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HUB NUMBER OF GENERALIZED TRANSFORMATION GRAPHS

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ABSTRACT. A *hub set* in a graph G is a set $S \subseteq V(G)$ such that any two vertices outside S are connected by a path whose all internal vertices are members of S. The minimum cardinality of hub set is called *hub number*. In this paper, we give results for the hub number of generalized transformation graphs.

1. INTRODUCTION

In 2006, Walsh [11] have defined hub number of a graph to study a network related problem. A graph which represents the buildings in a large industrial complex, with an edge between two buildings if it is an easy walk from one to the other.

Let G be a graph with vertex set $V(G)$ and S be a hubset in a graph G such that $S \subseteq V(G)$ and let $x, y \in V(G)$. An S-path between x and y is a path where all intermediate vertices are from S. A set $S \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) \setminus S$ there is an S-path in G between x and y. The minimum cardinality of hub set is called *hub number* and is denoted by $h(G)$ [11].

The above definition of hub number of a graph G is applied on this problem by Walsh. The corporation is considering implementing a rapid transit system (RTS), and wants to place its stations in buildings so that to travel between two nonadjacent buildings. The corporation would like to implement this plan as cheaply as possible, which involves converting as few building as possible into transist stations. In this problem, what is the smallest size of a hub set in G?. We shall call this the hub number of G, and denoted by $h(G)$. This inspired us to study the hub number of generalized transformation graphs.

Grauman et al. [5] obtained the relationship between hub number, connected hub number and connected domination number of a graph. Cauresma et al. [4] obtained hub number of join, corona and Cartesian product of two connected graphs. Basavanagoud et al. [1, 2] obtained hub number of some wheel related graphs and hub number of generalized middle graphs. In this paper, we obtain hub number of generalized transformation graphs.

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2. Preliminaries

All graphs considered in this paper are nontrivial, connected, simple and undirected graphs. Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$. Thus $|V(G)| = n$ and $|E(G)| = m$ where, n and m are called *order* and *size* of graph G respectively. The pendant vertices are called leaf vertices of graph G, it is denoted by l. An edge incident to a pendant vertex l of a graph G is called pendant edge and denoted by l_e. The *complement* of a graph G [6] is denoted by $\overline{G}$ whose vertex set is $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. The *line graph* $L(G)$ of a graph G [10] is the graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G have a vertex in common. The *subdivision graph* $S(G)$ of a graph G [6] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is an edge of G incident with it. The *partial complement of subdivision graph* $\overline{S}(G)$ of a graph G [7] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and the other is an edge of G not incident with it. For undefined terminology and notations refer [6].

2.1. **Generalized transformation graphs G^{xy}.** Sampathkumar and Chikkodimath [9] defined the semitotal-point graph $T_2(G)$ as the graph whose vertex set is $V(G) \cup E(G)$, and where two vertices are adjacent if and only if (i) they are adjacent vertices of G or (ii) one is a vertex of G and other is an edge of G incident with it. Inspired by this definition, Basavanagoud et al. [3] introduced some new graphical transformations. These generalize the concept of semitotal-point graph.

Let $G = (V, E)$ be a graph, and let α, β be two elements of $V(G) \cup E(G)$. We say that associativity of α and β is $+$ if they are adjacent or incident in G, otherwise $-$. Let xy be a 2-permutation of the set $\{+, -\}$.

The *generalized transformation graph* G^{xy} [3], is a graph whose vertex set is $V(G) \cup E(G)$, and α, $\beta \in V(G^{xy})$. The vertices α and β are adjacent in G^{xy} if and only if (a) and (b) holds:
(a) α, $\beta \in V(G)$, α, β are adjacent in G if $x = +$ and α, β are not adjacent in G if $x = -$.
(b) $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $y = +$ and α, β are not incident in G if $y = -$.

One can obtain the four graphical generalized transformation of graphs as G^{++}, G^{+-}, G^{-+} and G^{--}. An example of generalized transformation graphs and their complements are depicted in Figure 1. Note that, G^{++} is just the *semitotal-point graph* of G, which was introduced in [9]. The vertex v of G^{xy} corresponding to a vertex v of G is referred to as a *point vertex*. The vertex e of G^{xy} corresponding to an edge e of G is referred to as a *line vertex*. Recently, Ramane et al. [8] studied generalized transformation graphs.

