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NEW APPROACH FOR ANALYSIS OF SOLUTION OF CERTAIN FRACTIONAL SUM-DIFFERENCE EQUATIONS

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Abstract. This paper investigates a class of nonlinear fractional sum–difference equations in a discrete setting. By converting the considered problem into an equivalent fractional summation equation, fixed point techniques are effectively applied to analyze the behavior of solutions. The existence of solutions is established through the Leray–Schauder alternative, whereas uniqueness is derived using a discrete form of Bihari's inequality. The obtained results extend several existing works related to fractional discrete systems under suitable growth conditions. Moreover, the developed approach offers an efficient framework for studying nonlinear fractional dynamical models with memory effects arising in discrete fractional calculus.

Keywords. Fractional sum–difference equations; Discrete fractional calculus; Leray–Schauder alternative; Bihari's inequality; Existence and uniqueness; Nonlinear systems
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1. Introduction and preliminaries

Discrete fractional calculus has gained significant attention because of its wide applications in dynamical systems, control theory, and mathematical modeling. Atici et al.[1] derived the Euler–Lagrange equation in discrete fractional variational calculus and studied a Gompertz fractional difference model for tumor growth. Basic existence, uniqueness and boundary value results for fractional difference equations were established in [2, 3, 4].Using fixed-point techniques,Chen et al.[5] and Ibrahim and Jalab [6] investigated nonlinear fractional difference equations and related boundary value problems.

The qualitative behavior of fractional discrete systems has also been widely studied. Deshpande et al.[7] analyzed chaotic properties of fractional maps and showed improved stability compared with integer-order systems, while [8] examined uncertain fractional forward difference trajectories. Positive solutions under nonlocal conditions were obtained in [9], and discrete fractional Dirichlet-type

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problems were discussed in [10]. Anh et al.[11] derived variation-of-constants formulas using the Z-transform, whereas Jonnalagadda et al.[?] studied existence, uniqueness, boundedness, and stability. Stability criteria for delayed and feedback systems were presented in [13].

Further developments on Caputo-type fractional operators and nonlinear boundary value problems can be found in [14, 15, 16, 17]. Applications to physical systems such as mass–spring–damper models were reported in [18]. Mozyrska [19] studied fractional differences using L-transform methods, while [20] introduced the nabla discrete Sumudu transform and Mittag–Leffler eigenfunctions. A unified framework for fractional sums and differences is given in [21]. For standard terminology and preliminaries, see [4]. The results on the problem of existence and uniqueness of solutions of differential equations of fractional order have been discussed by some authors, which can be found in [22, 23, 24]. The main objective of this paper is to study the existence and uniqueness of solutions for a class of fractional difference equations. The analysis is carried out by applying the Leray–Schauder alternative together with Bihari's inequality in a discrete framework. The analysis is carried out by applying the Leray–Schauder alternative together with Bihari's inequality in a discrete framework. Recent studies have introduced new methods for analyzing divergent integrals involving submultiplicative functions and developed a discrete symplectic Dirac operator framework using symplectic Clifford algebra, providing applications to discrete Laplacian factorization and representation theory [27, 28].

The novelty of the present work lies in transforming the considered nonlinear fractional sum–difference equation into an equivalent fractional summation equation and employing the Leray–Schauder alternative together with a discrete version of Bihari's inequality under weaker growth assumptions. The obtained existence and uniqueness results extend and generalize several related results available in the current literature. Furthermore, the developed approach provides an effective analytical framework for investigating nonlinear discrete fractional systems with memory effects.

We consider the following fractional difference equation of order α given by

$$\Delta^{\alpha}x(t) = \chi(t, x(t), x(t-1)),\ \forall\ t \in {N_a}^T\ (T > 0), \tag{1.1}$$

under the initial conditions

$$x(t-1) = \varphi(t)\ (-1 \leq t < 0), x(0) = x_0, \tag{1.2}$$

Where $\chi \in C([0,T] \times R^n \times R^n, R^n)$ and $\varphi(t)$ is a continuous function for $0 \leq t < 1, \lim_{t\to 1-0}\varphi(t)$ exists, for which we denote by $\varphi(1-0) = c_0$ If we consider the solutions of (1.1) for $t \in [0,T]$, we get a function $x(t-1)$ which is unable to define as a solution for $0 \leq t < 1$. Hence, we need to impose some condition, if $T < 1$, the problem is reduced to fractional ordinary differential equation

$$\Delta^{\alpha}x(t) = \chi(t, x(t), \varphi(t)), \tag{1.3}$$

for $, 0 \leq t < 1$ with the second condition in (1.2). Here, it is necessary to obtain the solutions of equation (1.1) for $0 \leq t \leq T$, so that, we assume, in the sequel,

$T \not< 1$. The integral equations which are equivalent to (1.1) -(1.2) are

$$x(t) = x_0 + \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} \frac{\Delta^n \chi(t, x(s), \varphi(s))}{(t-s-1)^{-(n-\alpha-1)}} \tag{1.4}$$

for , $0 \le t < 1$ and

$$x(t) = x_0 + \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{1-(n-\alpha)} \frac{\Delta^n \chi(t, x(s), \varphi(s))}{(t-s-1)^{-(n-\alpha-1)}} + \frac{1}{\Gamma(n-\alpha)} \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} \frac{\Delta^n \chi(t, x(s), \varphi(s-1))}{(t-s-1)^{-(n-\alpha-1)}} \tag{1.5}$$

for $0 \le t \le T$.

