

THE PARADOX OF OVER-PARAMETERIZATION IN LEARNING ALGORITHM

FABRIZIO MORLANDO¹

ABSTRACT. Traditional learning theory fails to explain why over-parameterized networks generalize. We propose a physical synthesis treating learning as a dissipative process. By integrating Differential Geometry, Dynamical Systems, and Algorithmic Stability, we show generalization depends on dynamical stability rather than architecture. Utilizing the Polyak-Lojasiewicz condition and Restricted Strong Convexity, we establish that gradient flow acts as a contractive operator, preventing trajectory divergence. Finally, McDiarmid's inequalities convert this contraction into rigorous probabilistic bounds. Our framework proves that over-parameterization facilitates geometric properties that inherently regularize the learning process, ensuring robust generalization despite high model capacity.

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1. INTRODUCTION

A long-standing paradox in classical Machine Learning Theory is: why models with a high number of parameters do not suffer from catastrophic overfitting by complexity? Classical bounds suggest that the generalization gap scales with the model's capacity to fit arbitrary labels. Traditional learning theory, built upon the Vapnik-Chervonenkis (VC) framework [1], predicts that generalization error should scale with model complexity. Specifically, for a hypothesis class \mathcal{H} with VC dimension d , the generalization gap is bounded by $\mathcal{O}(\sqrt{d/n})$ where n is the sample size. This theory suggests a fundamental trade-off: complex models (with large d) should overfit small datasets. Modern deep neural networks routinely violate this principle. Networks with millions or billions of parameters, far exceeding the number of training examples, achieve excellent generalization performance. For instance, a ResNet with $d \approx 10^7$ parameters trained on ImageNet with $n \approx 10^6$ samples should, by classical theory, completely overfit. Yet it generalizes remarkably well. This phenomenon cannot be explained by traditional capacity-based arguments. The issue is not that the bounds are loose;

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* Corresponding author

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rather, the *conceptual framework* is inadequate. The VC dimension of a deep network grows with its parameter count, yet empirical performance improves with over-parameterization. We propose a fundamental reframing: *generalization is a property of the learning trajectory, not of the model architecture*. Instead of asking: how many functions can this model represent? one should ask how does the optimization algorithm explore the hypothesis space? This shift draws inspiration from three key observations: gradient descent does not find arbitrary global minima but rather converges to solutions with specific geometric properties such as flat minima or low-rank structures [2]; the information bottleneck principle [5] suggests that learning consists of two distinct phases, starting with data fitting and followed by the compression of representations to discard noise; and finally, the fact that small perturbations to the training set lead to only minor changes in the learned model [6, 8] indicates that the algorithm effectively “forgets” individual data points. This paper establishes a rigorous connection between four traditionally separate domains:

- (1) Differential Geometry: The Polyak-Lojasiewicz (PL) condition [9] characterizes loss landscape geometry. However, in over-parameterized settings, the PL condition alone is insufficient to guarantee stability. We therefore introduce *Restricted Strong Convexity* (RSC) as a local geometric property to provide the local curvature necessary for trajectory stability.
- (2) Dynamical Systems: Under the RSC assumption, gradient flow acts as a contractive operator. This ensures that two separate learning trajectories (starting from slightly different datasets S and $S^{(i)}$) do not diverge, but instead stay exponentially close in the parameter space.
- (3) Algorithmic Stability: The contraction induced by RSC directly implies uniform stability in Bousquet’s sense [6].
- (4) Concentration of Measure: McDiarmid’s inequality [10] converts this algorithmic stability into probabilistic generalization bounds. It guarantees that the generalization gap concentrates tightly around its expectation, confirming that the model’s performance on unseen data is a predictable outcome of the learning physics.

The core insight is that these are not independent perspectives but rather sequential steps in a single logical derivation:

Geometry (PL) \longrightarrow Dynamics (Contraction) \longrightarrow Stability \longrightarrow Generalization.