Proposition 2.1. [11] *The hub number of*
i) The path P_n, $h(P_n) = n - 2$,
ii) The cycle C_n, $h(C_n) = n - 3$,

iii) The complete graph K_n, $h(K_n) = 0$,
iv) The tree T, $h(T) = n - l$.

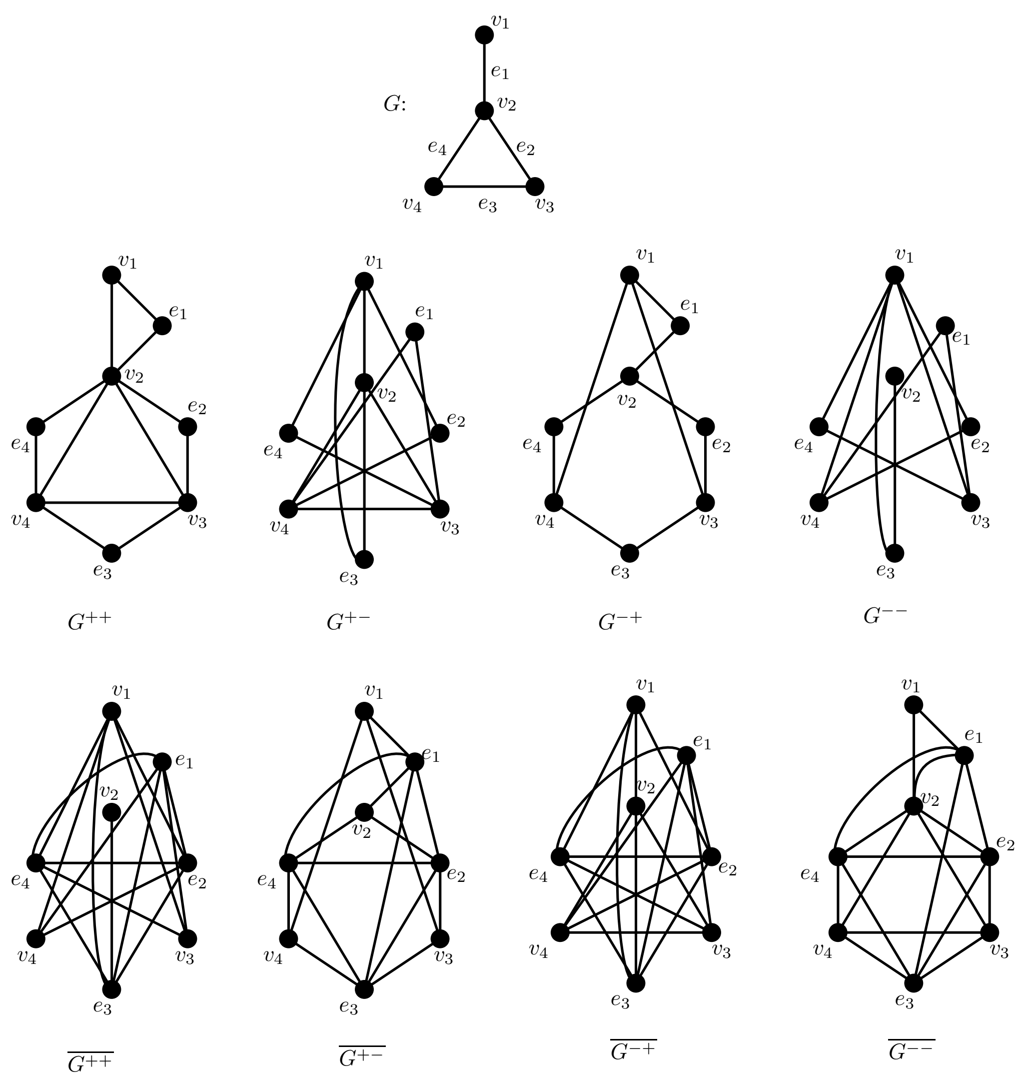

FIGURE 1. Graph G, its generalized transformation graph G^{xy} and their complements $\overline{G^{xy}}$.