Note: Throughout this work, the discrete interval is defined by

$$\mathbb{N}_{a,T} = \{a, a+1, a+2, \dots, T\}.$$

Definition 1.1. let $\alpha > o$,The α^{th} fractional sum of f is defined by

(1) $\Delta^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s)$

Definition 1.2. let $\alpha > 0$ and $n-1 < \alpha < n$,whrer m denotes a positive integer,$\lceil \alpha \rceil, \left\lceil \dot{(.)} \right\rceil$ ceiling of number.The Caputo-like fractional difference operator of order $\alpha > o$ is defined by

(1) $\Delta_c^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} \Delta^n u(s)$

where Δ^n is the n^{th} forward difference operator.The fractional Caputo-like difference operator Δ^α maps functions defined on N_a to the functions defined on $N_{a+n-\alpha}$.

2. Main results

Lemma 2.1. *(Leray–Schauder alternative, [25]). Suppose $\chi : E \to E$ is a completely continuous operator. If $\mu(\chi) = \{x \in E : x = \lambda\, \chi(x) for some, 0 < \lambda < 1\}$. then, either χ has at least one fixed point or the set $\mu(\chi)$ is unbounded..*

Lemma 2.2. [26] *Let $u(t), p(t) \in C(R_+, R_+), R_+ = [0, \infty)$. Let $\psi(u)$ is a continuous, non-decreasing function defined on $R_+, \psi(u) > 0$ for $u > 0$ and $\psi(0) = 0$. If*

$$u(t) \le c + \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} \frac{\Delta^n p(s) \psi(u(s))}{(t-s-1)^{-(n-\alpha-1)}}$$

for $t \in R_+$, where $c \ge 0$ is a constant, then for $0 \le t \le t_1$,

$$u(t) \le \psi^{-1} \left[\psi(c) + \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} \frac{\Delta^n p(s)}{(t-s-1)^{-(n-\alpha-1)}} \right],$$

where

$$\psi(r) = \sum_{s=a+r_0}^{r-(n-\alpha)} \frac{1}{\psi(s)} \frac{1}{\zeta^*}, \quad r > 0, r_0 > 0,$$

ψ^{-1} is the inverse function of ψ and $T_1 \in R_+$ be chosen so that

$$\psi(c) + \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} \frac{\Delta^n p(s)}{(t-s-1)^{-(n-\alpha-1)}} \in Dom(\psi^{-1}),$$

for all $t \in R_+$ lying in the interval $0 \leq t \leq t_1$.

Theorem 2.3. *If the function χ in (1.1) satisfies the condition*

$$| \Delta^n \chi(t,x,y) | \leq h(t) \left[f(| x |) + f(| y |) \right] \tag{2.1}$$

where $h \in C([0.T], R)$ and $f : R_+ \to (0, \infty)$ is continuous and non-decreasing function. Then (1.1) -(1.2) has a solution $x(t)$ defined on $[0.T](T > 0)$ provided T satisfies

$$\sum_{s=a}^{T-(n-\alpha)} \left[h(s) + h(s+1) \right] \frac{1}{\zeta^*} < \sum_{s=\beta}^{\infty} \frac{1}{f(s)} \frac{1}{\zeta^*} \tag{2.2}$$

where

$$\beta = | x_0 | + \sum_{s=a}^{1-(n-\alpha)} h(s) f(| \varphi(s) |) \frac{1}{\zeta^*}. \tag{2.3}$$

Proof. **Claim I.** By using Lemma (2.1), we establish the priori bounds on the solutions of the problem

$$\Delta^\alpha x(t) = \lambda \chi(t, x(t), x(t-1)), \tag{2.4}$$

under the initial conditions (1.2) for $\lambda \in (0,1)$. If $x(t)$ is a solution of (2.4)-(1.2), then we have to consider the following two cases.
Case 1: If $0 \leq t < 1$, then by the hypotheses, we have

$$| x(t) | = | x_0 + \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} \frac{\lambda \Delta^n \chi(t, x(s), \varphi(s))}{(t-s-1)^{-(n-\alpha-1)}} |$$

$$\leq | x_0 | + \sum_{s=a}^{t-(n-\alpha)} h(s) \left[f(| x(s) |) + f(| \varphi(s) |) \right] \frac{1}{| \Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)} |} \tag{2.5}$$

$$= \beta + \sum_{s=a}^{t-(n-\alpha)} h(s) f(| x(s) |) \frac{1}{\zeta^*}.$$

where $\Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)} = \zeta$ and $| \zeta | = \zeta^*$.
If$u(t)$ is defined by the right hand side of (1.2), then $u(0) = \beta, | x(t) | \leq u(t)$ and

$$\Delta^\alpha u(t) = h(t) f(| x(t) |) \leq h(t) f(u(t));$$

that is ,

$$\frac{\Delta^\alpha u(t)}{f(u(t))} \leq h(t). \tag{2.6}$$

Integration of (2.6) from 0 to t $(0 \leq t < 1)$, together with the change of variable, and the condition (2.2) gives

$$\sum_{s=\beta}^{u(t)-(n-\alpha)} \frac{1}{f(s)}\frac{1}{\zeta^*} \leq \sum_{s=a}^{t-(n-\alpha)} h(s)\frac{1}{\zeta^*} \leq \sum_{s=a}^{1-(n-\alpha)} h(s)\frac{1}{\zeta^*} < \sum_{s=\beta}^{\infty}\int_{\beta}^{\infty} \frac{1}{f(s)}\frac{1}{\zeta^*}. \tag{2.7}$$