This chain reveals that generalization is an inevitable consequence of dissipative learning dynamics, independent of the model’s representational capacity. While individual components of this framework exist in the literature, they have rarely been synthesized. Most researchers treat these as separate tools: statisticians use McDiarmid’s inequality but ignore ODEs, dynamical systems theorists study contraction but not generalization, deep learning researchers use PL only for convergence speed, not for stability. We prove that these are not independent perspectives but sequential steps in one unified theory. Bousquet and Elisseeff [6] established that algorithmic stability implies generalization, but did not connect this to optimization dynamics. Hardt et al. [8] studied the stability of stochastic

gradient descent but focused on convex settings and did not bridge to concentration inequalities. The PL condition has been used to analyze convergence rates [16] but not systematically linked to generalization via stability. Tishby’s information bottleneck [5] provides an intuitive picture but lacks the mathematical rigor of our dynamical approach. Our novelty lies in constructing the complete logical chain and showing that the PL condition, much weaker than convexity, suffices to guarantee generalization in non-convex settings typical of deep learning. Section 3 establishes mathematical foundations and key definitions. Section 4 presents the four-step logical chain with detailed proofs. Section 5 discusses implications, limitations, and connections to related work. Section 6 concludes. Our analysis focused on continuous gradient flow. Extending to SGD requires analyzing the effect of gradient noise. Recent work [8] shows that small learning rates and mini-batching maintain similar stability properties. The paradox of over-parameterization is one of the most important open questions in deep learning theory. Our approach provides a satisfying resolution: it’s not about counting parameters, it’s about the dynamics of how we find them.

2. APPLICATIONS IN COMPUTER SCIENCE

The theoretical synthesis presented in this work offers practical blueprints for several emerging fields in computer science. In Federated Learning, where data is decentralized and heterogeneous, our stability-based bounds provide a rigorous framework for ensuring that local model updates do not diverge, maintaining global generalization despite the lack of centralized data access. In the realm of AI Safety and Robustness, the connection between Gradient Flow contraction and McDiarmid’s concentration provides a metric to quantify a model’s sensitivity to adversarial perturbations or “data poisoning,” where the PL condition acts as a certificate of reliable convergence. Furthermore, in Neural Architecture Search (NAS), our results suggest shifting the search objective from mere accuracy to “dynamical stability,” allowing for the automated design of large-scale networks that are inherently resistant to overfitting. Finally, this framework provides a formal justification for continual learning strategies, where maintaining algorithmic stability is essential to prevent catastrophic forgetting during sequential task acquisition.

3. PRELIMINARIES AND DEFINITIONS

3.1. The Learning Setup. Let $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ be the instance space, where \mathcal{X} is the input space and \mathcal{Y} is the label or target space. We assume a fixed but unknown distribution \mathcal{D} over \mathcal{Z} . A learning algorithm \mathcal{A} receives a training set $S = \{z_1, \dots, z_n\}$ drawn i.i.d. from \mathcal{D} and outputs a hypothesis $w_S \in \mathcal{W}$, where $\mathcal{W} \subseteq \mathbb{R}^d$ is the parameter space.

Definition 3.1. A loss function $\ell : \mathcal{W} \times \mathcal{Z} \rightarrow [0, M]$ measures prediction error. We define: $R(w) = \mathbb{E}_{z \sim \mathcal{D}}[\ell(w, z)]$ as the *popular risk* and $\hat{R}_S(w) = \frac{1}{n} \sum_{i=1}^n \ell(w, z_i)$ as the *empirical risk*.

Definition 3.2. The *generalization gap* of an algorithm \mathcal{A} on dataset S is:

$$\text{Gen}(S) = R(w_S) - \hat{R}_S(w_S) \quad (3.1)$$

where $w_S = \mathcal{A}(S)$ is the output of the algorithm.

In the sequel, we seek to bound $\mathbb{E}_S[\text{Gen}(S)]$ and show concentration around this expectation. The concept of algorithmic stability, introduced by Bousquet and Elisseeff [6], measures how sensitive an algorithm is to perturbations of the training set.

Definition 3.3. For a dataset $S = \{z_1, \dots, z_n\}$ and index $i \in [n]$, define $S^{(i)} = \{z_1, \dots, z_{i-1}, z'_i, z_{i+1}, \dots, z_n\}$ where z'_i is drawn independently from \mathcal{D} . This represents replacing the i -th example with a fresh sample.

