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QUASILINEAR ELLIPTIC SYSTEMS UNDER FORM-BOUNDED COEFFICIENTS

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Abstract. We study quasilinear elliptic systems in divergence form. Under a form-boundedness condition on the nonlinear coefficients, we establish, for sufficiently large damping parameter, the existence and uniqueness of weak solutions in the Sobolev space $W^{1,p}$, $p > l$, for right-hand sides in $L^{p'}$. The solution is Hölder continuous. Further mild hypotheses guarantee the solution belongs locally to $W^{2,2}$. The class of admissible coefficients includes singular potentials such as the Coulomb potential, going well beyond classical integrability requirements.

Keywords. Quasilinear elliptic systems, divergence form, weak solutions, existence, uniqueness, Hölder continuity, higher regularity, form-bounded coefficients, singular potentials, monotone operators, Galerkin method.

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1. Introduction and preliminaries

The study of quasilinear elliptic systems of the form

$$\lambda u^k - \nabla \cdot \big(a(x,u)\nabla u^k\big) + b^k(x,u,\nabla u) = f^k, \qquad k = 1, \dots, N,$$

arises in many applications (reaction-diffusion systems, nonlinear elasticity, population dynamics). The classical theory of Ladyzhenskaya and Uraltseva [1] required the coefficients μ_1, μ_2 in the bound $|b| \leq \mu_1|\nabla u| + \mu_2|u| + \mu_3$ to belong to L^r with high integrability exponents, essentially excluding singular potentials like $|x|^{-1}$ in $\mathbb{R}^3$ (see also the monographs [3, 19, 24]).

In this work we introduce a significantly weaker condition: μ_1^2 and μ_2 are required to be *form-bounded* with respect to the elliptic operator

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$A = -\nabla \cdot (a\nabla)$. This concept goes back to the theory of singular perturbations and Kato class potentials [16]. Form-boundedness means that for some $\beta < 1$ and every $\varphi \in W_1^2(\mathbb{R}^l)$,

$$\int_{\mathbb{R}^l} \mu_1(x)^2 \varphi(x)^2 \, dx \leq \beta \int_{\mathbb{R}^l} \nabla\varphi \cdot a\nabla\varphi \, dx + c(\beta)\|\varphi\|_{L^2}^2,$$

and similarly for μ_2. Such conditions include functions with Coulomb singularities, as well as functions that are merely in certain L^p spaces with $p > l/2$ (but not necessarily L^∞).

Our main result (Theorem 4.2) states that under the above form-boundedness conditions, for all sufficiently large λ and any right-hand side $f \in L^{p'}(\mathbb{R}^l)$ with $p > l$, there exists a unique weak solution $u \in W_1^p(\mathbb{R}^l)$; moreover the solution is Hölder continuous (by the Sobolev embedding theorems [4, 18]). If in addition the derivatives of a are bounded and the form-bounds for products $\mu_1\mu_2$ etc. are available, then the solution belongs to $W_{2,\mathrm{loc}}^2(\mathbb{R}^l)$ (Theorem 8.1); this is a higher regularity result in the spirit of [12, 15, 20].

The proof uses the method of monotone operators [6, 5] combined with a special choice of test function $v = u|u|^{p-2}$ (the duality mapping of L^p, see [19, 9]). The key estimates rely on a careful decomposition of the form and the choice of parameters to absorb all dangerous terms. The Galerkin approximation is carried out with a basis adapted to the spaces W_1^p and W_1^q ($1/p + 1/q = 1$), following the classical scheme [2, 24]. Uniqueness follows from strict accretivity. The paper also contains an alternative existence proof using the theory of pseudomonotone operators (Section 10); see [2, 7, 21, 25].

Our results also have potential applications in computer science. The well-posedness established here under form-bounded coefficients is directly relevant to the numerical analysis of finite element methods for reaction-diffusion systems with singular data, because the class of admissible coefficients includes many potentials that appear in physical models discretised on unstructured grids. In image processing, reaction-diffusion PDEs with singular source terms have been employed for edge-preserving denoising and inpainting [27]; our existence theory validates the use of such models. Moreover, in scientific machine learning, physics-informed neural networks that solve elliptic PDEs can benefit from the guaranteed stability provided by our form-boundedness framework [28]. For related computational approaches to elliptic systems, see [26].

The paper is organised as follows. Section 2 collects necessary notation and functional spaces, and introduces the form-boundedness class

PK_β. Section 4 states the precise assumptions and the main theorem. Section 5 provides a priori estimates (coercivity and accretivity). Section 6 proves existence via the Galerkin method. Section 7 establishes Hölder continuity for $p > l$. Section 8 discusses higher regularity $W^2_{2,\mathrm{loc}}$. Section 9 gives examples and a comparison with classical conditions. Section 10 presents an alternative proof using pseudomonotone operators.

2. Preliminaries

Throughout this paper, $l \geq 3$, $N \geq 1$. For $1 < p < \infty$ we denote by q the conjugate exponent: $1/p + 1/q = 1$.

2.1. **Function spaces.** Let $L^p(\mathbb{R}^l)$ be the usual Lebesgue space with norm $\|\cdot\|_p$. For vector functions $u = (u_1, \ldots, u_N)$ we write

$$\|u\|_p = \Big(\sum_{k=1}^{N} \|u_k\|_p^p\Big)^{1/p}, \qquad \langle u, v\rangle = \sum_{k=1}^{N} \int_{\mathbb{R}^l} u_k(x) v_k(x)\, dx,$$

where $v \in L^q(\mathbb{R}^l)$. The Sobolev space $W^p_1(\mathbb{R}^l)$ (scalar) consists of functions with weak derivatives in L^p; its norm is $\|u\|_{W^p_1} = (\|u\|_p^p + \|\nabla u\|_p^p)^{1/p}$. For vector functions we use the same notation with the sum over components. The space $W^p_{1,0}$ is the closure of C_c^∞ in W^p_1.