From this inequality,we deduce that, there is a constant Q_1 independent of $\lambda \in (0,1)$ such that $u(t) \leq Q_1$ for $0 \leq t < 1$ and hence $\mid x(t) \mid \leq Q_1$.
Case 2: If $1 \leq t \leq T$, then by the hypotheses, we have

$$\begin{aligned} \mid x(t) \mid =\mid x_0 &+ \sum_{s=a}^{1-(n-\alpha)} \frac{\lambda\Delta^n\chi(t,x(s),\varphi(s))}{\Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}} \\ &+ \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} \frac{\lambda\Delta^n\chi(t,x(s),x(s-1))}{\Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}} \mid \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\leq \mid x_0 \mid + \sum_{s=a}^{t-(n-\alpha)} h(s)\left[f(\mid x(s)\mid) + f(\mid \varphi(s)\mid)\right]\frac{1}{\mid \Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}\mid} \\ &+ \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} h(s)\left[f(\mid x(s)\mid) + f(\mid x(s-1)\mid)\right]\frac{1}{\mid \Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}\mid} \end{aligned} \tag{2.9}$$

$$\begin{aligned} &= \mid x_0 \mid + \sum_{s=a}^{1-(n-\alpha)} h(s)f(\mid \psi(s)\mid)\frac{1}{\mid \zeta \mid} + \sum_{s=a}^{1-(n-\alpha)} h(s)f(\mid x(s)\mid)\frac{1}{\mid \zeta \mid} \\ &+ \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} h(s)f(\mid x(s)\mid)\frac{1}{\zeta^*} + \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} h(s)f(\mid x(s-1)\mid)\frac{1}{\zeta^*} \\ &= \beta + \sum_{s=a}^{t-(n-\alpha)} h(s)f(\mid x(s)\mid)\frac{1}{\zeta^*} + I_1 \end{aligned}$$

where

$$I_1 = \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} h(s)f(\mid x(s-1)\mid)\frac{1}{\zeta^*}. \tag{2.10}$$

By the change of variable, from (2.10), we have

$$I_1 = \sum_{\sigma=a}^{t-1-(n-\alpha)} h(\sigma+1)f(\mid x(\sigma)\mid)\frac{1}{\zeta^*} \leq \sum_{\sigma=a}^{t-(n-\alpha)} h(\sigma+1)f(\mid x(\sigma)\mid)\frac{1}{\zeta^*} \tag{2.11}$$

Using this inequality in (2.9), we get

$$\mid x(t) \mid \leq \beta + \sum_{s=a}^{t-(n-\alpha)} \left[h(s) + h(s+1)\right] f(\mid x(s)\mid). \tag{2.12}$$

If $v(t)$ be defined by the right hand side of (2.12), then
$v(0)=\beta, \mid x(t) \mid \leq v(t)$ and

$$\Delta^{\alpha} v(t)=[h(t)+h(t+1)]\, f(\mid x(t)\mid) \leq [h(t)+h(t+1)]\, f(v(t));$$

that is ,

$$\frac{\Delta^{\alpha} v(t)}{f(v(t))} \leq [h(t)+h(t+1)]\,. \tag{2.13}$$

Integration of (2.13) from 0 to t, $1 \leq t \leq T$, the change of variable, and the condition (2.2) give

$$\begin{aligned}
\sum_{s=\beta}^{v(t)-(n-\alpha)} \frac{1}{f(s)} \frac{1}{\zeta^{*}} &\leq \sum_{s=a}^{t-(n-\alpha)} [h(t)+h(t+1)]\, \frac{1}{\zeta^{*}} \\
&\leq \sum_{s=a}^{T-(n-\alpha)} [h(t)+h(t+1)]\, \frac{1}{\zeta^{*}} \\
&< \sum_{s=\beta}^{\infty} \frac{1}{f(s)} \frac{1}{\zeta^{*}}
\end{aligned} \tag{2.14}$$

From (2.14) we conclude that there is a constant Q_2 independent of $\lambda \in (0,1)$ such that $v(t) \leq Q_2$ and hence $\mid x(t) \mid \leq Q_1$ for $1 \leq t \leq T$. Let $Q=\max Q_1, Q_2$. Obviously,$\mid x(t) \mid \leq Q$ for $t \in [0,T]$ and consequently,$\parallel x(t) \parallel = \sup\{\mid x(t) \mid : t \in [0,b]\} \leq Q$.

Claim II. χ is continuous and uniformly bounded.

We define $B=C([0,T],R^{n})$ to be the Banach space of all continuous functions from $[0,T]$ into R^{n} endowed with sup norm defined above. We rewrite the problem (1.1) -(1.2) as follows. If $y \in B$ and $x(t)=y(t)+x_0$, then

$$y(t)=\frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} \frac{\Delta^{n} \chi(s, y(t)+x_0, \varphi(s))}{(t-s-1)^{-(n-\alpha-1)}},$$

for $0 \leq t < 1$ and

$$y(t)=\frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{1-(n-\alpha)} \frac{\Delta^{n} \chi(s, y(s)+x_0, \varphi(s))}{(t-s-1)^{-(n-\alpha-1)}}$$

$$+\frac{1}{\Gamma(n-\alpha)} \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} \frac{\Delta^{n} \chi(s, y(s)+x_0, y(s-1)+x_0),}{(t-s-1)^{-(n-\alpha-1)}}$$

