase space and then draw until when the orbit intersects the plane $\dot{\eta} \geq 0$. Finally, we get the surfaces of section in $\xi - \dot{\xi}$ -plane by known Mathematica software for various values of the variation parameters used and given in figure (5) (a, b, c, d). Sub-figure (a), (b), (c) and (d) represent the Poincaré surfaces of section for the various values of

FIGURE 2. Locations of parking points in $\xi - \eta$ -plane.

$\alpha_2 = 1$ (brown), 0.4 (purple), 0.8 (red) and 1.2 (green) respectively. We observed from here that these surfaces are symmetrical about both axes and also there is no chaos.

6. STABILITY STATES

In this section, we shall perform the stability states for each parking points numerically. Since $L_{1,2}$ and $L_{3,4}$ are symmetrical about η -axis as well as L_5 and L_6 are symmetrical about ξ -axis. Therefore it is sufficient to show the stability of $L_{1,2,5}$. Following the procedure given by [10], we can write the characteristic polynomial of the Eq. (2.5) in the planar case as:

$$\lambda^4 + A_3 \lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0 = 0, \quad (6.1)$$

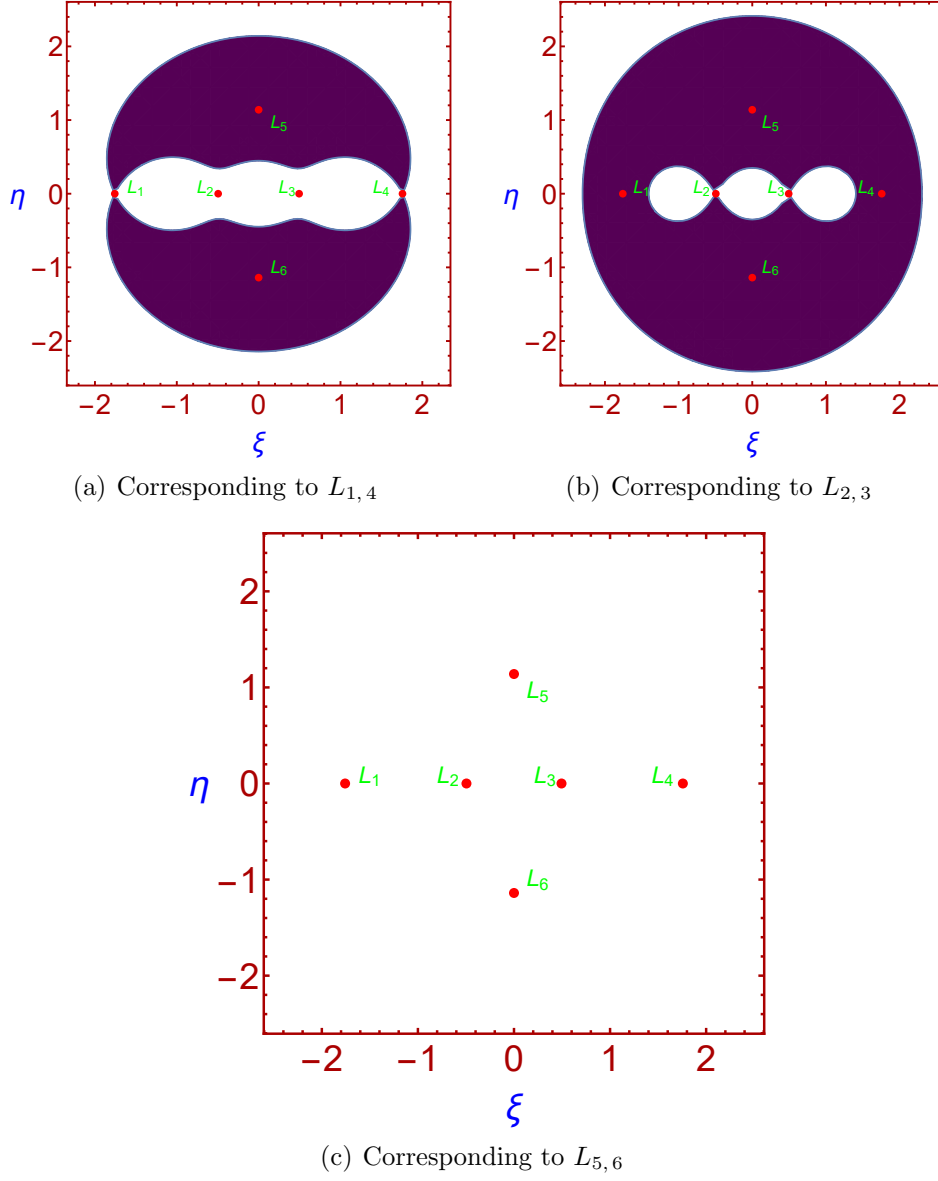


FIGURE 3. Allowed regions of motion at $\alpha_1 = 0$ & $\alpha_2 = 1$ in $\xi - \eta$ -plane.