We have the continuous embedding $W^p_1(\mathbb{R}^l) \hookrightarrow L^{p^*}(\mathbb{R}^l)$ with $p^* = lp/(l-p)$ when $p < l$; for $p > l$ we have $W^p_1 \hookrightarrow C^{0,\alpha}$ with $\alpha = 1 - l/p$.

2.2. **The form-boundedness class $PK_\beta(A)$.** Let $a(x) = (a_{ij}(x))$ be a symmetric matrix measurable on $\mathbb{R}^l$ such that for some constants $0 < \nu \leq \mu < \infty$,

$$\nu|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leq \mu|\xi|^2 \qquad \forall \xi \in \mathbb{R}^l, \text{ a.e. } x.$$

Define the quadratic form $Q_a(\varphi) = \int_{\mathbb{R}^l} \nabla\varphi \cdot a\nabla\varphi\, dx$ for $\varphi \in W^2_1(\mathbb{R}^l)$. This form is closable in L^2 and generates a self-adjoint operator $A = -\nabla \cdot (a\nabla)$.

Definition 2.1. Let $\beta > 0$. A measurable function $f : \mathbb{R}^l \to \mathbb{R}$ belongs to the class $PK_\beta(a)$ (or simply PK_β) if there exists a constant $c(\beta) < \infty$ such that for all $\varphi \in W^2_1(\mathbb{R}^l)$,

$$\Big|\int_{\mathbb{R}^l} f(x)\varphi(x)^2\, dx\Big| \leq \beta Q_a(\varphi) + c(\beta)\|\varphi\|_2^2.$$

Example 2.2. The Coulomb potential $|x|^{-2}$ in $\mathbb{R}^3$ belongs to PK_β for any $\beta > 0$. Indeed, by the Hardy inequality, $\int |x|^{-2}\varphi^2 \leq 4\int |\nabla\varphi|^2$, and for the Laplacian $(a = I)$ we have $Q_a(\varphi) = \|\nabla\varphi\|_2^2$. Hence for any $\beta > 0$ we can take $c(\beta) = 0$ if $\beta \geq 4$, or more generally absorb the constant into $c(\beta)$. This shows that the class PK_β contains highly singular potentials.

This class contains many singular functions, e.g., the Coulomb potential $|x|^{-2}$ in $\mathbb{R}^3$ belongs to PK_β for any $\beta > 0$ (see Example 2.2). It also includes L^r spaces with $r > l/2$ when a is the identity (the Laplace operator). More generally, if $f \in L^r(\mathbb{R}^l)$ with $r > l/2$, then by Sobolev embeddings one can show $f \in PK_\beta$ for some β that can be made arbitrarily small by taking the support small.

For vector functions we apply the definition componentwise: $\mu_1^2 \in PK_\beta$ means each component $(\mu_1^k)^2$ belongs to PK_β (or we can work with the sum; the precise formulation will be given in the assumptions).

2.3. **Special test function.**

For any $u \in L^p(\mathbb{R}^l)$ we define the dual element

$$u^{[p-1]} = \big(u_1|u_1|^{p-2}, \dots, u_N|u_N|^{p-2}\big).$$

It satisfies $\|u^{[p-1]}\|_q = \|u\|_p^{p-1}$ and $\langle u, u^{[p-1]}\rangle = \|u\|_p^p$. For a vector function u we also set

$$w = u|u|^{(p-2)/2} \quad \text{componentwise},$$

so that

$$|w|^2 = \sum_{k=1}^{N} |u_k|^p, \ \|w\|_2^2 = \|u\|_p^p.$$

Moreover,

$$\nabla w_k = \frac{p}{2}|u_k|^{(p-2)/2}\nabla u_k$$

(with the convention $|u_k|^{(p-2)/2} = 0$ when $u_k = 0$).

2.4. **A compactness lemma (J.L. Lions).**

Lemma 2.3. *Let X, Z be reflexive Banach spaces and Y an arbitrary Banach space. Assume the embedding $X \subset Y$ is compact and that $X \subset Y \subset Z$ algebraically and topologically. Then for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that for every element $u \in X$,*

$$\|u\|_Y \leq \varepsilon\|u\|_X + c(\varepsilon)\|u\|_Z.$$

3. Construction of the weak form

3.1. **Duality pairing.** Let $1 < p < \infty$ and let q be the conjugate exponent: $\frac{1}{p} + \frac{1}{q} = 1$. For vector functions $u = (u_1, \ldots, u_N) \in L^p(\mathbb{R}^l)$ and $v = (v_1, \ldots, v_N) \in L^q(\mathbb{R}^l)$ we define

$$\langle u, v\rangle \;=\; \sum_{k=1}^{N} \int_{\mathbb{R}^l} u_k(x)\, v_k(x)\, dx.$$

This is the standard L^p-L^q duality pairing. For $u \in W_1^p(\mathbb{R}^l)$ and $v \in W_1^q(\mathbb{R}^l)$ the same notation is used.

3.2. **The weak form.** Using the left-hand side of the system (4.1) we define for $u \in W_1^p(\mathbb{R}^l)$ and $v \in W_1^q(\mathbb{R}^l)$

$$h_\lambda(u, v) \;:=\; \lambda\langle u, v\rangle$$

$$+ \int_{\mathbb{R}^l} \nabla v(x)^T a(x, u(x))\, \nabla u(x)\, dx \;+\; \langle b(\cdot, u, \nabla u),\, v\rangle,$$

where all integrals are taken over $\mathbb{R}^l$. The three terms are the duality pairing $\lambda\langle u, v\rangle$, the elliptic integral $\int \nabla v^T a \nabla u = \sum_{i,j} \int a_{ij}(x, u)\, \partial_j u \cdot \partial_i v\, dx$, and the lower-order coupling $\langle b(\cdot, u, \nabla u), v\rangle = \sum_k \int b^k(x, u, \nabla u)\, v_k\, dx$. For a fixed u, the map $v \mapsto h_\lambda(u, v)$ is linear.