3. Hub number of generalized transformation graphs

Theorem 3.1. *Let T be tree of order n. Then*

$$h(T^{++}) = \begin{cases} 0 & if\ n = 2, \\ n - l & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of T^{++} and $e_1, e_2, ..., e_m$ be line vertices of T^{++}. We consider the following cases:
Case 1: If $n = 2$, P_2^{++} is isomorphic to complete graph K_3. By Proposition 2.1, $h(P_2^{++}) = 0$.

Case 2: If $n \geq 3$, then choose nonpendant vertices, which is required to form S-path in T^{++}. Thus, $S = \{v_1, v_2, v_3, ..., v_{n-l}\}$ gives the minimum hub set. Therefore, $|S| = n - l$. Hence, $h(T^{++}) = n - l$. □

Theorem 3.2. *Let G be cycle or complete graph of order $n \geq 3$. Then*

$$h(G^{++}) = n - 1.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of G^{++} and $e_1, e_2, ..., e_m$ be line vertices of G^{++}. Let $S \subset V(G^{++})$. Choose a set S containing $n - 1$ adjacent vertices in such way that it forms required S-path in G^{++}. Therefore, $|S| = n - 1$. Hence, $h(G^{++}) = n - 1$. □

Theorem 3.3. *Let T be tree of order n. Then*

$$h(T^{+-}) = \begin{cases} 3 & if\ n = 3, 4, \\ 4 & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of T^{+-} and $e_1, e_2, ..., e_m$ be line vertices of T^{+-}. We consider the following cases:
Case 1: If $n = 3$, P_3^{+-} is isomorphic to P_5. By Proposition 2.1, $h(P_3^{+-}) = 3$.
Case 2: If $n = 4$, then choose nonpendant vertices v_1, v_2 and an edge e_1 which is incident to both nonpendant vertices in T. Thus, $S = \{v_1, v_2, e_1\}$ gives the minimum hub set. Therefore, $|S| = 3$. Hence, $h(T^{+-}) = 3$.
Case 3: If $n \geq 5$, then choose two pendant vertices v_1, v_2 and two edges e_1 and e_2 which are incident to vertices v_1 and v_2 in T. Thus, $S = \{v_1, v_2, e_1, e_2\}$ gives the minimum hub set. Therefore, $|S| = 4$. Hence, $h(T^{+-}) = 4$. □

Theorem 3.4. *Let C_n be cycle graph of order $n \geq 3$. Then*

$$h(C_n^{+-}) = 3.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of C_n^{+-} and $e_1, e_2, ..., e_m$ be line vertices of C_n^{+-}. We consider the following cases:
Case 1: If $n = 3$, C_3^{+-} is isomorphic to crown graph CW_3. Therefore, $h(C_3^{+-}) = 3$ (since $h(CW_n) = n$ [1]).
Case 2: If $n = 4$, then choose any three vertices $\{v_1, v_2, v_3\}$ in C_4, which forms a S-path in C_4^{+-}. Therefore, $h(C_4^{+-}) = 3$.
Case 3: If $n \geq 5$, then choose a set S such that it contains any two nonadjacent edges e_1, e_2 and a vertex v_1 which is nonincident to edges e_1 and e_2 in C_n. Thus, $S = \{e_1, e_2, v_1\}$ gives the minimum hub set. Therefore, $|S| = 3$. Hence, $h(C_n^{+-}) = 3$. □

Theorem 3.5. *Let K_n be complete graph of order $n \geq 4$. Then*

$$h(K_n^{+-}) = 3.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of K_n^{+-} and $e_1, e_2, ..., e_m$ be line vertices of K_n^{+-}. We consider the following cases:
Case 1: If $n = 4$, then choose three vertices $\{v_1, v_2, v_3\}$ in K_4, which forms a S-path in K_4^{+-}. Therefore, $h(K_4^{+-}) = 3$.
Case 2: If $n \geq 5$, then choose a set S such that it contains any two nonadjacent

vertices v_1, v_2 and an edge e_1 which is nonincident to vertices v_1 or v_2 in K_n. Thus, $S = \{v_1, v_2, e_1\}$ gives the minimum hub set. Therefore, $|S| = 3$. Hence, $h(K_n^{+-}) = 3$. □

