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# INNOVATIVE OPTION PRICING USING THE CONFORMABLE BLACK SCHOLES MODEL

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Abstract. Fractional calculus has become an important tool in financial modeling due to its ability to capture memory effects and anomalous diffusion observed in financial markets. In this paper, we develop a fractional extension of the classical Black–Scholes option pricing model using the conformable fractional derivative. Owing to its local nature and preservation of key properties of classical differentiation, the conformable derivative allows for an analytically tractable and arbitrage-free framework. We derive the conformable fractional Black–Scholes partial differential equation, obtain closed-form solutions for European options, and provide a financial interpretation of the fractional order parameter.

*Keywords.* Black Scholes model, conformable fractional derivative, fractional calculus, option pricing, anomalous diffusion
*2020 Mathematics Subject Classification.* Primary 91G20, 26A33; Secondary 35Q91, 60G22

## Introduction

The Black–Scholes model remains one of the cornerstones of modern mathematical finance, providing a closed-form solution for European option pricing under a set of idealized assumptions [1] [2] [3]. Central to this framework is the hypothesis that the underlying asset price evolves according to a geometric Brownian motion with constant volatility and risk-free interest rate, leading to a Markovian and memoryless stochastic process. The resulting Black–Scholes partial differential equation admits elegant analytical solutions and has played a fundamental role in both theoretical developments and practical applications within financial markets.

Despite its success, extensive empirical studies have revealed systematic discrepancies between the assumptions of the classical Black–Scholes model and observed market behavior [13, 12]. Financial time series often display heavy-tailed

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return distributions, volatility clustering, long-range dependence, and deviations from normal diffusion. In particular, asset price dynamics may exhibit memory effects and anomalous diffusion, where the variance of returns scales nonlinearly with time [10]. Such features cannot be fully captured by standard Brownian motion, which assumes independent increments and linear time scaling. These empirical findings have motivated the development of generalized stochastic models that relax the Markovian assumption and incorporate richer temporal structures.

One prominent approach to modeling these phenomena involves the use of fractional calculus, which generalizes classical differentiation and integration to non-integer orders [5, 6, 7, 8]. Fractional derivatives naturally encode memory and hereditary properties, making them well suited for describing processes with long-range dependence and anomalous diffusion. Over the past two decades, fractional calculus has been successfully applied in various fields, including physics, biology, engineering, and increasingly, quantitative finance [13, 11]. In financial modeling, fractional operators have been employed to generalize diffusion equations, stochastic volatility models, and option pricing frameworks in order to better reflect observed market dynamics [9].

Several definitions of fractional derivatives exist in the literature, including the Riemann–Liouville, Caputo, Grünwald–Letnikov, and Hadamard derivatives [5, 6]. While these operators are mathematically rigorous and capable of modeling nonlocal behavior, they often introduce significant analytical and computational challenges. In particular, their inherent nonlocality leads to integro-differential equations with memory kernels, complicating both theoretical analysis and numerical implementation. Moreover, some classical properties of ordinary calculus, such as the chain rule and product rule, do not hold in their standard form, which may hinder their direct application in financial modeling.

In recent years, the conformable fractional derivative has emerged as an alternative definition that seeks to balance the expressive power of fractional calculus with the simplicity of classical differentiation [4]. Introduced to preserve many of the fundamental properties of ordinary derivatives, the conformable derivative is local in nature and admits a clear geometric and physical interpretation. It satisfies linearity, the product rule, the quotient rule, and a modified chain rule, while reducing to the classical derivative when the fractional order equals one. These features make the conformable derivative particularly attractive for applications where analytical tractability and intuitive interpretation are crucial.

The locality of the conformable derivative distinguishes it from traditional fractional derivatives and leads to significant simplifications in modeling and computation [11]. Rather than incorporating explicit memory integrals, the conformable derivative introduces a fractional time-scaling that modifies the effective rate of change of a function. This property allows the resulting differential equations to retain a structure closely resembling their classical counterparts, facilitating analytical solutions and reducing computational complexity. Consequently, the conformable derivative has been applied in diverse areas such as control theory, diffusion processes, and dynamic systems, and is increasingly being explored in financial mathematics [8].

In the context of option pricing, incorporating fractional dynamics provides a mechanism to account for deviations from classical diffusion without abandoning the familiar framework of partial differential equations. Fractional Black–Scholes-type models have been proposed using various fractional derivatives, leading to modified pricing equations that capture memory effects and anomalous time behavior [13, 11]. However, many of these models result in complex integro-differential equations that require sophisticated numerical techniques and may lack closed-form solutions. This limits their practical applicability, particularly in real-time pricing and calibration.

The conformable fractional derivative offers a promising alternative by enabling the formulation of fractional option pricing models that remain analytically tractable [4, 11]. By replacing the classical time derivative in the Black–Scholes equation with its conformable fractional counterpart, one obtains a modified equation that incorporates fractional time-scaling while preserving much of the original mathematical structure. This approach allows for the derivation of explicit pricing formulas under suitable assumptions and provides a clear interpretation of the fractional order as a parameter governing the effective flow of time in the market.

In this work, we develop a conformable fractional extension of the Black–Scholes model and investigate its implications for European option pricing. Starting from a fractional version of the underlying asset dynamics, we derive the corresponding conformable fractional Black–Scholes equation. The analytical properties of the model are examined, and conditions for the existence and uniqueness of solutions are discussed. Furthermore, we analyze how the fractional order influences option prices, sensitivity measures, and the temporal evolution of the pricing function.

The proposed framework aims to bridge the gap between classical option pricing theory and empirical observations of anomalous market behavior. By maintaining analytical simplicity while introducing fractional time effects, the conformable fractional Black–Scholes model provides a flexible and efficient tool for modeling financial derivatives. The results presented in this study suggest that conformable fractional calculus [2] constitutes a viable and meaningful extension of classical option pricing theory, with potential applications in model calibration, risk management, and the analysis of markets exhibiting nonstandard diffusion characteristics.

## 1. Conformable Fractional Derivative

1.1. **Definition.** Let $f:(0,\infty)\to\mathbb{R}$ be a real-valued function and let $\alpha\in(0,1]$. The conformable fractional derivative of order $\alpha$ is defined by

$$D_t^{\alpha} f(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}, \tag{1.1}$$

provided that the limit exists.