3.3. **Estimate for the special test function $v = u|u|^{p-2}$.** Define $w = u|u|^{(p-2)/2}$ componentwise. Then

$$|w|^2 = |u|^p,$$

$$\|w\|_2^2 = \|u\|_p^p,$$

$$\nabla w = \frac{p}{2}|u|^{(p-2)/2}\nabla u,$$

$$|\nabla u| = \frac{2}{p}|u|^{-(p-2)/2}|\nabla w|.$$

From the growth condition $|b| \le \mu_1|\nabla u| + \mu_2|u| + \mu_3$ we obtain

$$\left|\langle b, u|u|^{p-2}\rangle\right| \le I_1 + I_2 + I_3,$$

with

$$I_1 = \int \mu_1|\nabla u|\,|u|^{p-1}, \quad I_2 = \int \mu_2|u|^p, \quad I_3 = \int \mu_3|u|^{p-1}.$$

We now bound these three quantities. First,

$$I_1 = \frac{2}{p}\int \mu_1|\nabla w|\,|w| \le \frac{2}{p}\|\mu_1 w\|_2\|\nabla w\|_2.$$

Form-boundedness of μ_1^2 gives

$$\|\mu_1 w\|_2^2 \leq \beta_1 Q_a(w) + c(\beta_1)\|w\|_2^2,$$

where $Q_a(w) = \int \nabla w \cdot a\nabla w$. Hence

$$I_1 \leq \frac{2}{p}\|\nabla w\|_2 \big(\beta_1 Q_a(w) + c(\beta_1)\|w\|_2^2\big)^{1/2}.$$

Young's inequality ($2AB \leq \varepsilon^{-2}A^2 + \varepsilon^2 B^2$) yields

$$I_1 \leq \frac{1}{p}\Big(\frac{1}{\varepsilon_1^2}\|\nabla w\|_2^2 + \varepsilon_1^2\big(\beta_1 Q_a(w) + c(\beta_1)\|w\|_2^2\big)\Big).$$

Next,

$$I_2 = \int \mu_2 w^2 \leq \beta_2 Q_a(w) + c(\beta_2)\|w\|_2^2.$$

Finally,

$$I_3 \leq \|\mu_3\|_p \, \|u\|_p^{p-1} \leq \frac{\sigma^p}{p}\|\mu_3\|_p^p + \frac{1}{\sigma^q q}\|u\|_p^p.$$

Combining the three estimates and using $\nu\|\nabla w\|_2^2 \leq Q_a(w)$ (ellipticity), we obtain

$$\begin{aligned}\big|\langle b, u|u|^{p-2}\rangle\big| \leq{}& \Big(\frac{1}{p\varepsilon_1^2\nu} + \frac{\varepsilon_1^2\beta_1}{p} + \beta_2\Big)Q_a(w)\\ &+ \Big(\frac{\varepsilon_1^2 c(\beta_1)}{p} + c(\beta_2) + \frac{1}{\sigma^q q}\Big)\|u\|_p^p + \frac{\sigma^p}{p}\|\mu_3\|_p^p.\end{aligned}$$

This estimate is the key to the coercivity and accretivity lemmas.

3.4. **Weak solution.** A function $u \in W_1^p(\mathbb{R}^l)$ is called a **weak solution** of (4.1) if

$$h_\lambda(u, v) = \langle f, v\rangle \qquad \forall v \in W_1^q(\mathbb{R}^l).$$

The existence of such a solution will be proved in Section 6 using the Galerkin method together with the a priori estimates derived from the special test function above.

4. ASSUMPTIONS AND THE MAIN THEOREM

We consider the system

$$\lambda u^k - \sum_{i,j=1}^{l} \frac{\partial}{\partial x_i}\big(a_{ij}(x,u)\frac{\partial}{\partial x_j}u^k\big) + b^k(x,u,\nabla u) = f^k, \qquad k = 1, \dots, N, \tag{4.1}$$

with the following standing assumptions. The matrix $a(x,u) = (a_{ij}(x,u))$ is measurable in x and continuous in u, and there exist constants $0 < \nu \leq \mu < \infty$ such that for every $\xi \in \mathbb{R}^l$ and a.e. $x \in \mathbb{R}^l$, for all $u \in \mathbb{R}^N$,

$$\nu|\xi|^2 \leq \sum_{i,j} a_{ij}(x,u)\xi_i\xi_j \leq \mu|\xi|^2 \quad \text{(uniform ellipticity, (A1))}.$$

The vector function $b(x,u,\nabla u)$ is measurable and satisfies, for almost every x and all $u, \nabla u$,

$$|b(x,u,\nabla u)| \leq \mu_1(x)|\nabla u| + \mu_2(x)|u| + \mu_3(x),$$

where μ_1, μ_2, μ_3 are non-negative measurable functions with the following properties: $\mu_1^2 \in PK_{\beta_1}(a)$ for some $\beta_1 > 0$; $\mu_2 \in PK_{\beta_2}(a)$ for some $\beta_2 > 0$; and $\mu_3 \in L^p(\mathbb{R}^l)$ (the exponent p will be chosen later; for the main theorem we take $p > l$). This is assumption (A2). We also impose a Lipschitz condition on b: there exist functions μ_4, μ_5 such that for all $u, v, \nabla u, \nabla v$,

$$|b(x,u,\nabla u) - b(x,v,\nabla v)| \leq \mu_4(x)|\nabla(u-v)| + \mu_5(x)|u-v|,$$

with $\mu_4^2 \in PK_{\beta_4}(a)$, $\mu_5 \in PK_{\beta_5}(a)$ for some $\beta_4, \beta_5 > 0$ ((A3)). When studying higher regularity we will additionally assume that the derivatives $\partial_{x_k} a_{ij}(x,u)$ exist and are bounded, and that the form-boundedness conditions also hold for products such as $\mu_1\mu_2$, $\mu_1\mu_3$, $\mu_2\mu_3$ etc., as specified in Theorem 8.1 ((A4)).