for $1 \leq t \leq T$, $y(0)=y_0=0$ if and only if $x(t)$ satisfies (1.1) -(1.2).
If $B_0=\{y \in B : y_0=0\}$ and $\chi : B_0 \to B_0$ is defined by

$$\chi y(t)=\frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} \frac{\Delta^{n} \chi(s, y(t)+x_0, \varphi(s))}{(t-s-1)^{-(n-\alpha-1)}}, \tag{2.15}$$

for $0 \leq t < 1$ and

$$\chi y(t) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{1-(n-\alpha)} \frac{\Delta^n \chi(s, y(s) + x_0, \varphi(s))}{(t-s-1)^{-(n-\alpha-1)}}$$

$$+ \frac{1}{\Gamma(n-\alpha)} \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} \frac{\Delta^n \chi(s, y(s) + x_0, y(s-1) + x_0),}{(t-s-1)^{-(n-\alpha-1)}} \tag{2.16}$$

for $1 \leq t \leq T$. Then χ is clearly continuous. Now, we will prove that χ is uniformly bounded. If $\{b_m\}$ is a bounded sequence in B_0, that is,$\| b_m \| \leq b \ \forall m$, where $b > 0$ is a constant. If $N = \sup \{h(t) : t \in [0, T]\}$. then We have to consider following the two cases:

Case 1: If $0 \leq t < 1$, then by (2.15) and hypotheses, we have

$$| \chi b_m(t) | \leq \sum_{s=a}^{t-(n-\alpha)} \left| \frac{\Delta^n \chi(s, b_m(s) + x_0, \varphi(s))}{\Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}} \right|$$

$$\leq \sum_{s=a}^{t-(n-\alpha)} h(s) \left[f(| b_m(s) | + | x_0 |) + f(| \varphi(s) |) \right] \frac{1}{| \Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)} |}$$

$$\leq \sum_{s=a}^{1-(n-\alpha)} h(s) f(| \varphi(s) |) \frac{1}{| \zeta |} + \sum_{s=a}^{t-(n-\alpha)} h(s) f(b + | x_0 |) \frac{1}{| \zeta |} \tag{2.17}$$

$$\leq \gamma + \sum_{s=a}^{1-(n-\alpha)} N f(b + | x_0 |) \frac{1}{\zeta^*}$$

where

$$\gamma = \sum_{s=a}^{1-(n-\alpha)} h(s) f(| \varphi(s) |) \frac{1}{| \zeta |}. \tag{2.18}$$

Case 2: If $1 \leq t \leq T$, then by (2.16) and the hypotheses, we have

$$| \chi b_m(t) | \leq \sum_{s=a}^{1-(n-\alpha)} \left| \frac{\Delta^n \chi(s, b_m(s) + x_0, \varphi(s))}{\Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}} \right|$$

$$+ \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} \left| \frac{\Delta^n \chi(s, b_m(s) + x_0, b_m(s-1) + x_0)}{\Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}} \right|$$

$$\leq \sum_{s=a}^{1-(n-\alpha)} h(s) \left[f(| b_m(s) | + | x_0 |) + f(| \varphi(s) |) \right] \frac{1}{| \Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)} |}$$

$$+ \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} h(s) \left[f(| b_m(s) | + | x_0 |) + f(| b_m(s-1) + x_0 |) \right] \frac{1}{| \Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)} |}$$

$$= \sum_{s=a}^{1-(n-\alpha)} h(s) f(| \varphi(s) |) \frac{1}{\zeta^*} + \sum_{s=a}^{1-(n-\alpha)} h(s) f(b_m(s) + | x_0 |) \frac{1}{\zeta^*}$$

$$+\sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} h(s)f(b_m(s)+\mid x_0\mid)\frac{1}{\zeta^*}+\sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} h(s)f(b_m(s-1)+\mid x_0\mid)\frac{1}{\zeta^*}$$

$$=\gamma+\sum_{s=a}^{t-(n-\alpha)} h(s)f(b_m(s)+\mid x_0\mid)\frac{1}{\zeta^*}+I_2, \tag{2.19}$$

where γ is given by (2.12), and

$$I_2=\sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} h(s)f(b_m(s-1)+\mid x_0\mid)\frac{1}{\zeta^*} \tag{2.20}$$

By the change of variable, we have

$$I_2=\sum_{\sigma=1-(n-\alpha)}^{t-1-(n-\alpha)} h(\sigma+1)f(b_m(\sigma)+\mid x_0\mid)\frac{1}{\zeta^*}$$

$$\leq\sum_{\sigma=a}^{t-(n-\alpha)} h(\sigma+1)f(b_m(\sigma)+\mid x_0\mid)\frac{1}{\zeta^*} \tag{2.21}$$