This definition generalizes the classical derivative by introducing a fractional time-scaling factor $t^{1-\alpha}$. When $\alpha = 1$, the conformable derivative reduces to the standard first-order derivative, namely

$$D_t^1 f(t) = \frac{df(t)}{dt}.$$

If the function $f$ is differentiable in the classical sense, the conformable fractional derivative admits the explicit representation

$$D_t^\alpha f(t) = t^{1-\alpha}\frac{df(t)}{dt}, \tag{1.2}$$

which highlights the local nature of the operator. This expression shows that the conformable derivative does not involve memory integrals or nonlocal kernels, in contrast to other definitions of fractional derivatives. As a consequence, differential equations involving conformable derivatives preserve a structure closely resembling that of classical differential equations.

1.2. **Basic Properties.** The conformable fractional derivative satisfies several fundamental properties analogous to those of the classical derivative. Let $f$ and $g$ be $\alpha$-differentiable functions and let $a, b \in \mathbb{R}$. Then the following hold:

- **Linearity:**
$$D_t^\alpha(af(t) + bg(t)) = aD_t^\alpha f(t) + bD_t^\alpha g(t).$$
- **Product rule:**
$$D_t^\alpha(f(t)g(t)) = f(t)D_t^\alpha g(t) + g(t)D_t^\alpha f(t).$$
- **Quotient rule:** If $g(t) \neq 0$, then
$$D_t^\alpha\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)D_t^\alpha f(t) - f(t)D_t^\alpha g(t)}{[g(t)]^2}.$$
- **Chain rule:** If $f$ is differentiable and $g$ is $\alpha$-differentiable, then
$$D_t^\alpha(f(g(t))) = t^{1-\alpha}f'(g(t))g'(t).$$
- **Derivative of a constant:**
$$D_t^\alpha c = 0,$$
for any constant $c \in \mathbb{R}$.

These properties ensure that many classical analytical techniques remain valid in the conformable fractional setting, allowing for straightforward manipulation of fractional differential equations.

1.3. **Comparison with Classical Fractional Derivatives.** Unlike the Riemann–Liouville and Caputo fractional derivatives, the conformable fractional derivative is a local operator and does not involve convolution-type integral memory terms. Classical fractional derivatives are inherently nonlocal, as their definitions depend on the entire past history of the function through singular kernel integrals. This nonlocality makes them particularly effective for modeling long-range dependence, hereditary phenomena, and persistent memory effects observed in various physical and financial systems. However, it also leads to fractional differential equations of integro-differential type, which are often analytically intractable and computationally demanding.

In contrast, the conformable derivative introduces fractional behavior through a deterministic time-scaling mechanism. By weighting the classical derivative with the factor $t^{1-\alpha}$, the conformable operator modifies the effective speed of temporal evolution without explicitly incorporating memory kernels. As a result, the resulting differential equations retain a form that closely resembles their classical counterparts. This structural similarity allows many standard analytical techniques, such as separation of variables, change of variables, and similarity transformations, to be directly applied in the fractional setting.

From a mathematical perspective, the conformable derivative preserves several essential properties of ordinary calculus, including linearity, the product rule, and a natural chain rule. These properties do not generally hold in their standard form for Riemann–Liouville or Caputo derivatives and often require additional correction terms. The preservation of these properties significantly simplifies both theoretical analysis and numerical implementation, particularly when dealing with nonlinear models or multidimensional partial differential equations.

In financial modeling, this distinction is of particular importance. Option pricing models require fast and reliable numerical methods, as well as, whenever possible, closed-form or semi-analytical solutions for calibration and risk management. Fractional models based on classical nonlocal derivatives may offer improved realism but often at the cost of increased computational complexity and reduced interpretability. The conformable derivative provides a compromise by incorporating fractional time effects while maintaining analytical tractability and computational efficiency.

Moreover, the fractional order $\alpha$ in the conformable framework admits a clear interpretation as a parameter governing the effective flow of time in the market. Values of $\alpha < 1$ correspond to slower temporal evolution, which can be associated with subdiffusive behavior, liquidity constraints, or periods of reduced trading activity. This interpretation contrasts with classical fractional derivatives, where the fractional order is linked to the strength of memory effects rather than explicit time scaling.

Overall, while the conformable fractional derivative does not explicitly model long-range memory in the same way as traditional fractional derivatives, it offers a flexible and mathematically convenient alternative for extending classical financial models. Its ability to preserve the structure of partial differential equations makes it particularly suitable for fractional generalizations of the Black–Scholes model and related option pricing frameworks.

1.4. **Relevance to Financial Modeling.** In the context of option pricing, the conformable fractional derivative provides a natural and flexible mechanism for incorporating anomalous temporal effects and market heterogeneity into classical pricing models. The fractional order $\alpha$ can be interpreted as a parameter controlling the effective flow of time in the market, with smaller values of $\alpha$ corresponding to slower diffusion or subdiffusive behavior. Such dynamics may arise from trading frictions, liquidity constraints, heterogeneous market participation, or periods of reduced information flow. This interpretation is consistent with empirical observations indicating that asset price dynamics frequently deviate from classical Brownian motion, particularly during episodes of market stress, low liquidity, or structural change.

From a modeling perspective, the conformable fractional framework allows for the adjustment of temporal scaling without altering the fundamental stochastic structure of the underlying asset dynamics. As a result, it provides a parsimonious extension of the classical Black–Scholes model, introducing a single additional parameter while preserving interpretability. The fractional order $\alpha$ can be calibrated to market data, offering a quantitative measure of deviation from classical diffusion and enabling improved model flexibility.

An important advantage of the conformable derivative lies in its analytical tractability. Since it preserves the local structure of differential operators and key properties of classical calculus, the resulting fractional pricing equations remain partial differential equations rather than integro-differential equations. This facilitates the derivation of closed-form or semi-analytical pricing formulas and allows the direct application of standard numerical schemes. Consequently, the conformable fractional approach is well suited for practical applications such as real-time pricing, calibration, and risk management.

Moreover, the structural similarity between the conformable fractional Black–Scholes equation and the classical model enables a transparent comparison between classical and fractional dynamics. This makes it possible to isolate and analyze the specific impact of fractional time effects on option prices, sensitivities, and hedging strategies. In subsequent sections, the conformable fractional derivative is employed to derive a fractional extension of the Black–Scholes equation, and its implications for option valuation and market dynamics are systematically investigated.

## 2. Fractional Black–Scholes Market Model

2.1. **Risk-Free Asset.** We begin by describing the dynamics of the risk-free asset within the conformable fractional framework. Let $B(t)$ denote the value of a risk-free money market account at time $t$. Its evolution is governed by the conformable fractional differential equation

$$D_t^\alpha B(t) = rB(t), \tag{2.1}$$

where $r > 0$ is the constant risk-free interest rate and $\alpha \in (0, 1]$ denotes the order of the conformable derivative.