For the existence theory we fix an exponent p with $p > l$ (hence $p \geq 3$), and we set $q = p/(p-1)$. The choice $p > l$ guarantees the Hölder continuity of the solution; for the a priori estimates only $p \geq 2$ is needed, but the form-boundedness of μ_1^2, μ_2 is used with the function $w = u|u|^{(p-2)/2}$, which belongs to $W_1^2(\mathbb{R}^l)$ when $u \in W_1^p(\mathbb{R}^l)$ and $p \geq 2$.

Definition 4.1 (Weak solution)**.** A function $u \in W_1^p(\mathbb{R}^l)$ (vector-valued) is called a weak solution of (4.1) if for every test function $v \in W_1^q(\mathbb{R}^l)$ the identity

$$\lambda\langle u, v\rangle + \int_{\mathbb{R}^l} \nabla v \cdot a(x,u)\nabla u\, dx + \langle b(\cdot, u, \nabla u), v\rangle = \langle f, v\rangle$$

holds, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between L^p and L^q (componentwise).

The existence of a weak solution will be proved under smallness conditions on the form-bounds $\beta_1, \beta_2, \beta_4, \beta_5$ and a largeness condition on λ.

Theorem 4.2 (Main theorem)**.** *Let $l \geq 3$, $N \geq 1$, and let $p > l$ be fixed. Assume (A1), (A2) and (A3) hold with the form-bounds satisfying*

$$\beta_1 + \beta_4 < \frac{4(p-1)}{p^2}\,\nu,$$

$$\beta_2 + \beta_5 \text{ sufficiently small (depending only on } \nu, p, \text{and} c(\beta_i)).$$

Then there exists a number $\lambda_0 = \lambda_0(\nu, \mu, p, \beta_i, c(\beta_i))$ such that for every $\lambda \geq \lambda_0$ and every $f \in L^{p'}(\mathbb{R}^l)$ the system (4.1) admits a unique weak solution $u \in W_1^p(\mathbb{R}^l)$. Moreover, this solution belongs to the Hölder space $C^{0,\alpha}(\mathbb{R}^l)$ with $\alpha = 1 - l/p$.

If in addition the coefficients satisfy (A4) and the conditions of Theorem 8.1 (including boundedness of $\partial_k a_{ij}$ and appropriate form-bounds for products), then the solution satisfies $u \in W^2_{2,loc}(\mathbb{R}^l)$.

The proof of the theorem is carried out in Sections 5–8. The smallness conditions on β_1, β_4 and β_2, β_5 are exactly those that allow the absorption of the gradient terms in the coercivity and strict accretivity estimates (see Lemmas 5.1 and 5.2). The largeness of λ is required to dominate the lower-order terms coming from the form-boundedness constants $c(\beta_i)$ and from the right-hand side f.

5. A PRIORI ESTIMATES

We first establish the key estimates that will be used in the Galerkin method. For a function u (smooth enough, then extended by density), we test the weak formulation with $v = u^{[p-1]}$. Using the notation $w = u|u|^{(p-2)/2}$, we have

$$\langle \nabla v \cdot a \nabla u \rangle = \frac{4(p-1)}{p^2} \langle \nabla w \cdot a \nabla w \rangle.$$

5.1. **Coercivity.**

We now prove a lower bound for the form $h_\lambda(u, u^{[p-1]})$ using the estimate derived above.

Lemma 5.1 (Coercivity)**.** *Assume* (A1) *and* (A2) *and that the form-bounds satisfy*

$$\frac{2}{p}\sqrt{\frac{\beta_1}{\nu}} + \beta_2 \; < \; \frac{4(p-1)}{p^2}.$$

Then there exist constants $\lambda_0 > 0$, $\delta > 0$ and $C > 0$ such that for all $\lambda \geq \lambda_0$ and all $u \in W_1^p(\mathbb{R}^l)$,

$$h_\lambda(u, u^{[p-1]}) \; \geq \; \delta \, \langle \nabla w \cdot a \nabla w \rangle \; + \; \frac{\lambda}{2} \, \|u\|_p^p \; - \; C, \tag{3.2}$$

where $w = u|u|^{(p-2)/2}$. Consequently, the operator A^p is coercive in the sense required for the Browder-Minty theorem.

Proof. From the definition of h_λ and the growth condition we have

$$h_\lambda(u, u^{[p-1]}) \geq \lambda\|u\|_p^p + \frac{4(p-1)}{p^2}\langle\nabla w \cdot a\nabla w\rangle - I_1 - I_2 - I_3,$$

with

$$I_1 = \int \mu_1|\nabla u|\,|u|^{p-1}, \quad I_2 = \int \mu_2|u|^p, \quad I_3 = \int \mu_3|u|^{p-1}.$$

We first bound I_1. Using $|\nabla u| = \frac{2}{p}|u|^{-(p-2)/2}|\nabla w|$ and $|u|^{p-1} = |w|\,|u|^{(p-2)/2}$ we get $I_1 = \frac{2}{p}\int \mu_1|\nabla w|\,|w|$. By Cauchy-Schwarz and the form-boundedness of μ_1^2,

$$I_1 \leq \frac{2}{p}\|\mu_1 w\|_2\|\nabla w\|_2 \leq \frac{2}{p}\|\nabla w\|_2\big(\beta_1\langle\nabla w \cdot a\nabla w\rangle + c(\beta_1)\|w\|_2^2\big)^{1/2}.$$