Definition 3.4. An algorithm \mathcal{A} is β -uniformly stable if for all datasets $S, S^{(i)}$ differing in one element:

$$\sup_{z \in \mathcal{Z}} |\ell(\mathcal{A}(S), z) - \ell(\mathcal{A}(S^{(i)}), z)| \leq \beta. \quad (3.2)$$

Intuitively, a stable algorithm forgets individual training examples. If changing one data point causes a large change in the model's predictions, the algorithm has memorized that point (overfitting). Stability prevents this pathology. In contrast to the VC-dimension approach, which counts the number of representable functions, we treat Gradient Descent (GD) as a physical system evolving toward a state of lower empirical risk. Rather than analyzing discrete gradient descent updates $w_{t+1} = w_t - \eta \nabla \hat{R}_S(w_t)$, we study the continuous-time limit as $\eta \rightarrow 0$.

Definition 3.5. The *gradient flow* associated with empirical risk \hat{R}_S is the solution to the ordinary differential equation (ODE):

$$\frac{dw_t}{dt} = -\nabla \hat{R}_S(w_t), \quad w_0 = w_{\text{init}}. \quad (3.3)$$

Gradient flow can be viewed as the steepest descent path in the loss landscape. It provides a mathematically tractable approximation to gradient descent with small learning rate. When the learning rate η is sufficiently small, discrete GD closely follows the continuous trajectory [11]. In this view, training is analogous to a particle rolling down a potential energy surface $\hat{R}_S(w)$ with friction. The system dissipates energy and converges to a local minimum.

3.2. Contraction Mappings.

Definition 3.6. A mapping $T : \mathcal{W} \rightarrow \mathcal{W}$ is a *contraction* with rate $\lambda > 0$ if:

$$\|T(w) - T(w')\| \leq e^{-\lambda} \|w - w'\|, \quad \forall w, w' \in \mathcal{W}. \quad (3.4)$$

In the context of dynamical systems, if the gradient flow is contractive, trajectories originating from different initial conditions converge exponentially fast. This is the mathematical formalization of forgetting initial conditions.

Definition 3.7. Consider two gradient flows with different datasets:

$$\frac{dw_t}{dt} = -\nabla \hat{R}_S(w_t), \quad (3.5)$$

$$\frac{dw'_t}{dt} = -\nabla \hat{R}_{S^{(i)}}(w'_t). \quad (3.6)$$

Define the distance function $D(t) = \|w_t - w'_t\|$. If $\frac{d}{dt}D(t) < 0$, the trajectories contract.

3.3. The Polyak-Lojasiewicz Condition. The Polyak-Lojasiewicz (PL) condition [9] is a geometric property of the loss function that ensures convergence without requiring convexity.

Definition 3.8. A differentiable function $\mathcal{L} : \mathcal{W} \rightarrow \mathbb{R}$ satisfies the *PL condition* with parameter $\mu > 0$ if:

$$\frac{1}{2} \|\nabla \mathcal{L}(w)\|^2 \geq \mu (\mathcal{L}(w) - \mathcal{L}^*), \quad \forall w \in \mathcal{W}, \quad (3.7)$$

where $\mathcal{L}^* := \inf_{w \in \mathcal{W}} \mathcal{L}(w)$ is the global minimum value.

The PL condition is strictly weaker than strong convexity, in fact, strong convexity requires $\nabla^2 \mathcal{L}(w) \succeq \mu I$ (i.e., Hessian is positive definite). While the PL condition only requires that the gradient is large enough near non-optimal points. The loss can have multiple local minima and still satisfy PL. For instance, among the functions satisfying PL we have any strongly convex function. $\mathcal{L}(x) = x^2 + \sin^2(x)$ has infinitely many local minima but satisfies PL near the global minimum. Certain over-parameterized neural networks satisfy PL locally near minima [12]. Intuitively, the PL condition says that there are no flat regions except at optimal points. If you are far from the optimum, the gradient must be too large. This prevents the optimization from getting stuck in plateaus and ensures exponential convergence. The main limitation is that the PL condition is hard to verify empirically. Future work could develop practical tests for when μ is large in real networks.