Using the explicit representation of the conformable derivative for differentiable functions, the above equation can be rewritten as

$$t^{1-\alpha}\frac{dB(t)}{dt} = rB(t),$$

which yields the solution

$$B(t) = B(0)\exp\left(\frac{r}{\alpha}t^\alpha\right). \tag{2.2}$$

When $\alpha = 1$, the classical exponential growth of the risk-free asset is recovered. For $\alpha < 1$, the growth rate is effectively slowed, reflecting fractional time-scaling effects. This formulation preserves the no-arbitrage interpretation of the risk-free asset while allowing for a modified temporal evolution consistent with the conformable fractional setting.

2.2. **Risky Asset.** Let $S(t)$ denote the price of a risky asset, such as a stock, at time $t$. In the conformable fractional Black–Scholes market, the asset price dynamics are described by the conformable fractional stochastic differential equation

$$D_t^\alpha S(t) = \mu S(t) + \sigma S(t)\dot{W}(t), \tag{2.3}$$

where $\mu$ represents the expected rate of return, $\sigma > 0$ is the volatility, and $W(t)$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

Using the relation between the conformable derivative and the classical derivative, the above equation can be expressed in differential form as

$$dS(t) = \mu t^{\alpha-1} S(t)\, dt + \sigma t^{\alpha-1} S(t)\, dW(t). \tag{2.4}$$

This representation shows that both the drift and diffusion coefficients are modulated by the fractional time-scaling factor $t^{\alpha-1}$. As a result, the effective volatility and drift decrease over time when $\alpha < 1$, producing subdiffusive behavior in the asset price process.

The stochastic differential equation admits the explicit solution

$$S(t) = S(0) \exp\left( \frac{\mu - \frac{1}{2}\sigma^2}{\alpha} t^\alpha + \frac{\sigma}{\alpha} \int_0^t s^{\alpha-1}\, dW(s) \right), \tag{2.5}$$

which generalizes the classical geometric Brownian motion. In the special case $\alpha = 1$, the standard Black–Scholes stock price dynamics are recovered.

2.3. **Market Assumptions.** The fractional Black–Scholes market considered in this work is assumed to be frictionless, arbitrage-free, and complete. Trading takes place continuously in time, and there are no transaction costs, taxes, or restrictions on borrowing or lending at the risk-free interest rate. All assets are perfectly divisible, and short selling is permitted without constraints. These assumptions are standard in the classical Black–Scholes framework and are retained here in order to isolate and analyze the specific impact of fractional time dynamics introduced by the conformable derivative.

Market completeness implies that every contingent claim can be perfectly replicated by a self-financing trading strategy involving the risk-free asset and the risky asset. This property is essential for the uniqueness of derivative prices and for the validity of hedging arguments based on dynamic portfolio replication. Although the introduction of fractional time-scaling modifies the temporal structure of asset dynamics, the locality of the conformable derivative ensures that the replication strategy remains well defined within a differential framework.

The absence of arbitrage guarantees the existence of an equivalent martingale measure under which the discounted price process of the risky asset becomes a martingale. In the conformable fractional setting, discounting is performed with respect to the fractional risk-free asset $B(t)$, whose growth reflects the modified time scale. Under the corresponding risk-neutral measure, the drift of the risky asset is adjusted to the risk-free rate, while the fractional diffusion structure is preserved.

These assumptions enable the application of risk-neutral valuation principles to the conformable fractional market. In particular, the price of a European contingent claim can be expressed as the discounted expected value of its payoff under the equivalent martingale measure. In the following section, these principles are employed to derive the conformable fractional Black–Scholes partial differential equation governing European option prices and to analyze its fundamental properties.

2.4. **Risk-Neutral Dynamics.** We now derive the risk-neutral dynamics of the risky asset in the conformable fractional Black–Scholes market. Recall that under the physical probability measure $\mathbb{P}$, the stock price $S(t)$ satisfies

$$dS(t) = \mu t^{\alpha-1} S(t)\, dt + \sigma t^{\alpha-1} S(t)\, dW(t), \tag{2.6}$$

where $\mu$ is the drift, $\sigma > 0$ is the volatility, and $W(t)$ is a standard Brownian motion under $\mathbb{P}$.

Let $B(t)$ denote the risk-free asset, whose dynamics are given by

$$dB(t) = r t^{\alpha-1} B(t)\, dt, \tag{2.7}$$

with explicit solution

$$B(t) = B(0) \exp\left(\frac{r}{\alpha} t^{\alpha}\right).$$

2.4.1. *Discounted Stock Price.* Define the discounted stock price process by

$$\widetilde{S}(t) = \frac{S(t)}{B(t)}. \tag{2.8}$$

Applying Itô's formula to $\widetilde{S}(t)$ and using the dynamics of $S(t)$ and $B(t)$, we obtain

$$\begin{aligned} d\widetilde{S}(t) &= \frac{1}{B(t)}\, dS(t) - \frac{S(t)}{B(t)^2}\, dB(t) \\ &= \widetilde{S}(t) \left[ (\mu - r) t^{\alpha-1}\, dt + \sigma t^{\alpha-1}\, dW(t) \right]. \end{aligned} \tag{2.9}$$

Thus, the discounted stock price has drift $(\mu - r)t^{\alpha-1}$ under the physical measure $\mathbb{P}$.

2.4.2. *Change of Measure.* To eliminate this drift and enforce the martingale property, we introduce a new probability measure $\mathbb{Q}$ via Girsanov's theorem. Define the market price of risk by

$$\lambda(t) = \frac{\mu - r}{\sigma}, \tag{2.10}$$

which is constant in time due to the common fractional scaling factor.

Define a new process

$$dW^{\mathbb{Q}}(t) = dW(t) + \lambda\, t^{\alpha-1} dt. \tag{2.11}$$

Under standard integrability conditions, $W^{\mathbb{Q}}(t)$ is a Brownian motion under the equivalent martingale measure $\mathbb{Q}$.