Using (2.21) in (2.19), we have

$$\mid \chi b_m(t)\mid\leq\gamma+\sum_{s=a}^{t-(n-\alpha)} [h(s)+h(s+1)]f(b_m(s)+\mid x_0\mid)\frac{1}{\zeta^*}$$

$$\leq\gamma+\sum_{s=a}^{T-(n-\alpha)} 2Nf(b+\mid x_0\mid)\frac{1}{\zeta^*} \tag{2.22}$$

$$=\gamma+2NTf(b+\mid x_0\mid)\frac{1}{\zeta^*}.$$

From (2.17) and (2.22), it conclude that $\{\chi b_m\}$ is uniformly bounded.
Claim III. the sequence $\{\chi b_m\}$ is equicontinuous.
Suppose $\{b_m\}$ and N are as in claim II. We have to consider three cases.
Case 1: If t and $t^{'}$ are contained in $0\leq t<1$, then from (2.15), it follows that

$$\chi b_m(t)-\chi b_m(t^{'})=\frac{1}{\Gamma(n-\alpha)}\sum_{s=a}^{t-(n-\alpha)}\frac{\Delta^n\chi(s,b_m(s)+x_0,\varphi(s))}{(t-s-1)^{-(n-\alpha-1)}}$$

$$-\frac{1}{\Gamma(n-\alpha)}\sum_{s=a}^{t^{'}-(n-\alpha)}\frac{\Delta^n\chi(s,b_m(s)+x_0,\varphi(s))}{(t-s-1)^{-(n-\alpha-1)}}$$

$$=\frac{1}{\Gamma(n-\alpha)}\sum_{s=t^{'}-(n-\alpha)}^{t-(n-\alpha)}\frac{\Delta^n\chi(s,b_m(s)+x_0,\varphi(s))}{(t-s-1)^{-(n-\alpha-1)}} \tag{2.23}$$

By the equality (2.23) and hypotheses, we have

$$\mid\chi b_m(t)-\chi b_m(t^{'})\mid\leq\sum_{s=t^{'}-(n-\alpha)}^{t-(n-\alpha)}\mid\frac{\Delta^n\chi(s,b_m(s)+x_0,\varphi(s))}{\Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}}\mid$$

$$\leq \sum_{s=t^{\shortmid}-(n-\alpha)}^{t-(n-\alpha)} h(s)\left[f(\mid b_m(s)\mid + \mid x_0 \mid) + f(\mid \varphi(s)\mid)\right] \frac{1}{\mid \Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}\mid} \tag{2.24}$$

$$\leq \sum_{s=t^{\shortmid}-(n-\alpha)}^{t-(n-\alpha)} N\left[f(\mid b \mid + \mid x_0 \mid) + f(\mid c_0 \mid)\right] \frac{1}{\zeta^*}$$

$$= N\left[f(\mid b \mid + \mid x_0 \mid) + f(\mid c_0 \mid)\right] \mid t - t^{\shortmid} \mid .$$

Case 2: If t and $t^{\shortmid}$ are contained in $1 \leq t \leq T$, then from (2.16), it follows that

$$\chi b_m(t) - \chi b_m(t^{\shortmid}) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{1-(n-\alpha)} \frac{\Delta^n \chi(s, b_m(s)+x_0, \varphi(s))}{(t-s-1)^{-(n-\alpha-1)}}$$

$$+\frac{1}{\Gamma(n-\alpha)} \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} \frac{\Delta^n \chi(s, b_m(s)+x_0, b_m(s-1)+x_0)}{(t-s-1)^{-(n-\alpha-1)}}$$

$$-\frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{1-(n-\alpha)} \frac{\Delta^n \chi(s, b_m(s)+x_0, \varphi(s))}{(t-s-1)^{-(n-\alpha-1)}}$$

$$-\frac{1}{\Gamma(n-\alpha)} \sum_{s=1-(n-\alpha)}^{t^{\shortmid}-(n-\alpha)} \frac{\Delta^n \chi(s, b_m(s)+x_0, b_m(s-1)+x_0)}{(t-s-1)^{-(n-\alpha-1)}}$$

$$= \sum_{s=t^{\shortmid}-(n-\alpha)}^{t-(n-\alpha)} \frac{\Delta^n \chi(s, b_m(s)+x_0, b_m(s-1)+x_0)}{\Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}}.$$

By this equality and the hypotheses, we have

$$\mid \chi b_m(t) - \chi b_m(t^{\shortmid}) \mid \leq \sum_{s=t^{\shortmid}-(n-\alpha)}^{t-(n-\alpha)} \mid \frac{\Delta^n \chi(s, b_m(s)+x_0, b_m(s-1)+x_0)}{\Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}} \mid$$

$$\leq \sum_{s=t^{\shortmid}-(n-\alpha)}^{t-(n-\alpha)} h(s)\left[f(\mid b_m(s)\mid + \mid x_0 \mid) + f(\mid b_m(s)\mid + \mid x_0 \mid)\right] \frac{1}{\mid \Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}\mid} \tag{2.25}$$

$$= \sum_{s=t^{\shortmid}-(n-\alpha)}^{t-(n-\alpha)} N f(\mid b_m(s)\mid + \mid x_0 \mid)\frac{1}{\zeta^*} + I_3$$

where

$$I_3 = \sum_{s=t^{\shortmid}-(n-\alpha)}^{t-(n-\alpha)} N f(\mid b_m(s-1)\mid + \mid x_0 \mid)\frac{1}{\zeta^*}$$

By the change of variable, we have

$$I_3 = \sum_{\sigma=t^{'}-1-(n-\alpha)}^{t-1-(n-\alpha)} Nf(\mid b_m(\sigma) \mid + \mid x_0 \mid)\frac{1}{\zeta^*} \leq 2Nf(\mid b_m(\sigma) \mid + \mid x_0 \mid)(t-t^{'})\frac{1}{\zeta^*}. \tag{2.26}$$

Using the above inequality in (2.25), we get

$$\mid \chi b_m(t) - \chi b_m(t^{'}) \mid \leq 2Nf(\mid b_m(\sigma) \mid + \mid x_0 \mid)(t-t^{'})\frac{1}{\zeta^*}. \tag{2.27}$$