4. FROM GEOMETRY TO GENERALIZATION

We now present the four-step logical chain that connects geometric properties of the loss landscape to probabilistic generalization guarantees. Each step is a result that builds upon the previous ones.

4.1. From Geometry to Dynamics (The PL Effect). In neural networks, the loss function $\hat{R}_S(w)$ is non-convex. How can we guarantee that gradient flow converges to a good solution, let alone that trajectories contract? The solution is obtained via Polyak-Lojasiewicz condition provides a weaker-than-convexity assumption that still ensures exponential convergence to the global minimum.

Theorem 4.1. *Suppose \hat{R}_S satisfies the PL condition with parameter $\mu > 0$ and is L -smooth (i.e., $\|\nabla \hat{R}_S(w) - \nabla \hat{R}_S(w')\| \leq L\|w - w'\|$). Then the gradient flow*

$\frac{dw_t}{dt} = -\nabla \hat{R}_S(w_t)$ satisfies:

$$\hat{R}_S(w_t) - \hat{R}_S^* \leq e^{-2\mu t} \left(\hat{R}_S(w_0) - \hat{R}_S^* \right), \quad (4.1)$$

where $\hat{R}_S^* := \min_w \hat{R}_S(w)$ is the global minimum value.

Proof. Consider the time derivative of the loss. By definition of gradient flow:

$$\frac{d}{dt} \hat{R}_S(w_t) = \nabla \hat{R}_S(w_t)^\top \frac{dw_t}{dt} \quad (4.2)$$

$$= -\|\nabla \hat{R}_S(w_t)\|^2. \quad (4.3)$$

By the PL condition (Definition 3.8):

$$\|\nabla \hat{R}_S(w_t)\|^2 \geq 2\mu \left(\hat{R}_S(w_t) - \hat{R}_S^* \right). \quad (4.4)$$

Therefore,

$$\frac{d}{dt} \hat{R}_S(w_t) \leq -2\mu \left(\hat{R}_S(w_t) - \hat{R}_S^* \right). \quad (4.5)$$

This is a first-order linear ODE. Define $\Delta(t) = \hat{R}_S(w_t) - \hat{R}_S^*$. Then:

$$\frac{d\Delta(t)}{dt} \leq -2\mu\Delta(t). \quad (4.6)$$

Grönwall's inequality implies:

$$\Delta(t) \leq e^{-2\mu t} \Delta(0), \quad (4.7)$$

which completes the proof. \square

This theorem shows that the gradient flow does not wander in the loss landscape. Instead, it flows exponentially fast toward the global minimum, regardless of local non-convexities. The flow acts like a funnel guiding the system to optimal regions. In over-parameterized networks, the loss landscape often has many global minima forming a connected manifold [13]. The PL condition is more likely to hold in such settings because there are no isolated local minima that trap the optimization. over-parameterization helps create highways in parameter space that facilitate convergence.

4.2. From Dynamics to Stability (The Contraction Theorem). A fundamental observation regarding the geometry of the loss landscape is required. While the Polyak-Łojasiewicz (PL) condition ensures the exponential convergence of the loss function to a global minimum, it is *not* sufficient to guarantee the contraction of trajectories in the parameter space. The critical issue is that the PL condition allows for the existence of an entire manifold of global minima. Consequently, a small perturbation in the dataset can lead the gradient flow to converge toward a significantly different point on such a manifold, making it impossible to bound the distance $\|w_\infty - w'_\infty\|$ based solely on the PL property. To overcome this difficulty, we introduce an assumption of local strong convexity or, more precisely, the Restricted Strong Convexity (RSC). Without this additional geometric structure, the differential inequality governing the distance between trajectories

would lead to an exponential divergence e^{Lt} rather than contraction. The assumption of Restricted Strong Convexity (RSC) is particularly well-suited for over-parameterized models. Recent results in the Neural Tangent Kernel (NTK) literature [7] suggest that as the width of the network increases, the optimization trajectory remains within a region where the empirical risk behaves effectively as a strongly convex objective, thereby justifying the use of μ in the following stability bounds. This derivation follows the stability framework of Hardt et al. [8], with the critical distinction that their stability results are formally proven for convex and strongly convex functions. By explicitly assuming RSC, we bridge the gap between their convex analysis and the non-convex geometry of modern neural networks.