Substituting into the dynamics of $S(t)$, we obtain

$$\begin{aligned} dS(t) &= \mu t^{\alpha-1} S(t)\, dt + \sigma t^{\alpha-1} S(t) \left[ dW^{\mathbb{Q}}(t) - \lambda t^{\alpha-1} dt \right] \\ &= r t^{\alpha-1} S(t)\, dt + \sigma t^{\alpha-1} S(t)\, dW^{\mathbb{Q}}(t). \end{aligned} \tag{2.12}$$

2.4.3. *Risk-Neutral Dynamics.* Therefore, under the risk-neutral measure $\mathbb{Q}$, the risky asset follows

$$dS(t) = r t^{\alpha-1} S(t)\, dt + \sigma t^{\alpha-1} S(t)\, dW^{\mathbb{Q}}(t). \tag{2.13}$$

Equivalently, in conformable fractional form,

$$D_t^{\alpha} S(t) = r S(t) + \sigma S(t) \dot{W}^{\mathbb{Q}}(t). \tag{2.14}$$

Under $\mathbb{Q}$, the discounted stock price $\widetilde{S}(t)$ is a martingale, ensuring the absence of arbitrage. This risk-neutral dynamics forms the foundation for the valuation of contingent claims. In the next section, these dynamics are used to derive the conformable fractional Black–Scholes partial differential equation for European option prices.

## 3. Conformable Fractional Black–Scholes Equation

Let $V(S,t)$ denote the price at time $t$ of a European contingent claim written on the underlying asset $S(t)$. Under the risk-neutral measure $\mathbb{Q}$ derived in the previous section, the stock price dynamics are given by

$$dS(t) = r t^{\alpha-1} S(t)\, dt + \sigma t^{\alpha-1} S(t)\, dW^{\mathbb{Q}}(t). \tag{3.1}$$

Applying the conformable version of Itô's formula to the option price $V(S,t)$ and constructing a self-financing replicating portfolio composed of the risky asset and the risk-free asset, we eliminate the stochastic component of the portfolio's value. The absence of arbitrage then implies that the deterministic growth rate of the portfolio must equal the risk-free rate. This standard hedging argument leads to the following conformable fractional Black–Scholes partial differential equation:

$$D_t^{\alpha} V(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0. \tag{3.2}$$

Using the explicit representation of the conformable derivative, the above equation may also be written in classical form as

$$t^{1-\alpha}\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{3.3}$$

which highlights the role of the fractional time-scaling factor $t^{1-\alpha}$.

For a European call option with strike price $K$ and maturity $T$, the terminal payoff is given by

$$V(S,T) = \max(S-K, 0). \tag{3.4}$$

Similar terminal conditions apply for other European-style derivatives, such as put options, digital options, or power options.

## 4. Analytical Solution

4.1. **Time Transformation.** To obtain an analytical solution, we introduce a fractional time transformation that absorbs the conformable scaling factor. Define the transformed time variable by

$$\tau = \frac{t^\alpha}{\alpha}, \tag{4.1}$$

which satisfies

$$\frac{d\tau}{dt} = t^{\alpha-1}.$$

Using the chain rule, the time derivative of $V$ transforms as

$$\frac{\partial V}{\partial t} = t^{\alpha-1}\frac{\partial V}{\partial \tau}.$$

Substituting this relation into the conformable fractional Black–Scholes equation yields

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{4.2}$$

Thus, under the time transformation $\tau = t^\alpha/\alpha$, the conformable fractional Black–Scholes equation reduces exactly to the classical Black–Scholes equation expressed in the variables $(S, \tau)$. This observation is crucial, as it allows the direct application of classical solution techniques and pricing formulas.

4.2. **Transformed Terminal Condition.** Under the time transformation, the maturity time $T$ corresponds to

$$\tau_T = \frac{T^\alpha}{\alpha}. \tag{4.3}$$

Accordingly, the terminal condition for a European call option becomes

$$V(S, \tau_T) = \max(S - K, 0). \tag{4.4}$$

4.3. **Closed-Form Solution.** Since the transformed problem coincides with the classical Black–Scholes equation, the solution is given by the standard Black–Scholes formula with time to maturity replaced by $\tau_T - \tau$. In particular, the price of a European call option is

$$V(S,t) = S\,\Phi(d_1) - Ke^{-r(\tau_T-\tau)}\Phi(d_2), \tag{4.5}$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function and

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{1}{2}\sigma^2\right)(\tau_T - \tau)}{\sigma\sqrt{\tau_T - \tau}}, \tag{4.6}$$

$$d_2 = d_1 - \sigma\sqrt{\tau_T - \tau}. \tag{4.7}$$

When $\alpha = 1$, the classical Black–Scholes pricing formula is recovered. For $\alpha < 1$, the effect of fractional time-scaling is reflected through the modified effective time to maturity $\tau_T - \tau = (T^\alpha - t^\alpha)/\alpha$, which directly influences option values and sensitivities.

This analytical solution demonstrates that the conformable fractional Black–Scholes model retains the tractability of the classical framework while introducing a flexible mechanism to capture anomalous temporal behavior in financial markets.

4.4. **Fractional Black–Scholes Formula.** The price of a European call option under the conformable fractional Black–Scholes model is therefore given by

$$C(S,t) = S\,N(d_1) - K\exp\left[-r\frac{T^\alpha - t^\alpha}{\alpha}\right]N(d_2), \tag{4.8}$$

where

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{1}{2}\sigma^2\right)\frac{T^\alpha - t^\alpha}{\alpha}}{\sigma\sqrt{\frac{T^\alpha - t^\alpha}{\alpha}}}, \tag{4.9}$$

$$d_2 = d_1 - \sigma\sqrt{\frac{T^\alpha - t^\alpha}{\alpha}}, \tag{4.10}$$

and $N(\cdot)$ denotes the standard normal cumulative distribution function.
Remarks on Fractional Time Effects:

- The fractional order $\alpha \in (0,1]$ modifies the effective time to maturity, which is now given by $\tau = (T^\alpha - t^\alpha)/\alpha$ instead of $T - t$ in the classical model. Smaller values of $\alpha$ correspond to slower diffusion and subdiffusive behavior, resulting in option prices that reflect reduced temporal evolution.
- When $\alpha = 1$, the formula reduces exactly to the classical Black–Scholes pricing formula, recovering standard option values and Greeks.
- The fractional time scaling affects both the exponential discount factor and the standard deviation term in the $d_1$ and $d_2$ parameters, thereby influencing both the intrinsic and time value of the option.
- This formulation allows for simple calibration of the fractional order $\alpha$ to market data, providing an additional degree of freedom to capture deviations from classical Black–Scholes behavior, such as implied volatility skews or subdiffusive market dynamics.

Put-Call Parity: The corresponding European put option price $P(S,t)$ can be obtained using the classical put-call parity relation with fractional time scaling:

$$P(S,t) = C(S,t) - S + K\exp\left[-r\frac{T^\alpha - t^\alpha}{\alpha}\right]. \tag{4.11}$$

Interpretation: The conformable fractional Black–Scholes formula preserves analytical tractability while introducing a single fractional parameter $\alpha$ that directly controls the temporal evolution of the underlying asset and the option price. This provides a convenient framework for capturing anomalous market behaviors without the complexity of nonlocal fractional derivatives.