where

$$\begin{aligned}
 A_3 &= -2\alpha_1, \\
 A_2 &= 4\omega^2 + \frac{3\alpha_1^2}{2} - (V_{\xi\xi})^0 - (V_{\eta\eta})^0, \\
 A_1 &= -\alpha_1(4\omega^2 + \frac{\alpha_1^2}{2} - (V_{\xi\xi})^0 - (V_{\eta\eta})^0), \\
 A_0 &= \frac{\alpha_1^4}{16} + \frac{\alpha_1^2}{4}(4\omega^2 - (V_{\xi\xi})^0 - (V_{\eta\eta})^0) + (V_{\xi\xi})^0(V_{\eta\eta})^0 - ((V_{\xi\eta})^0)^2,
 \end{aligned} \tag{6.2}$$

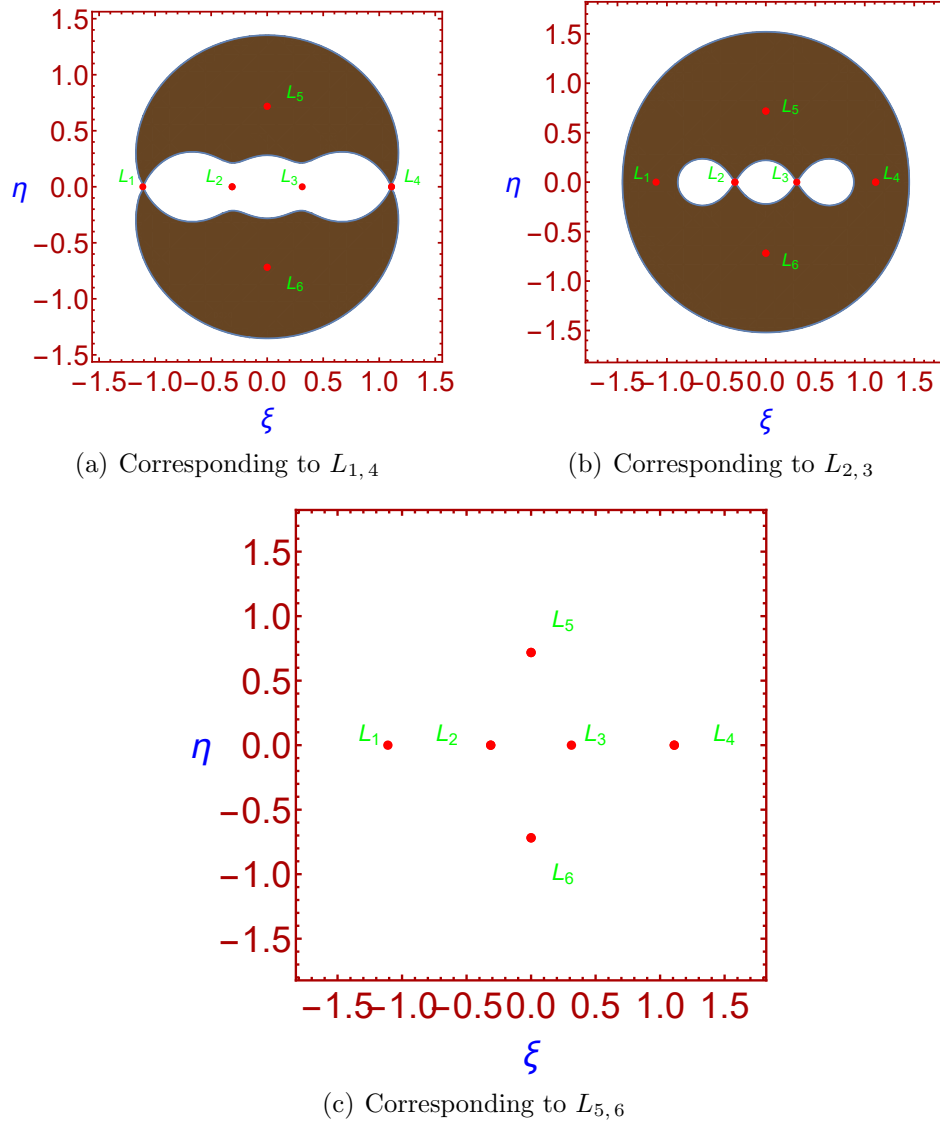
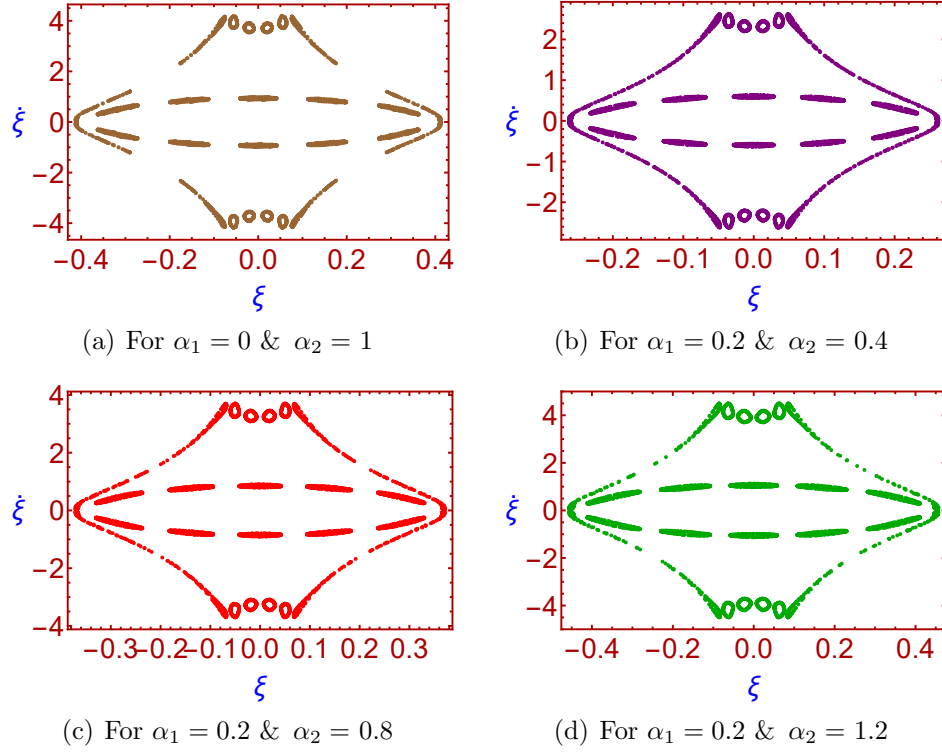