Theorem 4.2. *Consider the gradient flow trajectories w_t and w'_t originating from $w_0 = w'_0$ for the empirical risks \hat{R}_S and $\hat{R}_{S^{(i)}}$, respectively. Assume that:*

- (1) *The loss function $l(w, z)$ is L -smooth and ρ -Lipschitz in w for all z .*
- (2) *The empirical risk \hat{R}_S satisfies the Restricted Strong Convexity (RSC) condition with parameter $\mu > 0$ (the so-called RSC curvature) in a convex neighborhood \mathcal{K} containing both trajectories, such that for all $w, w' \in \mathcal{K}$:*

$$\langle \nabla \hat{R}_S(w) - \nabla \hat{R}_S(w'), w - w' \rangle \geq \mu \|w - w'\|^2. \quad (4.8)$$

Then, the distance between the trajectories $D(t) = \|w_t - w'_t\|$ is bounded for all $t \geq 0$ by:

$$\|w_t - w'_t\| \leq \frac{2\rho}{\mu n} (1 - e^{-\mu t}), \quad (4.9)$$

where n is the sample size. In particular, as $t \rightarrow \infty$, the trajectories converge to a steady-state separation $\|w_\infty - w'_\infty\| \leq \frac{2\rho}{\mu n}$.

Proof. To analyze the dynamical stability, we track the evolution of the squared Euclidean distance $D(t) = \|w_t - w'_t\|^2$. Following the dynamics of gradient flow, the time derivative is given by:

$$\frac{d}{dt} D(t) = -2(w_t - w'_t)^\top (\nabla \hat{R}_S(w_t) - \nabla \hat{R}_{S^{(i)}}(w'_t)). \quad (4.10)$$

We decompose the gradient term to isolate the effect of the dataset perturbation:

$$\frac{d}{dt} D(t) = -2\langle w_t - w'_t, \nabla \hat{R}_S(w_t) - \nabla \hat{R}_S(w'_t) \rangle - 2\langle w_t - w'_t, \nabla \hat{R}_S(w'_t) - \nabla \hat{R}_{S^{(i)}}(w'_t) \rangle. \quad (4.11)$$

By invoking the Restricted Strong Convexity of the empirical risk \hat{R}_S , the first term provides a negative feedback $-2\mu\|w_t - w'_t\|^2$, which is essential for stability. The second term represents the perturbation caused by replacing one sample in the dataset S . Since the loss is ρ -Lipschitz, the gradient difference is bounded by $\|\nabla \hat{R}_S(w'_t) - \nabla \hat{R}_{S^{(i)}}(w'_t)\| \leq \frac{2\rho}{n}$. Applying Cauchy-Schwarz inequality, we obtain the differential inequality:

$$\frac{d}{dt} D(t) \leq -2\mu D(t) + \frac{4\rho}{n} \sqrt{D(t)}. \quad (4.12)$$

Letting $u(t) = \sqrt{D(t)}$, the inequality simplifies to $\frac{du}{dt} \leq -\mu u + \frac{2\rho}{n}$. By applying Grönwall's Lemma, we find that the distance is bounded by $u(t) \leq \frac{2\rho}{\mu n}(1 - e^{-\mu t})$. This result ensures that the trajectories remain close in the parameter space, resolving the divergence issue inherent in the pure PL setting. \square

Without a positive μ , the trajectories could drift apart, leading to instability and overfitting. If μ is large, the algorithm is very stable; if μ is small, the algorithm becomes unstable. The following corollary shows that the Stability Parameter β , which measures how much the model changes when you remove one data point, is inversely proportional to the RSC curvature.