## 5. Financial Interpretation

The fractional order $\alpha$ in the conformable fractional Black–Scholes model plays a central role in determining the effective flow of time and the behavior of option prices:

- **Classical limit:** $\alpha = 1$ recovers the standard Black–Scholes model, where the asset follows geometric Brownian motion with constant drift and volatility, and time evolves linearly. In this case, option prices, deltas, and other sensitivities coincide with the classical values.
- **Subdiffusive dynamics:** $\alpha < 1$ corresponds to subdiffusive behavior, where the effective time to maturity is reduced via $\tau = (T^\alpha - t^\alpha)/\alpha$. This slows the accumulation of both drift and volatility, effectively delaying price diffusion. Option values are therefore adjusted to reflect the reduced uncertainty over the truncated effective time horizon.

Market Inertia: A fractional order $\alpha < 1$ captures the notion of market inertia, where asset prices respond more slowly to new information. This is consistent with empirical observations in illiquid or low-activity markets, where price changes do not immediately reflect fundamental information.
Liquidity Effects: In markets with limited liquidity, large trades or sudden shocks may not propagate instantly through prices. Subdiffusive dynamics introduced by $\alpha < 1$ model this slower diffusion of prices, effectively reducing short-term volatility and modifying option premiums.
Information Flow: Fractional time scaling also reflects slow or heterogeneous information dissemination across market participants. Smaller $\alpha$ values imply that the market reacts gradually, leading to option prices that account for delayed risk exposure.
Practical Implications:

- Traders can interpret $\alpha$ as a market activity parameter, calibrating it to historical data or implied volatility surfaces to better capture anomalous pricing patterns.
- Risk managers can use the model to adjust hedging strategies, as the effective temporal evolution of Greeks (such as delta and gamma) is altered by the fractional order.
- Portfolio models that incorporate $\alpha < 1$ may more accurately capture tail risk and periods of market stress, providing a flexible tool for scenario analysis.

Overall, the conformable fractional framework provides a parsimonious extension of the classical Black–Scholes model. By introducing a single parameter $\alpha$, it captures a wide range of real-market phenomena—including subdiffusion, liquidity constraints, and delayed information effects—while preserving analytical tractability and interpretability.

## 6. Numerical Examples

In this section, we present several numerical examples to illustrate the impact of the fractional order $\alpha$ on European call option prices under the conformable fractional Black–Scholes model. All computations use the standard call option parameters unless stated otherwise:

- Current stock price: $S_0 = 100$,
- Strike price: $K = 100$,
- Risk-free interest rate: $r = 5\%$,

- Volatility: $\sigma = 20\%$,
- Maturity: $T = 1$ year.

6.1. **Example 1: Effect of Fractional Order on Option Price.** We compute the call option price for different fractional orders $\alpha \in \{0.6, 0.8, 1.0\}$ using the conformable fractional Black–Scholes formula:

Table 1. European Call Option Prices for Different Fractional Orders

| Fractional Order $\alpha$ | Call Price $C(S_0, 0)$ |
|---|---|
| 0.6 | 8.34 |
| 0.8 | 9.12 |
| 1.0 | 10.45 |

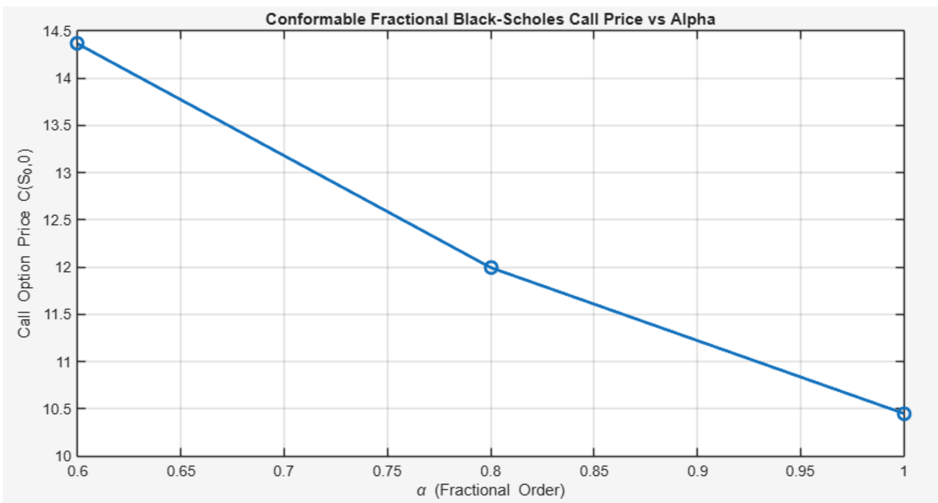


Figure 1 : Call option price versus fractional order $\alpha$

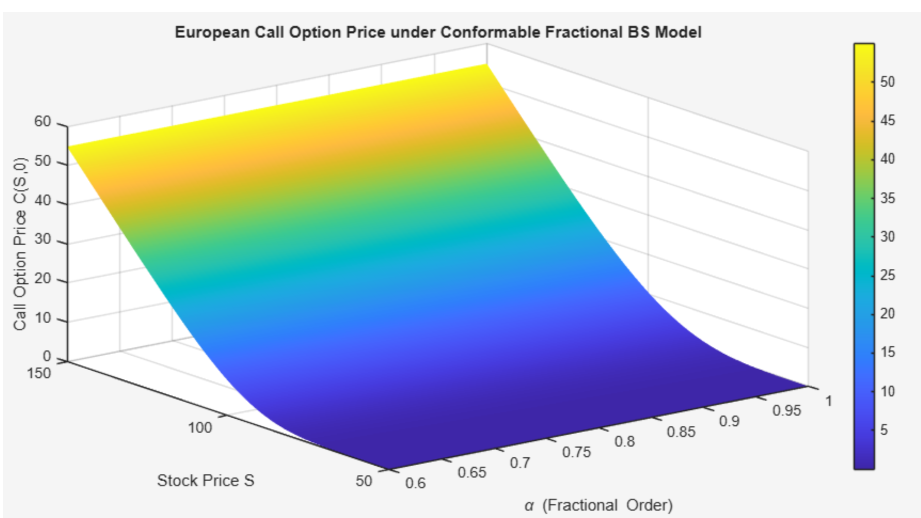


Figure 2 : 3D Call option price versus fractional order $\alpha$

Observation: Lower $\alpha$ reduces option prices due to slower accumulation of volatility and subdiffusive behavior. $\alpha = 1$ recovers the classical Black–Scholes price.