Theorem 3.6. *Let $K_{1,n}$ be star graph of order $n \geq 3$. Then*

$$h(K_{1,n}^{+-}) = 3.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of $K_{1,n}^{+-}$ and $e_1, e_2, ..., e_m$ be line vertices of $K_{1,n}^{+-}$. If $n \geq 3$, then choose a set S such that it contains central vertex v_i and any two vertices v_1, v_2 in $K_{1,n}$. Then for any two vertices $x, y \in K_{1,n}^{+-} \backslash S$, there exist an S-path between them in $K_{1,n}^{+-}$. Therefore S is a hub set of $K_{1,n}^{+-}$ which gives the minimum cardinality. Hence, $h(K_{1,n}^{+-}) = 3$. □

Theorem 3.7. *Let P_n be path graph of order n. Then*

$$h(P_n^{-+}) = \begin{cases} 1 & if\ n = 2, \\ 2 & if\ n = 3, \\ \lfloor \frac{n}{2} \rfloor & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of P_n^{-+} and $e_1, e_2, ..., e_m$ be line vertices of P_n^{-+}. We consider the following cases:
Case 1: If $n = 2$, P_2^{-+} is isomorphic to P_3. By Proposition 2.1, $h(P_2^{-+}) = 1$.
Case 2: If $n = 3$, P_3^{-+} is isomorphic to C_5. By Proposition 2.1, $h(P_3^{-+}) = 2$.
Case 3: If $n \geq 4$, then choose $\lfloor \frac{n}{2} \rfloor$ vertices in P_n, which forms a required S-path in P_n^{-+}. Thus, $S = \{v_1, v_2, v_3, ..., v_{\lfloor \frac{n}{2} \rfloor}\}$ gives the minimum hub set. Therefore, $|S| = \lfloor \frac{n}{2} \rfloor$. Hence, $h(P_n^{-+}) = \lfloor \frac{n}{2} \rfloor$. □

Theorem 3.8. *Let C_n be cycle graph of order n. Then*

$$h(C_n^{-+}) = \begin{cases} 3 & if\ n = 3, \\ \lceil \frac{n}{2} \rceil & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of C_n^{-+} and $e_1, e_2, ..., e_m$ be line vertices of C_n^{-+}. We consider the following cases:
Case 1: If $n = 3$, C_3^{-+} is isomorphic to C_6. By Proposition 2.1, $h(C_3^{-+}) = 3$.
Case 2: If $n = 4$, then choose any three vertices $\{v_1, v_2, v_3\}$ in C_4, which forms a S-path in C_4^{-+}. Therefore, $h(C_4^{-+}) = 3$.
Case 3: If $n \geq 5$, then choose $\lceil \frac{n}{2} \rceil$ vertices in C_n, which forms a required S-path in C_n^{-+}. Thus, $S = \{v_1, v_2, v_3, ...v_{\lceil \frac{n}{2} \rceil}\}$ gives the minimum hub set. Therefore, $|S| = \lceil \frac{n}{2} \rceil$. Hence, $h(C_n^{-+}) = \lceil \frac{n}{2} \rceil$. □

Theorem 3.9. *Let $K_{1,n}$ be star graph of order $n \geq 3$. Then*

$$h(K_{1,n}^{-+}) = 3.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of $K_{1,n}^{-+}$ and $e_1, e_2, ..., e_m$ be line vertices of $K_{1,n}^{-+}$. If $n \geq 3$, then choose central vertex v_i and any two vertices in $K_{1,n}$ which gives the minimum hub set. Thus, $S = \{v_i, v_1, v_2\}$. Therefore, $|S| = 3$. Hence, $h(K_{1,n}^{-+}) = 3$. □

Theorem 3.10. *Let T be tree graph of order $n \geq 4$. Then*

$$h(T^{-+}) = n + m - l - l_e.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of T^{-+} and $e_1, e_2, ..., e_m$ be line vertices of T^{-+}. If $n \geq 4$, then choose all nonpendant vertices $(n - l)$ and nonpendant edges $(m - l_e)$, which is incident to only nonpendant vertices in T, which forms a required S-path in T^{-+} and gives the minimum hub set. Thus, $S = \{v_1, v_2, v_3, ..., v_{n-l}, e_1, e_2, ..., e_{m-l_e}\}$. Therefore, $|S| = n + m - l - l_e$. Hence, $h(T^{-+}) = n + m - l - l_e$. □