Corollary 4.3. *Under the assumptions of Theorem 4.2, the gradient flow algorithm is β -uniformly stable with:*

$$\beta \leq \frac{2\rho^2}{\mu n}. \quad (4.13)$$

Proof. Uniform stability measures the maximum change in the loss function when one training point is replaced. By the Lipschitz continuity of the loss, we have $|l(w_\infty, z) - l(w'_\infty, z)| \leq \rho \|w_\infty - w'_\infty\|$. Substituting the upper bound for the parameter distance obtained in Theorem 4.2 as $t \rightarrow \infty$, specifically $\|w_\infty - w'_\infty\| \leq \frac{2\rho}{\mu n}$, we directly arrive at the stability bound $\beta \leq \frac{2\rho^2}{\mu n}$. \square

The stability constant β decreases as the geometric structure of the loss becomes more well-behaved (larger μ) and the sample size n increases, which dilutes the effect of any single perturbation, naturally improving stability.

4.3. From Stability to Concentration (McDiarmid's Inequality). We have established that the algorithm is stable (Corollary 4.3). How does this translate into a bound on the generalization gap? The solution is to use McDiarmid's inequality [10], a powerful concentration result, to show that the generalization gap concentrates around its expectation.

Lemma 4.4. *Let Z_1, \dots, Z_n be independent random variables taking values in a set \mathcal{Z} . Let $f : \mathcal{Z}^n \rightarrow \mathbb{R}$ satisfy the bounded differences property: there exist constants c_1, \dots, c_n such that for all $i \in [n]$ and all $z_1, \dots, z_n, z'_i \in \mathcal{Z}$:*

$$\sup_{z_1, \dots, z_n, z'_i \in \mathcal{Z}} |f(z_1, \dots, z_i, \dots, z_n) - f(z_1, \dots, z'_i, \dots, z_n)| \leq c_i. \quad (4.14)$$

Then, for any $\epsilon > 0$:

$$\mathbb{P}(f(Z_1, \dots, Z_n) - \mathbb{E}[f(Z_1, \dots, Z_n)] \geq \epsilon) \leq \exp\left(-\frac{2\epsilon^2}{\sum_{i=1}^n c_i^2}\right). \quad (4.15)$$

We point out that McDiarmid's inequality is a generalization of Hoeffding's inequality to functions of independent random variables. It states that if changing one input has a bounded effect on the output, then the function concentrates tightly around its expectation.

Proposition 4.5. *Define $\Phi(S) = R(w_S) - \hat{R}_S(w_S)$ as the generalization gap functional. If the algorithm \mathcal{A} is β -uniformly stable and the loss is bounded by M , the maximum loss, then:*

$$|\Phi(S) - \Phi(S^{(i)})| \leq 2\beta + \frac{M}{n}. \quad (4.16)$$

Proof. Expanding the difference:

$$\Phi(S) - \Phi(S^{(i)}) = \left[R(w_S) - \hat{R}_S(w_S) \right] - \left[R(w_{S^{(i)}}) - \hat{R}_{S^{(i)}}(w_{S^{(i)}}) \right] \quad (4.17)$$

$$= \left[R(w_S) - R(w_{S^{(i)}}) \right] - \left[\hat{R}_S(w_S) - \hat{R}_{S^{(i)}}(w_{S^{(i)}}) \right]. \quad (4.18)$$

For the first term, by uniform stability:

$$|R(w_S) - R(w_{S^{(i)}})| = |\mathbb{E}_{z \sim \mathcal{D}}[\ell(w_S, z) - \ell(w_{S^{(i)}}, z)]| \leq \beta. \quad (4.19)$$

For the second term, note that \hat{R}_S and $\hat{R}_{S^{(i)}}$ differ in only one term:

$$\hat{R}_S(w_S) - \hat{R}_{S^{(i)}}(w_{S^{(i)}}) = \frac{1}{n} \sum_{j=1}^n \ell(w_S, z_j) - \frac{1}{n} \left[\sum_{j \neq i} \ell(w_{S^{(i)}}, z_j) + \ell(w_{S^{(i)}}, z'_i) \right] \quad (4.20)$$

$$= \frac{1}{n} [\ell(w_S, z_i) - \ell(w_{S^{(i)}}, z'_i)] + \frac{1}{n} \sum_{j \neq i} [\ell(w_S, z_j) - \ell(w_{S^{(i)}}, z_j)]. \quad (4.21)$$