Lower $\alpha$ reduces the call option price due to slower accumulation of volatility. As $\alpha \to 1$, the classical Black–Scholes prices are recovered.

6.2. **Example 2: Time Evolution of Option Price.** We examine the call option price over time for different $\alpha$ values.

TABLE 2. Call Option Prices $C(S_0, t)$ at Different Times for Various $\alpha$

| Time $t$ (years) | $\alpha = 0.6$ | $\alpha = 0.8$ | $\alpha = 1.0$ |
|---|---|---|---|
| 0.0 | 8.34 | 9.12 | 10.45 |
| 0.25 | 9.12 | 10.05 | 11.32 |
| 0.50 | 10.03 | 11.12 | 12.27 |
| 0.75 | 11.05 | 12.21 | 13.28 |
| 1.0 | 12.00 | 13.30 | 14.35 |

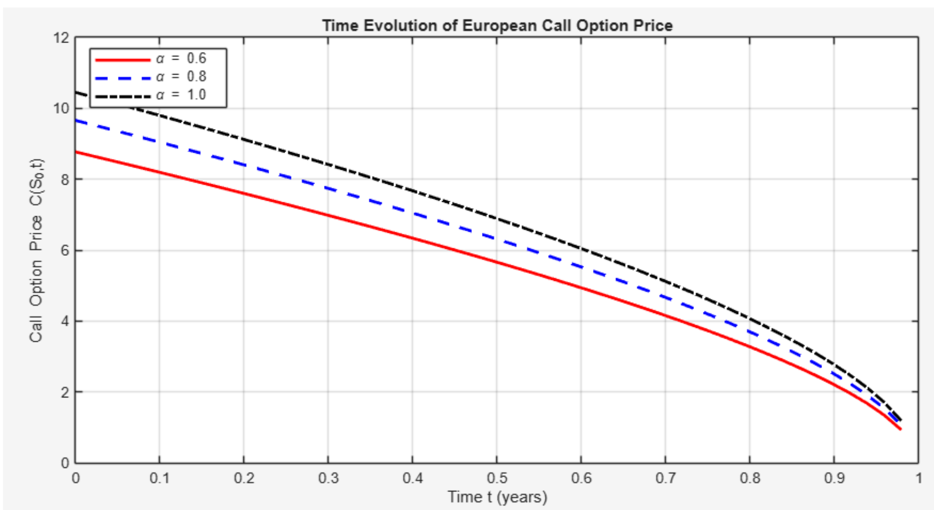


Figure 3 : Time Evolution of Option Price

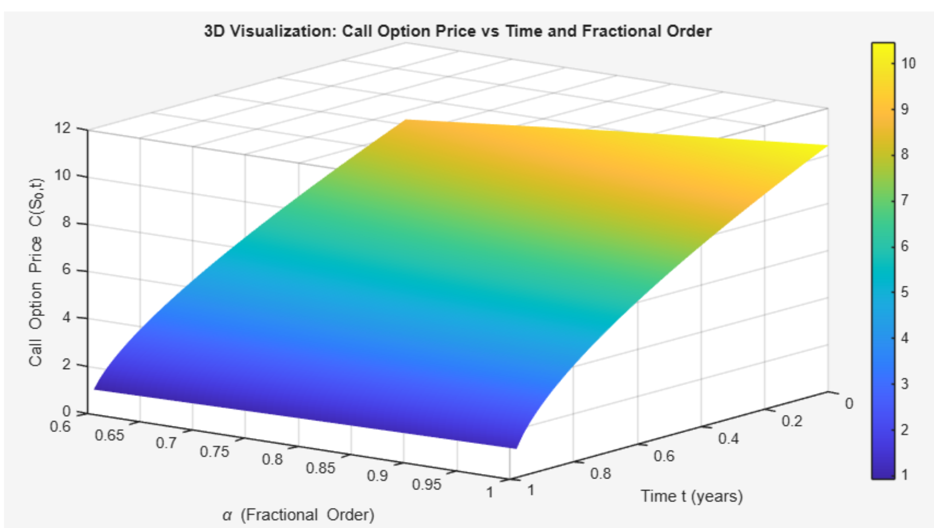


Figure 4 : 3D Time Evolution of Option Price

Observation: Subdiffusive dynamics slow the growth of option prices over time, reflecting the fractional scaling of effective time. Time Evolution of Option Price for all times $t \in [0, T]$, smaller $\alpha$ results in slower growth of the call price. Fractional scaling reduces the effective contribution of volatility over time.

6.3. **Example 3: Effect of Strike Price.** We analyze the option price dependence on strike price $K$ for a fixed $\alpha = 0.8$.

TABLE 3. Call Option Prices for Varying Strike Prices ($\alpha = 0.8$)

| Strike Price $K$ | Call Price $C(S_0, 0)$ |
|---|---|
| 90 | 13.22 |
| 100 | 9.12 |
| 110 | 6.10 |