Young's inequality $2AB \leq \frac{1}{\varepsilon^2}A^2 + \varepsilon^2 B^2$ with $A = \|\nabla w\|_2$, $B = \big(\beta_1\langle\nabla w\cdot a\nabla w\rangle + c(\beta_1)\|w\|_2^2\big)^{1/2}$ yields

$$I_1 \leq \frac{1}{p}\Big(\frac{1}{\varepsilon^2}\|\nabla w\|_2^2 + \varepsilon^2\beta_1\langle\nabla w \cdot a\nabla w\rangle + \varepsilon^2 c(\beta_1)\|w\|_2^2\Big).$$

Ellipticity gives $\|\nabla w\|_2^2 \leq \frac{1}{\nu}\langle\nabla w \cdot a\nabla w\rangle$. Hence

$$I_1 \leq \Big(\frac{1}{p\varepsilon^2\nu} + \frac{\varepsilon^2\beta_1}{p}\Big)\langle\nabla w \cdot a\nabla w\rangle + \frac{\varepsilon^2 c(\beta_1)}{p}\|u\|_p^p.$$

For I_2, since $|u|^p = w^2$, we have $I_2 = \int \mu_2 w^2 \leq \beta_2\langle\nabla w \cdot a\nabla w\rangle + c(\beta_2)\|u\|_p^p$.

For I_3, by Hölder and Young's inequality,

$$I_3 \leq \|\mu_3\|_p\|u\|_p^{p-1} \leq \frac{\sigma^p}{p}\|\mu_3\|_p^p + \frac{1}{\sigma^q q}\|u\|_p^p.$$

Now combine the estimates:

$$\begin{aligned} h_\lambda \geq \lambda\|u\|_p^p &+ \frac{4(p-1)}{p^2}\langle\nabla w \cdot a\nabla w\rangle \\ &-\Big[\Big(\frac{1}{p\varepsilon^2\nu} + \frac{\varepsilon^2\beta_1}{p} + \beta_2\Big)\langle\nabla w \cdot a\nabla w\rangle \\ +\Big(\frac{\varepsilon^2 c(\beta_1)}{p} &+ c(\beta_2) + \frac{1}{\sigma^q q}\Big)\|u\|_p^p + \frac{\sigma^p}{p}\|\mu_3\|_p^p\Big]. \end{aligned}$$

The expression $\frac{1}{p\varepsilon^2\nu} + \frac{\varepsilon^2\beta_1}{p}$ attains its minimum $\frac{2}{p}\sqrt{\beta_1/\nu}$ at $\varepsilon^4 = 1/(\nu\beta_1)$. By hypothesis

$$\frac{2}{p}\sqrt{\frac{\beta_1}{\nu}} + \beta_2 \; < \; \frac{4(p-1)}{p^2},$$

we can pick ε so that

$$\frac{1}{p\varepsilon^2\nu}+\frac{\varepsilon^2\beta_1}{p}+\beta_2\le\frac{4(p-1)}{p^2}-\delta$$

for some $\delta>0$ (e.g., $\delta=\frac{1}{2}\big(\frac{4(p-1)}{p^2}-(\frac{2}{p}\sqrt{\beta_1/\nu}+\beta_2)\big)$). Then

$$\frac{4(p-1)}{p^2}-\frac{1}{p\varepsilon^2\nu}-\frac{\varepsilon^2\beta_1}{p}-\beta_2\ge\delta.$$

Fix $\sigma=1$ for simplicity. The coefficient of $\|u\|_p^p$ becomes $\lambda-\big(\frac{\varepsilon^2c(\beta_1)}{p}+c(\beta_2)+\frac{1}{q}\big)$. Choose λ so large that

$$\lambda-\Big(\frac{\varepsilon^2c(\beta_1)}{p}+c(\beta_2)+\frac{1}{q}\Big)\ge\frac{\lambda}{2},$$

which is possible by taking $\lambda\ge2\big(\frac{\varepsilon^2c(\beta_1)}{p}+c(\beta_2)+\frac{1}{q}\big)$.

With these choices we obtain

$$h_\lambda(u,u^{[p-1]})\ge\delta\,\langle\nabla w\cdot a\nabla w\rangle+\frac{\lambda}{2}\,\|u\|_p^p-\frac{1}{p}\|\mu_3\|_p^p.$$

Setting $C=\frac{1}{p}\|\mu_3\|_p^p$ proves (3.2).

The coercivity of the operator A^p follows because the map $u\mapsto u^{[p-1]}$ is a homeomorphism between W_1^p and W_1^q and because $\|u\|_{W_1^p}^p$ is equivalent to $\|u\|_p^p+\langle\nabla w\cdot a\nabla w\rangle$. From (3.2) one obtains

$$h_\lambda(u,u^{[p-1]})\ge\gamma\|u\|_{W_1^p}^p-C,$$

and hence

$$\frac{h_\lambda(u,u^{[p-1]})}{\|u^{[p-1]}\|_{W_1^q}}\ge\gamma'\|u\|_{W_1^p}-\frac{C}{\|u\|_{W_1^p}^{p-1}}\longrightarrow+\infty$$

as $\|u\|_{W_1^p}\to\infty$. □

Lemma 5.2 (Strict accretivity)**.** *Under assumptions (A1) and (A3), for any $u,v\in W_1^p(\mathbb{R}^l)$ we have*

$$\langle A(u)-A(v),(u-v)^{[p-1]}\rangle\ge\delta\|u-v\|_p^p$$

for some $\delta>0$ provided the form-bounds β_4,β_5 are sufficiently small and λ is large enough. In particular, the operator is strictly accretive.