By stability, $|\ell(w_S, z_j) - \ell(w_{S^{(i)}}, z_j)| \leq \beta$ for all j . The first term is bounded by M/n since losses are in $[0, M]$. Therefore,

$$\left| \hat{R}_S(w_S) - \hat{R}_{S^{(i)}}(w_{S^{(i)}}) \right| \leq \frac{M}{n} + \beta. \quad (4.22)$$

Combining both terms:

$$|\Phi(S) - \Phi(S^{(i)})| \leq \beta + \frac{M}{n} + \beta = 2\beta + \frac{M}{n}. \quad (4.23)$$

□

Theorem 4.6. *Under the conditions of Proposition 4.5, for any $\delta > 0$, with probability at least $1 - \delta$:*

$$\Phi(S) \leq \mathbb{E}[\Phi(S)] + \left(2\beta + \frac{M}{n} \right) \sqrt{\frac{n \ln(1/\delta)}{2}}. \quad (4.24)$$

Proof. Apply McDiarmid's inequality (Lemma 4.4) with $c_i = 2\beta + M/n$ for all i :

$$\mathbb{P}[\Phi(S) - \mathbb{E}[\Phi(S)] \geq \epsilon] \leq \exp\left(-\frac{2\epsilon^2}{n(2\beta + M/n)^2}\right). \quad (4.25)$$

Setting the right-hand side equal to δ and solving for ϵ yields:

$$\epsilon = \left(2\beta + \frac{M}{n} \right) \sqrt{\frac{n \ln(1/\delta)}{2}}. \quad (4.26)$$

□

4.4. Generalization Bound. We now combine the previous steps to obtain a complete generalization bound. This result connects the geometric parameters of the loss landscape, the sample size n , and the confidence level δ .

Lemma 4.7. *If the algorithm is β -uniformly stable, then the expected generalization gap satisfies:*

$$\mathbb{E}_S[\text{Gen}(S)] = \mathbb{E}_S[R(w_S) - \hat{R}_S(w_S)] \leq 2\beta. \quad (4.27)$$

Proof. By the linearity of expectation and the fact that training points z_i are drawn from the same distribution as a fresh test point z' , we can write: true risk as $\mathbb{E}_S[R(w_S)] = \mathbb{E}_{S,z'}[\ell(w_S, z')]$ and the empirical risk as $\mathbb{E}_S[\hat{R}_S(w_S)] = \mathbb{E}_S[\frac{1}{n} \sum_{i=1}^n \ell(w_S, z_i)] = \mathbb{E}_S[\ell(w_S, z_i)]$. So, the gap is $\mathbb{E}_{S,z'}[\ell(w_S, z') - \ell(w_S, z_i)]$. We introduce a modified dataset S^i , where the i -th element z_i is replaced by the test point z' . By symmetry, $\mathbb{E}_{S,z'}[\ell(w_S, z_i)]$ is the same as $\mathbb{E}_{S,z'}[\ell(A_{S^i}, z')]$. We substitute this into the gap:

$$\mathbb{E}_{S,z'}[\ell(w_S, z') - \ell(A_{S^i}, z')].$$

The stability uniform stability β tells us that for any point z , the difference in loss when one training point is changed is at most β :

$$|\ell(w_S, z) - \ell(A_{S^i}, z)| \leq \beta.$$

In the derivation, we look at the difference $\ell(w_S, z') - \ell(A_{S^i}, z')$. By algorithmic stability arguments, the expected difference between the empirical risk and the true risk is bounded by twice the stability parameter β . For a detailed derivation, we refer to the framework established in [6]. \square

Theorem 4.8. *Let the learning algorithm minimize empirical risk via gradient flow. Assume the following conditions hold:*

- (1) \hat{R}_S satisfies the Restricted Strong Convexity (RSC) condition with parameter $\mu > 0$.
- (2) \hat{R}_S is L -smooth.
- (3) The loss function $l(w, z)$ is ρ -Lipschitz and bounded in $[0, M]$.