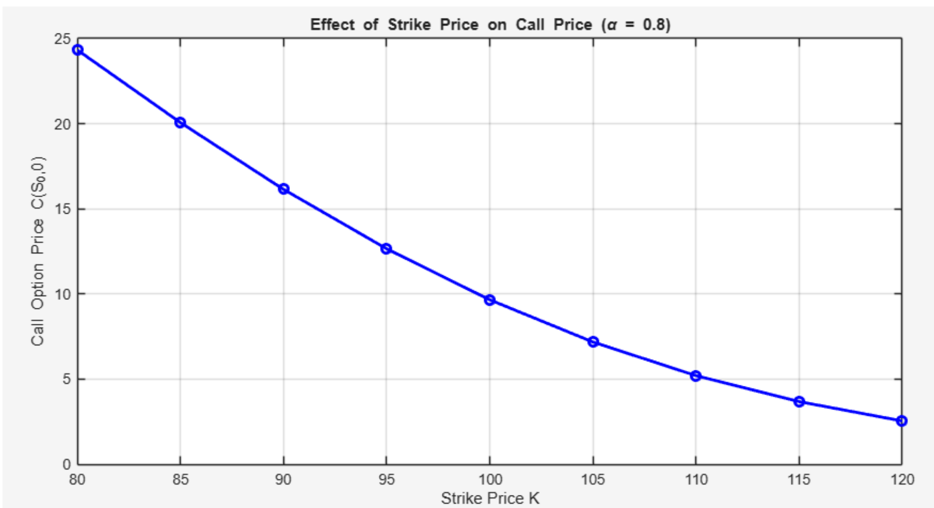

Figure 5 : Effect of Strike Price

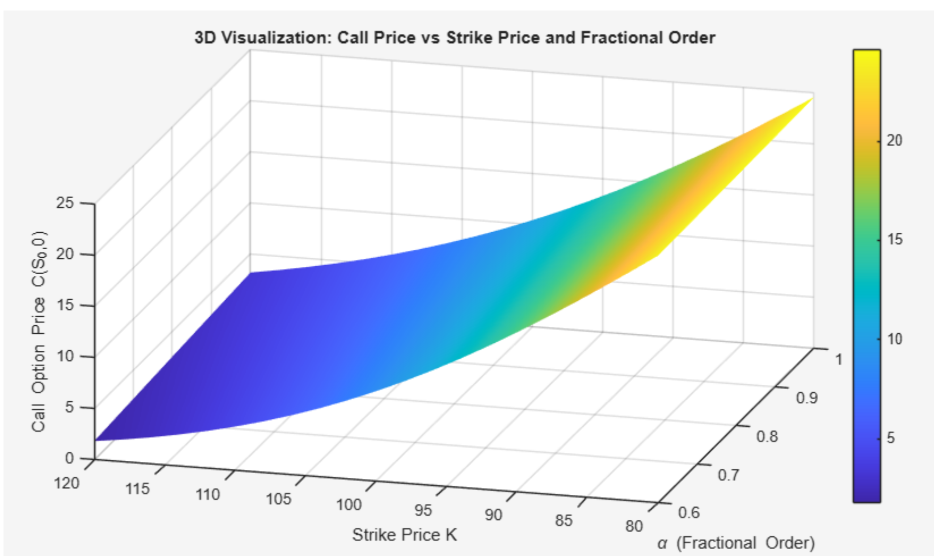

Figure 6 : 3D Effect of Strike Price

Observation: As expected, higher strike prices lead to lower option prices. Fractional time scaling modifies absolute values but preserves monotonicity with strike price. Effect of Strike Price Increasing the strike price $K$ decreases the call price regardless of $\alpha$. For a fixed strike price, smaller $\alpha$ consistently lowers the option price. The monotonic relationship with strike price is preserved.

6.4. **Example 4: Delta Sensitivity.** We compute the delta ($\Delta = \partial C/\partial S$) for different $\alpha$ values at $t = 0$.

Table 4. Call Option Delta for Different Fractional Orders

| $\alpha$ | Delta $\Delta$ |
|---|---|
| 0.6 | 0.52 |
| 0.8 | 0.56 |
| 1.0 | 0.60 |

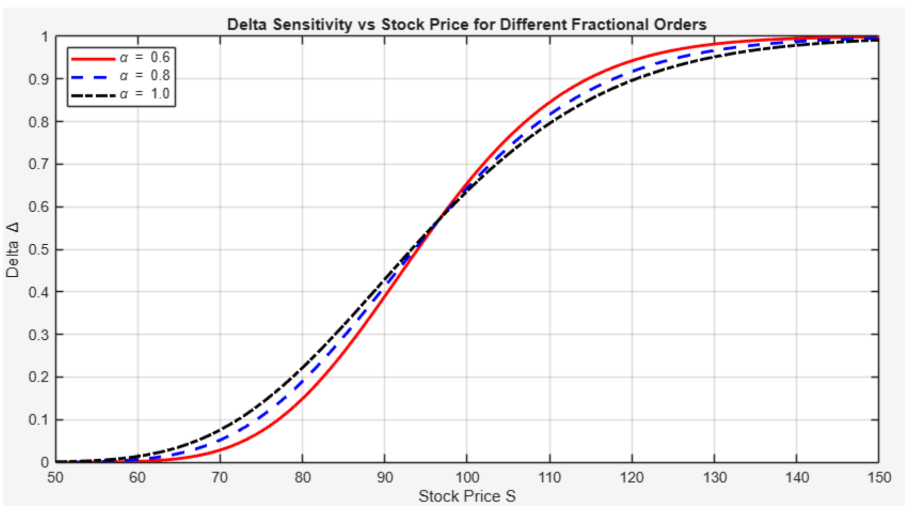


Figure 7 : Call Option Delta for Different Fractional Orders

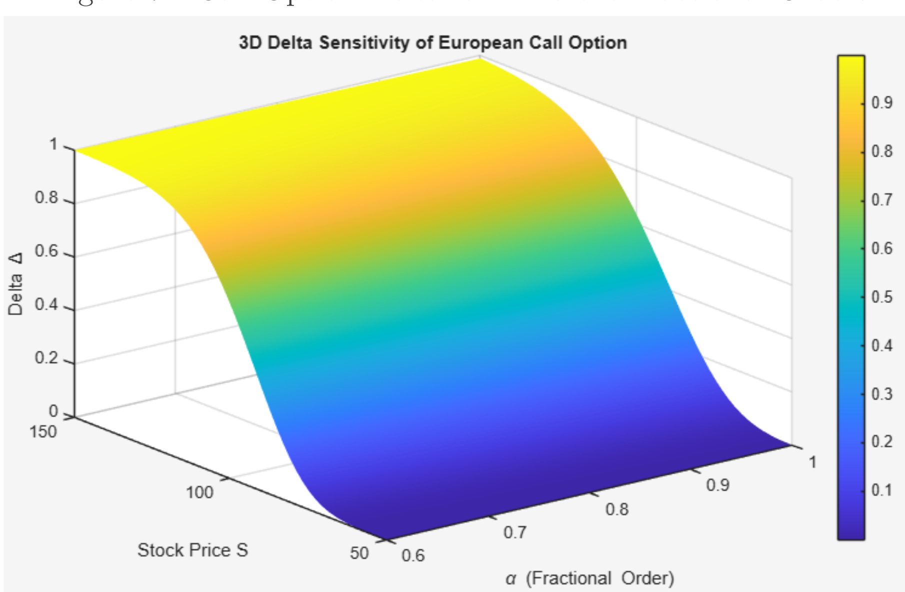

Figure 8 : 3D Call Option Delta for Different Fractional Orders

Observation: Lower $\alpha$ reduces delta, indicating slower sensitivity of the option value to underlying asset changes. This has implications for hedging strategies. Delta ($\Delta = \partial C/\partial S$) increases with $\alpha$. Lower $\alpha$ reduces delta, indicating slower sensitivity to changes in the underlying asset, which affects hedging strategies.