FIGURE 4. Allowed regions of motion at $\alpha_1 = 0.2$ & $\alpha_2 = 0.4$ in $\xi - \eta$ -plane.

the superscript zero represents the values of the derivatives of the potential function V corresponding to the parking point. If we ignore the variable mass then the odd coefficients A_3 and A_1 will vanish and then the reduced characteristic equations can be written as the quadratic equation in λ^2 and hence we can easily find the values of the roots. But analytical evaluation is slightly difficult in our case therefore we have numerically solved Eq. (6.1) corresponding to each parking points and given in tables (1, 2). From here we got that in both the cases (constant as well as variable mass) all the parking points are unstable. Our results confirm to the results obtained by [14].

FIGURE 5. Poincaré surfaces of section in $\xi - \dot{\xi}$ -plane.TABLE 1. The nature of parking points in $\xi\eta$ -plane at $\alpha_1 = 0$, $\alpha_2 = 1$.

<i>Parking Point</i>		<i>Roots</i>	<i>Nature</i>
$\xi - Co.$	$\eta - Co.$		
± 1.7576799791	0.0000000000	$\pm 1.6816423808 i$ ± 1.6907919469	<i>Unstable</i>
± 0.4946664910	0.0000000000	$\pm 4.0880772231 i$ ± 5.5247426529	<i>Unstable</i>
0.0000000000	± 1.1394282249	$\pm 0.6612296196 \pm 1.0306428139 i$	<i>Unstable</i>

7. CONCLUSION

The dynamical properties of the motion of the fourth variable mass body are studied under the influence of three spherical equal primaries placed on the same line. The evaluated equations of motion shown the effects of these perturbations. And then we have determined the quasi-Jacobi integral from where we have shown the allowed regions of motion for the infinitesimal body. The six parking points are shown here. Out of these six parking points four are collinear and two are

TABLE 2. The nature of parking points in $\xi\eta$ -plane at $\alpha_1 = 0.2$, $\alpha_2 = 0.4$.

<i>Parking Point</i>		<i>Roots</i>	<i>Nature</i>
$\xi - Co.$	$\eta - Co.$		
± 1.1099032534	0.0000000000	$0.0999999999 \pm 1.6844259713 i$ - 1.6072064597 1.8072064597	<i>Unstable</i>
± 0.3127622208	0.0000000000	$0.0999999999 \pm 4.0868094974 i$ - 5.4256775344 5.6256775344	<i>Unstable</i>
0.0000000000	± 0.7182492501	$\pm 0.6612296196 \pm 1.0306428139 i$	<i>Unstable</i>

non-collinear. The four collinear parking points lie on the ξ -axis while two non-collinear parking points lie on the η -axis. We also observed from there that as increase the values of the parameters, the parking points are moving away from the origin. On the other hand the Poincare surfaces of section is also one of the important dynamical properties. Here we observed that there is no chaos and the Poincaré surfaces of section obtained are symmetrical about both axes. In the last section, we have examined the stability states of the parking points and we found that all the parking points are unstable. And hence the perturbation parameters used, have great influence on the motion properties.

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¹ SHIVAJI COLLEGE, UNIVERSITY OF DELHI, DELHI, INDIA.
 Email address: abdullah@shivaji.du.ac.in

² INTERNATIONAL CENTER FOR ADVANCED INTERDISCIPLINARY RESEARCH (ICAIR), SANGAM VIHAR, NEW DELHI, INDIA.
 Email address: icairndin@gmail.com