Case 3: If t and $t^{'}$ are respectively contained in $[0,1)$ and $[1,T]$, then from (2.15) and (2.16), it follows that

$$\begin{aligned}
\chi b_m(t) - \chi b_m(t^{'}) &= \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} \frac{\Delta^n \chi(s, b_m(s)+x_0, \varphi(s))}{(t-s-1)^{-(n-\alpha-1)}} \\
&- \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{1-(n-\alpha)} \frac{\Delta^n \chi(s, b_m(s)+x_0, \varphi(s))}{(t-s-1)^{-(n-\alpha-1)}} \\
&- \frac{1}{\Gamma(n-\alpha)} \sum_{s=1-(n-\alpha)}^{t^{'}-(n-\alpha)} \frac{\Delta^n \chi(s, b_m(s)+x_0, b_m(s-1)+x_0)}{(t-s-1)^{-(n-\alpha-1)}} \\
&= -\frac{1}{\Gamma(n-\alpha)} \sum_{s=t-(n-\alpha)}^{a} \frac{\Delta^n \chi(s, b_m(s)+x_0, \varphi(s))}{(t-s-1)^{-(n-\alpha-1)}} \\
&- \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{1-(n-\alpha)} \frac{\Delta^n \chi(s, b_m(s)+x_0, \varphi(s))}{(t-s-1)^{-(n-\alpha-1)}} \\
&- \frac{1}{\Gamma(n-\alpha)} \sum_{s=1-(n-\alpha)}^{t^{'}-(n-\alpha)} \frac{\Delta^n \chi(s, b_m(s)+x_0, b_m(s-1)+x_0)}{(t-s-1)^{-(n-\alpha-1)}}
\end{aligned} \tag{2.28}$$

By (2.28) and using the hypotheses, we have

$$\begin{aligned}
\mid \chi b_m(t) - \chi b_m(t^{'}) \mid &= \sum_{s=t-(n-\alpha)}^{1-(n-\alpha)} \frac{\Delta^n \chi(s, b_m(s)+x_0, \varphi(s))}{\Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}} \mid \\
&+ \sum_{s=1-(n-\alpha)}^{t^{'}-(n-\alpha)} \mid \frac{\Delta^n \chi(s, b_m(s)+x_0, b_m(s-1)+x_0)}{\Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}} \mid \\
&\leq \sum_{s=t-(n-\alpha)}^{a} h(s)h(s)\left[f(\mid b_m(s) \mid + \mid x_0 \mid) + f(\mid \varphi(s) \mid)\right]\frac{1}{\zeta^*} \\
&+ \sum_{s=1-(n-\alpha)}^{t^{'}-(n-\alpha)} h(s)\left[f(\mid b_m(s) \mid + \mid x_0 \mid) + f(\mid b_m(s) \mid + \mid x_0 \mid)\right]\frac{1}{\zeta^*}
\end{aligned} \tag{2.29}$$

$$\leq \sum_{s=t-(n-\alpha)}^{1-(n-\alpha)} N\left[f(\mid b\mid+\mid x_0\mid)+f(\mid c_0\mid)\right]\frac{1}{\zeta^*}+\sum_{s=1-(n-\alpha)}^{t^{\prime}-(n-\alpha)} N\left[f(\mid b\mid+\mid x_0\mid)+f(\mid b\mid+\mid x_0\mid)\right]\frac{1}{\zeta^*}$$
$$\leq M(t^{\prime}-t),$$

where

$$M=\max\left\{N\left[f(\mid b\mid+\mid x_0\mid)+f(\mid c_0\mid),2Nf(\mid b\mid+\mid x_0\mid)\right]\right\}.$$

From (2.24), (2.27), (2.29), we conclude that $\{Fbm\}$ is equicontinuous. Hence the operator χ is completely continuous. By using Lemma (2.1), the operator χ has a fixed point in B_0. This implies that (1.1) -(1.2) has a solution.

Remark 2.4 Theorem 2.3 generalizes and extends several existing existence results available in the literature, particularly the work of Chen et al. [5], by studying nonlinear delayed fractional sum–difference equations under weaker growth and boundedness assumptions. The present approach combines the Leray–Schauder alternative with discrete Bihari-type inequalities to obtain existence results in a more general discrete fractional framework.

□

3. Analysis of Solution of Fractional Sum-Difference Equations

The present section focuses on establishing existence and uniqueness theorems for Fractional Sum–Difference Equations.

Theorem 3.1. *Assume (1.1) with $\chi\in C(R_+\times R^n\times R^n,R^n)$, under the initial conditions in (1.2). If*
(i) the function χ satisfies

$$\mid \Delta^n\chi(t,x,y)-\Delta^n\chi(t,\overline{x},\overline{y})\mid\leq h(t)[f(\mid x-\overline{x}\mid)+f(\mid y-\overline{y}\mid)], \tag{3.1}$$

where $h\in C(R_+,R_+)$, $f(u)$ is a continuous, non-decreasing function for $u\geq 0, f(0)=0$;
(ii)Let

$$G(r)=\sum_{s=a+r_0}^{r-(n-\alpha)}\frac{1}{f(s)}\frac{1}{\zeta^*},(0<r_0\leq r),$$

with G^{-1} being the inverse function of G and assume that $\lim_{r_0\to+0}G(r)=\infty$, for any fixed r.
Then (1.1) -(1.2) has at most one solution on R_+.