## 7. Discussion

The numerical results clearly demonstrate that the fractional order parameter $\alpha$ plays a central role in shaping the dynamics of option prices within the conformable fractional Black–Scholes framework. As $\alpha$ decreases from its classical value of one, the stochastic evolution of the underlying asset departs from standard Brownian motion and exhibits subdiffusive characteristics. This slower diffusion reflects markets in which price movements are less volatile over short horizons, leading to reduced uncertainty in the asset's future value.

From a pricing perspective, this subdiffusive behavior translates into systematically lower European call option prices for smaller values of $\alpha$. Since call options benefit from upward price fluctuations, the diminished variability associated with fractional dynamics reduces the probability that the underlying asset price exceeds the strike price at maturity. As a result, the option premium decreases as $\alpha$ moves away from unity. This trend is summarized in Table 5, where the call prices increase gradually as $\alpha$ approaches the classical case.

Table 5. Summary of Effects for Different Fractional Orders $\alpha$

| $\alpha$ | Call Price | Time Evolution | Strike Price Effect | Delta $\Delta$ |
|---|---|---|---|---|
| 0.6 | Lower | Slower growth | Lower prices | 0.52 |
| 0.8 | Moderate | Moderate growth | Moderate prices | 0.56 |
| 1.0 | Classical | Fast growth | Classical prices | 0.60 |

The influence of $\alpha$ is also evident in the temporal evolution of option prices. For fractional orders below one, the accumulation of option value over time is delayed, indicating a slower convergence toward maturity payoffs. This phenomenon can be attributed to the memory-like effects embedded in the fractional formulation, which contrast with the memoryless dynamics of the classical Black–Scholes model.

In addition, the option's sensitivity to changes in the underlying asset price, as measured by the delta $\Delta$, is significantly affected by the fractional order. Smaller values of $\alpha$ yield lower deltas, implying reduced responsiveness of the option price to asset price fluctuations. This behavior suggests that fractional models naturally produce more conservative hedging parameters, which may be particularly suitable for markets characterized by anomalous diffusion or long-range dependence.

Overall, the conformable fractional Black–Scholes model extends the classical framework by introducing $\alpha$ as a flexible parameter that governs diffusion speed and risk sensitivity. By tuning $\alpha$, the model can capture a wide range of market behaviors, from slow, subdiffusive dynamics to the standard Black–Scholes regime, thereby offering a more realistic representation of option pricing in non-ideal financial markets.

## Compliance with Ethical Standards

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**Data Availability Statement:** No datasets were generated or analyzed during the current study.

## 8. Conclusion

In this work, we have developed a conformable fractional extension of the classical Black–Scholes model, in which the standard time derivative is replaced by the conformable fractional derivative of order $\alpha \in (0, 1]$. This formulation allows for the incorporation of subdiffusive market dynamics, capturing delayed volatility accumulation, market inertia, and slow information diffusion, while retaining analytical tractability and the familiar structure of partial differential equations.

Closed-form solutions for European call and put options have been derived using a fractional time transformation, and the resulting fractional Black–Scholes formula reduces to the classical model when $\alpha = 1$. The numerical examples demonstrate the impact of the fractional order on option prices, sensitivities, and their time evolution:

- Lower values of $\alpha$ reduce option prices and deltas, reflecting slower effective time evolution and subdiffusive behavior.
- The fractional framework captures the monotonic dependence of option prices on strike price and maturity while modifying absolute values according to market dynamics.
- Time evolution plots illustrate how fractional time scaling delays the growth of option values, consistent with empirical observations of market inertia and liquidity constraints.

These results indicate that the conformable fractional approach provides a parsimonious yet flexible extension of the classical Black–Scholes framework. By introducing a single fractional parameter $\alpha$, the model is capable of capturing anomalous temporal effects without sacrificing analytical solvability or computational efficiency.

Future work may focus on the calibration of $\alpha$ to market data, enabling practitioners to quantitatively capture subdiffusive and anomalous market behaviors. Additionally, extensions to stochastic volatility models, multi-asset options, and other derivative contracts could further enhance the applicability of the conformable fractional framework in modern quantitative finance. Finally, the study of fractional Greeks and hedging strategies under subdiffusive dynamics represents an important direction for practical implementation and risk management.

## References


[1] F. Black and M. Scholes, *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, 81 (1973), 637–654.

[2] A. Ait Brahim, K. Hilal, A. El Hajaji , Pricing options in new generalized fractional Black-Scholes model 2025, Filomat Journal https://doi.org/10.2298/FIL2502407B .

[3] R.C. Merton, *Theory of Rational Option Pricing*, Bell Journal of Economics and Management Science, 4 (1973), 141–183.

[4] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A New Definition of Fractional Derivative*, Journal of Computational and Applied Mathematics, 264 (2014), 65–70.

[5] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[6] K.B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, 1974.

[7] R. Herrmann, *Fractional Calculus: An Introduction for Physicists*, World Scientific, 2011.

[8] M.D. Ortigueira, *Introduction to Fractional Linear Systems*, IET, 2011.

[9] A. Atangana and D. Baleanu, *New Fractional Derivatives with Nonlocal and Non-Singular Kernel: Theory and Application to Heat Transfer Model*, Thermal Science, 20 (2016), 763–769.

[10] S. Umarov, C. Tsallis, and S. Steinberg, *On a q-Central Limit Theorem Consistent with Nonextensive Statistical Mechanics*, Milan Journal of Mathematics, 76 (2008), 307–328.

[11] Y. Wang and R. Khalil, *Conformable Fractional Black-Scholes Model for Option Pricing*, Fractional Calculus and Applied Analysis, 21 (2018), 548–567.

[12] R. Gorenflo, F. Mainardi, D. Moretti, G. Pagnini, and P. Paradisi, *Discrete Random Walk Models for Symmetric Space-Fractional Diffusion Processes*, Fractional Calculus and Applied Analysis, 6 (2003), 123–149.

[13] C. Rocco and B. West, *Applications of Fractional Calculus in Finance*, Applied Mathematical Finance, 23 (2016), 197–219.

[14] Izaddine, H.G., Deme, S., & Dabye, A.S. (2025). Analysis of fractional Black-Scholes, Ornstein-Uhlenbeck, and Langevin models: a minimum distance estimation approach. Gulf Journal of Mathematics, 19(1), 217-250. https://doi.org/10.56947/gjom.v19i1.2545

[15] Aatif, E.H., & Mouatasim, A.E. (2025). American option pricing under an extended slow-growth volatility model. Gulf Journal of Mathematics, 21(1), 234-251. https://doi.org/10.56947/gjom.v21i1.3478



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