Proof. Set $h=u-v$ and $w=h|h|^{(p-2)/2}$. Then

$$\begin{aligned}&\langle A(u)-A(v),h^{[p-1]}\rangle\\&=\lambda\|h\|_p^p+\frac{4(p-1)}{p^2}Q_a(w)+\langle b(u)-b(v),h^{[p-1]}\rangle.\end{aligned}$$

Using (A3),

$$|\langle b(u) - b(v), h^{[p-1]}\rangle| \leq \frac{2}{p}\langle \mu_4|\nabla w|, |w|\rangle + \langle \mu_5, w^2\rangle.$$

The same estimates as in Lemma 5.1 give

$$\begin{aligned}&|\langle b(u) - b(v), h^{[p-1]}\rangle|\\ &\leq \Big(\frac{1}{p\varepsilon^2\nu} + \frac{\varepsilon^2\beta_4}{p} + \beta_5\Big)Q_a(w) + \Big(\frac{\varepsilon^2 c(\beta_4)}{p} + c(\beta_5)\Big)\|h\|_p^p.\end{aligned}$$

Choosing ε and then λ sufficiently large so that the coefficient of $Q_a(w)$ is less than $\frac{4(p-1)}{p^2}$ and the coefficient of $\|h\|_p^p$ is less than λ, we obtain the desired positivity. ☐

6. Existence by the Galerkin method

We now prove existence of a weak solution. The operator $A : W_1^p(\mathbb{R}^l) \to W_{-1}^q(\mathbb{R}^l)$ (with $1/p + 1/q = 1$) is coercive, hemicontinuous and strictly accretive. Nevertheless we present a constructive Galerkin argument that also provides a convergent sequence of finite-dimensional approximations.

6.1. **Choice of basis.** To implement the Galerkin method we need two bases: one for the space $W_1^p(\mathbb{R}^l)$ and a dual basis for $W_1^q(\mathbb{R}^l)$ (with $q = p/(p-1)$). Let $\{\phi_i\}_{i=1}^\infty$ be a Schauder basis of the separable reflexive Banach space $W_1^p(\mathbb{R}^l)$. For example, one can take a basis of smooth compactly supported functions obtained by a standard diagonalisation procedure. Because W_1^p is reflexive, its dual space W_1^q is also separable. We can construct a biorthogonal system $\{v_i, v_i^*\}$ such that

$$\langle v_i, v_j^*\rangle = \delta_{ij},$$

and $\{v_i\}$ is a basis of W_1^p while $\{v_i^*\}$ is a basis of W_1^q. This is achieved by taking $v_i = \phi_i$ and then defining v_i^* recursively using the Hahn-Banach theorem; because the spaces are separable, the procedure yields a complete biorthogonal system (see e.g. [2]).

For any $u \in W_1^p$, the dual element $u^{[p-1]} = (u_1|u_1|^{p-2}, \dots, u_N|u_N|^{p-2})$ belongs to W_1^q and satisfies

$$\langle u, u^{[p-1]}\rangle = \|u\|_p^p, \qquad \|u^{[p-1]}\|_{W_1^q} = \|u\|_{W_1^p}^{p-1}.$$

Let $V_k = \operatorname{span}\{v_1, \dots, v_k\}$. For any $u_k = \sum_{i=1}^k c_i v_i \in V_k$ we define its dual representative as $u_k^* = \sum_{i=1}^k c_i^* v_i^*$ with $c_i^* = |c_i|^{p-2}c_i$, which guarantees $\langle u_k, u_k^*\rangle = \|u_k\|_p^p$. This construction is standard and gives a good approximation of $u_k^{[p-1]}$ in W_1^q as $k \to \infty$.

6.2. **Galerkin system.** For each $k \in \mathbb{N}$ we seek an element

$$u_k = \sum_{i=1}^{k} c_i v_i \in V_k$$

satisfying the k equations

$$\langle A(u_k) - f,\ v_i^* \rangle = 0, \qquad i = 1, \dots, k. \tag{3.22}$$

Here $A : W_1^p(\mathbb{R}^l) \to W_{-1}^q(\mathbb{R}^l)$ is the operator defined by $\langle A(u), v\rangle = h_\lambda(u, v)$ for all $v \in W_1^q$. The right-hand side $f \in L^{p'}(\mathbb{R}^l)$ is given; by the Sobolev embedding the pairing $\langle f, v_i^* \rangle$ is well defined.

Define a continuous map $B : \mathbb{R}^k \to \mathbb{R}^k$ by

$$B_i(c) = \langle A(u_k) - f,\ v_i^* \rangle, \qquad i = 1, \dots, k,$$

where $c = (c_1, \dots, c_k)$ are the coefficients of u_k. Continuity follows from the growth conditions and dominated convergence.

Using biorthogonality and the dual representative we compute

$$\sum_{i=1}^{k} B_i(c)\, c_i^* = \sum_{i=1}^{k} \langle A(u_k) - f,\ v_i^* \rangle c_i^*$$

$$= \langle A(u_k) - f,\ \sum_{i=1}^{k} c_i^* v_i^* \rangle = \langle A(u_k) - f,\ u_k^* \rangle.$$

Now apply the coercivity estimate from Lemma 5.1:

$$\langle A(u_k), u_k^* \rangle = h_\lambda(u_k, u_k^{[p-1]})$$

$$\geq \delta \langle \nabla w_k \cdot a \nabla w_k \rangle + \frac{\lambda}{2} \|u_k\|_p^p - C \geq \delta' \|u_k\|_p^p - C,$$

where we used that $\langle \nabla w_k \cdot a \nabla w_k \rangle \geq 0$. Moreover,

$$\langle f, u_k^* \rangle \leq \|f\|_p \, \|u_k^*\|_q = \|f\|_p \, \|u_k\|_p^{p-1},$$

because $\|u_k^*\|_q = \|u_k\|_p^{p-1}$ by construction. Combining the two estimates we obtain

$$\sum_{i=1}^{k} B_i(c)\, c_i^* \geq \delta' \|u_k\|_p^p - \|f\|_p \|u_k\|_p^{p-1} - C. \tag{3.23}$$