Proof. If $x(t),y(t)$ are the two solutions of equation (1.1), under the initial conditions

$$x(t-1)=y(t-1)=\varphi(t),(0\leq t<1),x(0)=y(0)=x_0, \tag{3.2}$$

and if $u(t)=\mid x(t)-y(t)\mid, t\in R_+$, then We consider the following two cases.
Case 1: If $0\leq t<1$, then from the hypotheses, we have

$$u(t)\leq\sum_{s=a}^{t-(n-\alpha)}\mid \Delta^n\chi(s,x(s),\varphi(s))-\Delta^n\chi(s,\overline{y}(s),\overline{\varphi}(s))\mid\frac{1}{\mid\Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}\mid}$$

$$\leq \sum_{s=a}^{t-(n-\alpha)} h(s)f(\mid x(s)-y(s)\mid)\frac{1}{\mid \zeta \mid} \tag{3.3}$$
$$\leq \epsilon_1 + \sum_{s=a}^{t-(n-\alpha)} h(s)f(u(s))\frac{1}{\zeta^*},$$

where $\epsilon_1 > 0$ is sufficiently small constant. By Lemma (2.2) and (3.3) gives

$$\mid x(t)-y(t)\mid \leq G^{-1}[G(\epsilon_1) + \sum_{s=a}^{t-(n-\alpha)} h(s)\frac{1}{\zeta^*}] \tag{3.4}$$

Case 2: If $1 \leq t < \infty$, then from the hypotheses, we have

$$u(t) \leq \sum_{s=a}^{1-(n-\alpha)} \mid \Delta^n\chi(s,x(s),\varphi(s)) - \Delta^n\chi(s,y(s),\varphi(s))\mid \frac{1}{\mid \Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}\mid}$$

$$+ \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} \mid \Delta^n\chi(s,x(s),x(s-1)) - \Delta^n\chi(s,y(s),y(s-1))\mid \frac{1}{\mid \Gamma(n-\alpha)(t-s-1)^{-(n-\alpha-1)}\mid}$$

$$\leq \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} h(s)f(u(s))\frac{1}{\zeta^*} + \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} h(s)\left[f(u(s)) + f(u(s-1))\right]\frac{1}{\zeta^*} \tag{3.5}$$

$$= \sum_{s=a}^{1-(n-\alpha)} h(s)f(u(s))\frac{1}{\zeta^*} + \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} h(s)f(u(s))\frac{1}{\zeta^*} + \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} h(s)f(u(s-1))\frac{1}{\zeta^*}$$
$$= \sum_{s=a}^{t-(n-\alpha)} h(s)f(u(s))\frac{1}{\zeta^*} + I_4,$$

where

$$I_4 = \sum_{s=1-(n-\alpha)}^{t-(n-\alpha)} h(s)f(u(s-1))\frac{1}{\zeta^*} \tag{3.6}$$

$$I_4 \leq \sum_{s=a}^{t-(n-\alpha)} h(s+1)f(u(s))\frac{1}{\zeta^*} \tag{3.7}$$

Using (3.7) in (3.5), we get

$$u(t) \leq \sum_{s=a}^{t-(n-\alpha)} \left[h(s)+h(s+1)\right]f(u(s))\frac{1}{\zeta^*} \leq \epsilon_2 + \sum_{s=a}^{t-(n-\alpha)} \left[h(s)+h(s+1)\right]f(u(s))\frac{1}{\zeta^*}, \tag{3.8}$$

where $\epsilon_2 > 0$is sufficiently small constant. By Lemma (2.2) and (3.8) gives

$$\mid x(t)-y(t)\mid \leq G^{-1}[G(\epsilon_2) + \sum_{s=a}^{t-(n-\alpha)} [h(s)+h(s+1)]\frac{1}{\zeta^*}]. \tag{3.9}$$

To apply the estimations in (3.4), (3.9) to the uniqueness problem, we use the notation $G(r, r_0)$ instead of $G(r)$ and impose the assumption $\lim_{r_0 \to +0} G(r, r_0) =$

$+\infty$, for fixed r, then we obtain $\lim_{r_0\to+0} G^{-1}(r,r_0) = 0$. From (3.4), (3.9), it follows that $\mid x(t) - y(t) \mid\leq 0$ for $t \in R_+$ and hence $x(t) = y(t)$ on R_+. Thus, there is at most one solution to (1.1) -(1.2) on R_+.

□

4. Hyers–Ulam Stability Analysis

In this section, we establish the Hyers–Ulam stability result for the nonlinear fractional sum–difference equation

$$\Delta^\alpha x(t) = \chi(t, x(t), x(t-1)), \qquad t \in \mathbb{N}_{0,T}, \quad 0 < \alpha < 1, \tag{4.1}$$

subject to the initial conditions

$$x(t-1) = \phi(t), \qquad x(0) = x_0. \tag{4.2}$$

Definition 4.1. The fractional sum–difference equation is said to be Hyers–Ulam stable if, for every $\varepsilon > 0$ and every function $y(t)$ satisfying

$$|\Delta^\alpha y(t) - \chi(t, y(t), y(t-1))| \leq \varepsilon, \qquad t \in \mathbb{N}_{0,T}, \tag{4.3}$$

there exists an exact solution $x(t)$ of the problem such that

$$|y(t) - x(t)| \leq C\varepsilon, \qquad t \in \mathbb{N}_{0,T}, \tag{4.4}$$

where $C > 0$ is a constant independent of ε.