Theorem 3.11. *Let K_n be complete graph of order $n \geq 4$. Then*

$$h(K_n^{-+}) = 2n - 2.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of K_n^{-+} and $e_1, e_2, ..., e_m$ be line vertices of K_n^{-+}. If $n \geq 4$, then choosing $(n-1)$ adjacent vertices and $(n-1)$ edges in K_n, which forms a hub set in K_n^{-+}. Thus, $S = \{v_1, v_2, v_3, ..., v_{n-1}, e_1, e_2, e_3, ..., e_{n-1}\}$. Therefore, $|S| = 2n - 2$. Hence, $h(K_n^{-+}) = 2n - 2$. □

Theorem 3.12. *Let T be tree of order $n \geq 4$. Then*

$$h(T^{--}) = 2.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of T^{--} and $e_1, e_2, ..., e_m$ be line vertices of T^{--}. Let $S \subset V(T^{--})$. Then choose any two pendant vertices of T are sufficient to form S-path in T^{--}, as every other vertices of T^{--} can be reached by these two points. Thus, $|S| = 2$. Therefore, $h(T^{--}) = 2$. □

Theorem 3.13. *Let C_n be cycle graph of order $n \geq 4$. Then*

$$h(C_n^{--}) = \begin{cases} 3 & if\ n = 4, 5, \\ 2 & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of C_n^{--} and $e_1, e_2, ..., e_m$ be line vertices of C_n^{--}. We consider the following cases:

Case 1: If $n = 4$, then choose any three vertices $\{v_1, v_2, v_3\}$ in C_4, which forms a S-path in C_4^{--}. Therefore, $h(C_4^{--}) = 3$.

Case 2: If $n = 5$, then choose any three vertices $\{v_1, v_2, v_3\}$ in that two are adjacent in C_5, which forms a S-path in C_5^{--}. Therefore, $h(C_5^{--}) = 3$.

Case 3: If $n \geq 6$, choose a set S such that it contains any two nonadjacent vertices v_1 and v_2 in C_n which forms a S-path in C_n^{--}. Thus, $|S| = \{v_1, v_2\}$ which gives the minimum hub set. Therefore, $h(C_n^{--}) = 2$. □

Theorem 3.14. *Let K_n be complete graph of order $n \geq 5$. Then*

$$h(K_n^{--}) = n.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of K_n^{--} and $e_1, e_2, ..., e_m$ be line vertices of K_n^{--}. Let $S \subset V(K_n^{--})$. Then choose all vertices in K_n, which forms a S-path in K_n^{--}. Thus, $S = \{v_1, v_2, v_3, ..., v_n\}$ gives the minimum hub set. Therefore, $|S| = n$. Hence, $h(K_n^{--}) = n$. □

Theorem 3.15. *Let T be tree graph of order $n \geq 4$. Then*

$$h(\overline{T^{++}}) = 2.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of $\overline{T^{++}}$ and $e_1, e_2, ..., e_m$ be line vertices of $\overline{T^{++}}$. If $n \geq 4$, then choose any two pendant edges e_1 and e_2 in T which forms a required S-path in $\overline{T^{++}}$ and gives the minimum hub set. Thus, $S = \{e_1, e_2\}$. Therefore, $|S| = 2$. Hence, $h(\overline{T^{++}}) = 2$. □

Theorem 3.16. *Let G be cycle or complete graph of order $n \geq 3$. Then*

$$h(\overline{G^{++}}) == \begin{cases} 3 & if\ n = 3, \\ 2 & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of $\overline{G^{++}}$ and $e_1, e_2, ..., e_m$ be line vertices of $\overline{G^{++}}$. We consider the following cases:
Case 1: If $n = 3$, $\overline{G^{++}}$ is isomorphic to crown graph CW_3. Therefore, $h(\overline{G^{++}}) = 3$ (since $h(CW_n) = n$ [1]).
Case 2: If $n \geq 4$, then choose a set S such that it contains any two nonadjacent edges e_1 and e_2 in G. Thus, $|S| = \{e_1, e_2\}$ which gives the minimum hub set in $\overline{G^{++}}$. Therefore, $h(\overline{G^{++}}) = 2$. □