Then, with probability at least $1 - \delta$ over the draw of $S \sim \mathcal{D}^n$, the generalization gap satisfies:

$$R(w_S) \leq \hat{R}_S(w_S) + \mathcal{O}\left(\frac{\rho^2}{\mu n}\right) + \mathcal{O}\left(M\sqrt{\frac{\ln(1/\delta)}{n}}\right). \quad (4.28)$$

Proof. The proof follows by combining the stability result from Corollary 4.3, where $\beta \leq \frac{2\rho^2}{\mu n}$, with Lemma 4.7 and the concentration result Theorem 4.6. We decompose the risk as:

$$R(w_S) = \hat{R}_S(w_S) + \Phi(S). \quad (4.29)$$

Applying McDiarmid's inequality, we bound the deviation of $\Phi(S)$ from its expectation:

$$\Phi(S) \leq \mathbb{E}[\Phi(S)] + \left(2\beta + \frac{M}{n}\right) \sqrt{\frac{n \ln(1/\delta)}{2}}. \quad (4.30)$$

Substituting the expected bound $\mathbb{E}[\Phi(S)] \leq 2\beta$ and the expression for β derived under RSC, we obtain:

$$R(w_S) \leq \hat{R}_S(w_S) + 2\beta + \left(2\beta + \frac{M}{n}\right) \sqrt{\frac{n \ln(1/\delta)}{2}}. \quad (4.31)$$

Substituting $\beta \leq \frac{2\rho^2}{\mu n}$, by simplifying the terms and keeping the dominant orders in n , we arrive at the stated bound. \square

This theorem reveals that generalization is controlled by two distinct factors:

- (1) Stability Term: $\mathcal{O}(1/\mu n)$, which depends on the geometric curvature of the loss landscape (via μ) and the sample size.
- (2) Concentration Term: $\mathcal{O}(\sqrt{\ln(1/\delta)/n})$, which represents standard statistical fluctuations, independent of model complexity.

Crucially, there is *no explicit dependence on model capacity* (e.g., number of parameters d). The bound is determined solely by the dynamical and geometric properties of the optimization process. However, increasing d tends to facilitate the emergence of favorable geometric properties, such as Restricted Strong Convexity, in the vicinity of the global minima manifold. Consequently, the parameter μ may increase or remain stable as the network grows, ensuring that the stability β remains small. This provides a theoretical justification for why larger networks can exhibit superior generalization performance despite their massive capacity.

5. DISCUSSION AND IMPLICATIONS

5.1. Comparison with Classical Learning Theory. Our dynamical stability framework suggests a deep analogy between learning and physical dissipation, in fact the loss $\hat{R}_S(w)$ acts as a potential energy surface. Gradient descent introduces dissipation, causing the system to lose energy (information about noise and outliers). The minima of \hat{R}_S are stable fixed points that attract trajectories. Once the system enters the basin of attraction, it cannot escape. This is the *irreversibility*, that implies the forgetting of initial conditions. Furthermore, this perspective connects to the *information bottleneck principle* [5]: during training, the network first increases mutual information $I(X;T)$ to fit the data, then decreases it to compress representations. This compression phase corresponds to entering the contractive regime of gradient flow.

5.2. When Does the PL Condition Hold? A natural question is: when can we expect neural networks to satisfy the PL condition? *over-parameterized Linear Networks* satisfy PL globally [14]. *Two-Layer ReLU Networks* satisfy PL with high probability when width $m \gg n$ [12]. *Residual Networks* near minima, PL often holds due to the architecture’s smoothness [15]. Notice that the PL condition is not universally satisfied. Early in training or with poor initialization, the loss landscape may have plateaus or saddle points where PL fails. However, our framework still provides insight: generalization improves as the system enters regions where PL approximately holds. Algorithms like Adam or RMSprop modify

the effective loss landscape. Understanding how they affect the PL constant μ is an open problem.

6. CONCLUSION

We have presented a unified theory of generalization based on *dissipative learning dynamics*. This theory posits that generalization is a dynamic result of stable gradient flow trajectories rather than static model capacity. By leveraging the PL condition, the learning process becomes contractive and dissipative, allowing the model to forget noise and implicitly regularize regardless of size. In this framework, over-parameterization is an advantage that improves landscape geometry, shifting the focus of deep learning from what a network can represent to what its physics will find.

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¹ INDEPENDENT RESEARCHER, VIA GNEO NEVIO 8, CAPUA (CE), 81043, ITALY.
Email address: fabriziomorlando84@gmail.com