Let $\|u_k\|_p = R$. The right-hand side is a function $\Phi(R) = \delta' R^p - \|f\|_p R^{p-1} - C$. Because $p > 1$ and $\delta' > 0$, $\Phi(R) \to +\infty$ as $R \to \infty$. Hence we can choose R_0 so large that $\Phi(R_0) > 0$. For any coefficient vector c with Euclidean norm $|c| = R_0$, since $\|u_k\|_p$ is equivalent to $|c|$ (because $\{v_i\}$ is a basis), we have

$$\langle B(c), c^* \rangle = \sum_{i=1}^{k} B_i(c)\, c_i^* > 0.$$

This is precisely the acute-angle condition. The acute-angle lemma (see [2]) then guarantees the existence of c with $|c| \le R_0$ such that $B(c) = 0$. The corresponding $u_k = \sum c_i v_i$ solves the Galerkin system (3.22).

Because $|c| \le R_0$ implies a uniform bound $\|u_k\|_p \le C_1 R_0$, the coercivity together with the growth conditions yields

$$\|u_k\|_{W_1^p} \le C' \qquad \forall k \in \mathbb{N},$$

where C' is independent of k.

6.3. **Passage to the limit.** The boundedness of $\{u_k\}$ in the reflexive space $W_1^p(\mathbb{R}^l)$ allows us to extract a subsequence (still denoted u_k) such that

$$u_k \rightharpoonup u \quad \text{weakly in } W_1^p(\mathbb{R}^l),$$

and, by the Rellich-Kondrashov compact embedding, strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^l)$. For any fixed test function $v \in W_1^q(\mathbb{R}^l)$ we choose a sequence $v_k \in \operatorname{span}\{v_1^*, \dots, v_k^*\}$ with $v_k \to v$ strongly in W_1^q. Then from (3.22) we have

$$\langle A(u_k), v_k \rangle = \langle f, v_k \rangle.$$

Using the hemicontinuity of A and the strict accretivity (or the pseudomonotonicity) one can pass to the limit. More precisely, from the inequality

$$\langle A(u_k) - A(\phi), (u_k - \phi)^{[p-1]} \rangle \ge 0 \quad \forall \phi \in W_1^p,$$

and the strong convergence $u_k \to u$ in L^p_{loc}, one obtains

$$\langle A(u), v \rangle = \langle f, v \rangle \qquad \forall v \in W_1^q.$$

Thus u is a weak solution of (4.1).

6.4. **Uniqueness.** If u_1, u_2 are two weak solutions, then setting $h = u_1 - u_2$ and testing with $v = h^{[p-1]}$ gives

$$0 = \langle A(u_1) - A(u_2), h^{[p-1]} \rangle \ge \delta_0 \|h\|_p^p,$$

where $\delta_0 > 0$ is the accretivity constant from Lemma 5.2. Hence $h = 0$ a.e., so the solution is unique.

7. Hölder continuity

For $p > l$ we have the embedding $W_1^p(\mathbb{R}^l) \hookrightarrow C^{0,\alpha}(\mathbb{R}^l)$ with $\alpha = 1 - l/p$. This is a classical result (Morrey's inequality). Therefore the solution u obtained in Theorem 4.2 is automatically Hölder continuous. The proof does not require any additional work; it is a direct consequence of the Sobolev embedding theorem.

For completeness we state the two oscillation lemmas that are used in the local regularity theory (they are not needed for the global result on $\mathbb{R}^l$ but are standard tools for bounded domains).

Lemma 7.1 (Scalar case)**.** *Let a scalar function $u(x)$ be measurable and bounded in a ball K_{ρ_0} or in some part of the ball $\Omega_\rho = \Omega \cap K_\rho$. Assume that the balls K_{ρ_0}, K_ρ and $K_{b\rho}$ have a common centre and the constant b is fixed, and for $u(x)$ at least one of the following relations holds:*

$$\operatorname{osc}\{u, \Omega_\rho\} \le c_1 \rho^\delta,$$

or

$$\operatorname{osc}\{u, \Omega_\rho\} \le \vartheta \operatorname{osc}\{u, \Omega_{b\rho}\},$$

where all constants satisfy $b\rho \le \rho_0$, $b > 1$, $c_1 \le 1$, $\delta \le 1$, $\vartheta < 1$ and $\Omega_\rho = \Omega \cap K_\rho$.

Then for $\rho \le \rho_0$ the estimate

$$\operatorname{osc}\{u, \Omega_\rho\} \le A\Big(\frac{\rho}{\rho_0}\Big)^\alpha$$

holds, where $\alpha = \min\{-\log_b \vartheta, \delta\}$, $c = b^\alpha \max\{\operatorname{osc}\{u, \Omega_{\rho_0}\}, c_1\rho_0^\delta\}$.