Theorem 3.17. *Let T be tree of order n. Then*

$$h(\overline{T^{+-}}) = \begin{cases} 1 & if\ n = 2, \\ 2 & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of $\overline{T^{+-}}$ and $e_1, e_2, ..., e_m$ be line vertices of $\overline{T^{+-}}$. We consider the following cases:
Case 1: If $n = 2$, $\overline{P_2^{+-}}$ is isomorphic to P_3. By Proposition 2.1, $h(\overline{P_2^{+-}}) = 3$.
Case 2: If $n \geq 3$, then choose a vertex v_1 and next choose an edge e_1 which is incident to v_1 in T, which forms a required S-path in $\overline{T^{+-}}$. Thus, $S = \{v_1, e_1\}$ gives the minimum hub set. Therefore, $|S| = 2$. Hence, $h(\overline{T^{+-}}) = 2$. □

Theorem 3.18. *Let C_n be cycle graph of order $n \geq 3$. Then*

$$h(\overline{C_n^{+-}}) = 3.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of $\overline{C_n^{+-}}$ and $e_1, e_2, ..., e_m$ be line vertices of $\overline{C_n^{+-}}$. We consider the following cases:
Case 1: If $n = 3$, then choose all the three line vertices in $\overline{C_3^{+-}}$ which gives the S-path. Therefore, $|S| = 3$. Hence, $h(\overline{C_3^{+-}}) = 3$.
Case 2: If $n \geq 4$, then choose any two adjacent vertices v_1, v_2 of C_n and an edge e_1 which is incident to v_1 and v_2 in C_n which gives the minimum hub set to forms S-path in $\overline{C_n^{+-}}$. Thus, $S = \{v_1, v_2, e_1\}$ gives the minimum hub set. Therefore, $|S| = 3$. Hence, $h(\overline{C_n^{+-}}) = 3$. □

Theorem 3.19. *Let K_n be complete graph of order $n \geq 4$. Then*

$$h(\overline{K_n^{+-}}) = \left\lceil \frac{n}{2} \right\rceil.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of $\overline{K_n^{+-}}$ and $e_1, e_2, ..., e_m$ be line vertices of $\overline{K_n^{+-}}$. Choose $\lceil \frac{n}{2} \rceil$ nonadjacent edges in K_n, which forms a S-path in $\overline{K_n^{+-}}$. Thus, $S = \{e_1, e_2, e_3, ..., e_{\lceil \frac{n}{2} \rceil}\}$ gives the minimum hub set. Therefore, $|S| = \lceil \frac{n}{2} \rceil$. Hence, $h(\overline{K_n^{+-}}) = \lceil \frac{n}{2} \rceil$. □

Theorem 3.20. *Let T be tree of order $n \geq 3$. Then*

$$h(\overline{T^{-+}}) = 2.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of $\overline{T^{-+}}$ and $e_1, e_2, ..., e_m$ be line vertices of $\overline{T^{-+}}$. If $n \geq 3$, then choose any two pendant edges e_1 and e_2 in T, which is sufficient to forms a required S-path in $\overline{T^{-+}}$. Thus, $S = \{e_1, e_2\}$ gives the minimum hub set. Therefore, $|S| = 2$. Hence, $h(\overline{T^{-+}}) = 2$. □

Theorem 3.21. *Let C_n be cycle of order $n \geq 3$. Then*

$$h(\overline{C_n^{-+}}) = 2.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of $\overline{C_n^{-+}}$ and $e_1, e_2, ..., e_m$ be line vertices of $\overline{C_n^{-+}}$. Let $S \subset V(\overline{C_n^{-+}})$. Choose a set S containing any two nonadjacent edges e_1 and e_2 in C_n which forms required S-path in $\overline{C_n^{-+}}$. Therefore, $|S| = 2$. Hence, $h(\overline{C_n^{-+}}) = 2$. □

Theorem 3.22. *Let $K_{1,n}$ be star graph of order $n \geq 3$. Then*

$$h(\overline{K_{1,n}^{-+}}) = 3.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of $\overline{K_{1,n}^{-+}}$ and $e_1, e_2, ..., e_m$ be line vertices of $\overline{K_{1,n}^{-+}}$. Let $S \subset V(\overline{K_{1,n}^{-+}})$. Choose a set S containing a central vertex v_i and vertex v_1, next choose an edge e_1 which is nonincident to vertex v_1 in $K_{1,n}$, which gives the required S-path in $\overline{K_{1,n}^{-+}}$. Thus, $S = \{v_i, v_1, e_1\}$. Therefore, $|S| = 3$. Hence, $h(\overline{K_{1,n}^{-+}}) = 3$. □