Theorem 4.2. *Assume that the nonlinear function*

$$\chi : \mathbb{N}_{0,T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

is continuous and satisfies the Lipschitz condition

$$|\chi(t, x_1, y_1) - \chi(t, x_2, y_2)| \leq L\left(|x_1 - x_2| + |y_1 - y_2|\right), \tag{4.5}$$

for some constant $L > 0$ and for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Further, assume that

$$\frac{2LT^\alpha}{\Gamma(\alpha+1)} < 1. \tag{4.6}$$

Then the fractional sum–difference equation is Hyers–Ulam stable.

Proof. Let $y(t)$ be an approximate solution satisfying

$$|\Delta^\alpha y(t) - \chi(t, y(t), y(t-1))| \leq \varepsilon. \tag{4.7}$$

Then there exists a function $g(t)$ such that

$$|g(t)| \leq \varepsilon \tag{4.8}$$

and

$$\Delta^\alpha y(t) = \chi(t, y(t), y(t-1)) + g(t). \tag{4.9}$$

Using the fractional summation operator, we obtain

$$\begin{aligned} y(t) = x_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} \chi(s, y(s), y(s-1)) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} g(s). \end{aligned} \tag{4.10}$$

Let $x(t)$ be the exact solution of the problem. Then

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} \chi(s, x(s), x(s-1)). \tag{4.11}$$

Subtracting the above equations, we get

$$\begin{aligned} |y(t) - x(t)| \le \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} |\chi(s, y(s), y(s-1)) \\ - \chi(s, x(s), x(s-1))| \\ + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} |g(s)|. \end{aligned} \tag{4.12}$$

Using the Lipschitz condition, we obtain

$$\begin{aligned} |y(t) - x(t)| \le \frac{L}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left(|y(s) - x(s)| \right. \\ \left. + |y(s-1) - x(s-1)| \right) \\ + \frac{\varepsilon}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)}. \end{aligned} \tag{4.13}$$

Let

$$M(t) = \sup_{0 \le s \le t} |y(s) - x(s)|.$$

Then

$$M(t) \le \frac{2L}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} M(t) + \frac{\varepsilon T^\alpha}{\Gamma(\alpha+1)}. \tag{4.14}$$

Therefore,

$$M(t) \le \frac{2LT^\alpha}{\Gamma(\alpha+1)} M(t) + \frac{\varepsilon T^\alpha}{\Gamma(\alpha+1)}. \tag{4.15}$$

Hence,

$$\left(1 - \frac{2LT^\alpha}{\Gamma(\alpha+1)}\right) M(t) \le \frac{\varepsilon T^\alpha}{\Gamma(\alpha+1)}. \tag{4.16}$$

Since

$$\frac{2LT^\alpha}{\Gamma(\alpha+1)} < 1,$$

we obtain

$$M(t) \le \frac{T^\alpha}{\Gamma(\alpha+1) - 2LT^\alpha} \varepsilon. \tag{4.17}$$

Thus,

$$|y(t) - x(t)| \le C\varepsilon, \tag{4.18}$$

where

$$C = \frac{T^\alpha}{\Gamma(\alpha+1) - 2LT^\alpha}. \tag{4.19}$$

Hence, the problem is Hyers–Ulam stable. □

Remark 4.3. The obtained Hyers–Ulam stability result guarantees that every approximate solution of the nonlinear fractional sum–difference equation remains close to an exact solution. This property is important in applications involving numerical approximation, perturbation analysis, and discrete dynamical systems with memory.

5. Example

5.1. **Application to Population Dynamics.** Consider the nonlinear fractional sum–difference equation

$$\Delta^\alpha x(t) = \frac{t+1}{10+t^2}\left(\frac{|x(t)|}{1+|x(t)|} + \frac{|x(t-1)|}{1+|x(t-1)|}\right), \quad 0<\alpha<1, \tag{5.1}$$

with initial conditions

$$x(t-1) = \phi(t), \qquad x(0) = x_0. \tag{5.2}$$

Here, $x(t)$ represents the population density at time t, while the delayed term $x(t-1)$ describes the memory effect from the previous generation.

Define

$$\chi(t,x,y) = \frac{t+1}{10+t^2}\left(\frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}\right). \tag{5.3}$$

Since

$$\frac{|x|}{1+|x|} \le |x|, \qquad \frac{|y|}{1+|y|} \le |y|, \tag{5.4}$$

we obtain

$$|\chi(t,x,y)| \le \frac{t+1}{10+t^2}\left(|x|+|y|\right). \tag{5.5}$$

Let

$$h(t) = \frac{t+1}{10+t^2}, \qquad f(u) = u. \tag{5.6}$$

Then,

$$|\chi(t,x,y)| \le h(t)\left[f(|x|) + f(|y|)\right], \tag{5.7}$$

which verifies condition (2.1) of Theorem 2.3.

Further, the function

$$h(t) = \frac{t+1}{10+t^2}$$

is continuous and bounded on $[0,T]$. Hence, all assumptions of Theorem 2.3 are satisfied.

Therefore, by Theorem 2.3, the considered fractional sum–difference equation admits at least one solution on $[0,T]$.

The above model is useful in biological population dynamics where the present population growth depends not only on the current state but also on previous population states due to hereditary and memory effects.

6. Conclusion

We studied the new approach of analysis of solution of initial value problem of fractional order subjected to initial conditions, by using the Bihari's integral inequality and Leray-Schauder alternative. The fractional derivatives are described in the Caputo sense instead of Riemann Liouville, also we obtained the solutions of the difference differential equations in new sense.

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