Theorem 3.23. *Let K_n be complete graph of order $n \geq 3$. Then*

$$h(\overline{K_n^{-+}}) = 2.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of $\overline{K_n^{-+}}$ and $e_1, e_2, ..., e_m$ be line vertices of $\overline{K_n^{-+}}$. Let $S \subset V(\overline{K_n^{-+}})$. Choose a set S containing a vertex v_1 and next choose an edge e_1 which is nonincident to vertex v_1 in K_n, which gives the minimum hub set in $\overline{K_n^{-+}}$. Thus, $S = \{v_1, e_1\}$. Therefore, $|S| = 2$. Hence, $h(\overline{K_n^{-+}}) = 2$. □

Theorem 3.24. *Let G be path or cycle of order n. Then*

$$h(\overline{G^{--}}) = \begin{cases} 0 & if\ n = 2, \\ 1 & if\ n = P_3, \\ 2 & if\ G = C_3, \\ \lceil \frac{n}{2} \rceil & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of $\overline{G^{-+}}$ and $e_1, e_2, ..., e_m$ be line vertices of $\overline{G^{-+}}$. We consider the following cases:

Case 1: If $n = 2$, $\overline{P_2^{--}}$ is isomorphic to K_3. By Proposition 2.1, $h(\overline{P_2^{--}}) = 0$.

Case 2: If $n = 3$, then arises two graphs P_3 and C_3. If $G = P_3$, then choose a central vertex of P_3 is adjacent to every other vertex in $\overline{G^{--}}(P_3)$. Thus, $h(\overline{P_3^{--}}) = 1$. If $G = C_3$, then choose any two edges in C_3 which gives S-path in $\overline{C_3^{--}}$. Thus, $h(\overline{C_3^{--}}) = 2$.
Case 3: If $n \geq 4$, then each line vertex e_i is adjacent to incident point vertices and $(m-1)$ line vertices. Now choose $S = \{e_i | e_i \in E(\overline{G^{--}})\}$ such that e_i has maximum degree in $\overline{G^{--}}$, i.e, $d_{\overline{G^{--}}}(e_i) = max\{m+1 | e_i \in V(\overline{G^{--}})\}$ then each e_i is adjacent to $m+1$ vertices in $\overline{G^{--}}$. Choosing $S = \{e_1, e_2, ..., e_{\lceil \frac{n}{2} \rceil}\}$ gives the minimum hub set. Thus, $|S| = \lceil \frac{n}{2} \rceil$. Therefore, $h(\overline{G^{--}}) = \lceil \frac{n}{2} \rceil$. □

Theorem 3.25. *Let T be tree of order $n \geq 3$. Then*

$$h(\overline{T^{--}}) = n - l.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of $\overline{T^{--}}$ and $e_1, e_2, ..., e_m$ be line vertices of $\overline{T^{--}}$. Let $S \subset V(\overline{T^{--}})$. Choose a nonpendant vertices in T are sufficient to forms a S-path in $\overline{T^{--}}$. Thus, $S = \{v_1, v_2, v_3, ..., v_{n-l}\}$ which gives the minimum hub set in $\overline{T^{--}}$. Therefore, $|S| = n - l$. Hence, $h(\overline{T^{--}}) = n - l$. □

Theorem 3.26. *Let K_n be complete graph of order $n \geq 3$. Then*

$$h(\overline{K_n^{--}}) = 2.$$

Proof. Let $v_1, v_2, ..., v_n$ be point vertices of $\overline{K_n^{--}}$ and $e_1, e_2, ..., e_m$ be line vertices of $\overline{K_n^{--}}$. Let $S \subset V(\overline{K_n^{--}})$. Choose a set S containing a vertex v_1 and next choose an edge e_1 which is incident to vertex v_1 in K_n, which gives the minimum hub set in $\overline{K_n^{--}}$. Thus, $S = \{v_1, e_1\}$. Therefore, $|S| = 2$. Hence, $h(\overline{K_n^{--}}) = 2$. □

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