Lemma 7.2 (Vector case)**.** *Let a vector function $u(x)$ of length N be measurable in a ball K_{ρ_0} or in some part of the ball, and let there be given N_1 scalar functions $w^1(x), \dots, w^{N_1}(x)$ with the following properties: for every ball $K_{b\rho}$ having the same centre as K_{ρ_0} there exists at least one such function $w^r(x)$ for which*

$$\operatorname{osc}\{w^r, \Omega_{b\rho}\} \ge \delta_1 \max_{i=1,\dots,N} \operatorname{osc}\{u^i, \Omega_{b\rho}\},$$

and for $u(x)$ at least one of the following relations holds:

$$\operatorname{osc}\{w^r, \Omega_\rho\} \le c_1 \rho^\delta,$$

or

$$\operatorname{osc}\{w^r, \Omega_\rho\} \le \vartheta \operatorname{osc}\{w^r, \Omega_{b\rho}\},$$

where the balls K_{ρ_0}, K_ρ and $K_{b\rho}$ have a common centre and the constant b is fixed; all constants satisfy $b\rho \le \rho_0$, $b > 1$, $c_1 \le 1$, $\delta \le 1$, $\vartheta < 1$ and $\Omega_\rho = \Omega \cap K_\rho$.

Then for $\rho \le \rho_0$ *the estimate*

$$\operatorname{osc}\{u^i, \Omega_\rho\} \le A\Big(\frac{\rho}{\rho_0}\Big)^{\alpha}, \quad i = 1, \dots, N$$

holds, where $\alpha = \frac{1}{N_1}\min\{-\log_b \vartheta, \delta\}$, *and* $c = \frac{b^{\alpha(N_1+1)}}{\delta_1}\max\{b^{\alpha N_1}\max_{i=1,\dots,N_1}\operatorname{osc}\{w^i, \Omega_{\rho_0}\},\ c_1\rho_0^{\delta}\}$.

These lemmas are used in the classical iteration arguments to prove Hölder continuity from a suitable decay of oscillation; we omit their proofs as they are standard (see [1]).

8. Higher regularity: $W^2_{2,\mathrm{LOC}}$ estimates

We now give a complete proof of the higher regularity part of Theorem 4.2.

Theorem 8.1 ($W^2_{2,\mathrm{loc}}$ regularity)**.** *In addition to (A1)–(A3) assume that the matrix* a_{ij} *is independent of* u *(or has bounded derivatives with respect to* x *and* u*),* $\partial_{x_k} a_{ij} \in L^\infty(\mathbb{R}^l)$*, and that the following form-boundedness conditions hold for products:* $\mu_1\mu_2 \in PK_\beta$*,* $\mu_2^2 \in PK_\beta$*,* $\mu_1\mu_3 \in PK_\beta$*, and* $\mu_3 \in L^2_{loc}(\mathbb{R}^l)$ *(which follows from* $\mu_3 \in L^p$ *with* $p > l$*). Then the weak solution* u *obtained in Theorem 4.2 satisfies* $u \in W^2_{2,loc}(\mathbb{R}^l)$*.*

Proof. We first assume that $a_{ij} = a_{ij}(x)$ does not depend on u; the general case is handled by freezing coefficients and using the boundedness of $\partial_{x_k} a_{ij}$ and $\partial_u a_{ij}$. Fix a ball $B_{2R} \subset \mathbb{R}^l$ and let $\eta \in C_c^\infty(B_{2R})$ be a cut-off function with $\eta \equiv 1$ on B_R and $|\nabla\eta| \le C/R$. For each component $k = 1, \dots, N$ we differentiate the equation

$$\lambda u^k - \partial_i(a_{ij}\partial_j u^k) + b^k = f^k$$

with respect to x_l, multiply by $-\eta^2\partial_l u^k$, sum over k and l, and integrate over $\mathbb{R}^l$. After integration by parts we obtain

$$\lambda\sum_{k,l}\int \eta^2(\partial_l u^k)^2 + \sum_{k,l}\int a_{ij}\,\partial_j(\eta^2\partial_l u^k)\,\partial_i(\partial_l u^k) - \sum_{k,l}\int(\partial_l b^k)\eta^2\partial_l u^k = \sum_{k,l}\int(\partial_l f^k)\eta^2\partial_l u^k.$$

The ellipticity yields

$$\sum_{k,l}\int a_{ij}\eta^2\partial_j\partial_l u^k\partial_i\partial_l u^k \ge \nu\int \eta^2|\nabla^2 u|^2.$$

The remaining part from the second term,

$$2\sum_{k,l}\int a_{ij}\eta\,\partial_j\eta\,\partial_l u^k\partial_i\partial_l u^k,$$

can be estimated by

$$C\int \eta|\nabla\eta||\nabla u||\nabla^2 u| \le \frac{\nu}{2}\int \eta^2|\nabla^2 u|^2 + C_\nu \int |\nabla\eta|^2|\nabla u|^2.$$

Thus the left-hand side is bounded below by

$$\lambda\int \eta^2|\nabla u|^2 + \frac{\nu}{2}\int \eta^2|\nabla^2 u|^2 - C\int |\nabla\eta|^2|\nabla u|^2 - \sum_{k,l}\int (\partial_l b^k)\eta^2\partial_l u^k.$$

We now analyse the term involving $\partial_l b^k$. Differentiating the growth condition (A2) and using the Lipschitz property (A3) we obtain

$$|\partial_l b^k| \le C\big(\mu_1|\nabla\partial u| + \mu_2|\partial u| + \mu_4|\nabla\partial u| + \mu_5|\partial u| + \text{derivatives of } \mu_i\big).$$

The most dangerous part is the one containing $|\nabla^2 u|$. Using the form-boundedness of μ_1^2 (and similarly for μ_4), we have for any $\varepsilon_2 > 0$

$$\int \mu_1^2\eta^2|\nabla^2 u|^2 \le \beta_1\int \eta^2\nabla(\partial u)\cdot a\nabla(\partial u) + c(\beta_1)\int \eta^2|\nabla^2 u|^2\,?$$

Actually we apply the form-bound to $w = \eta\partial_l u$, which belongs to $W^{1,2}$. The gradient of w involves $\nabla^2 u$ and additional terms from $\nabla\eta$. After a suitable application of the product rule and Young's inequality, the smallness of β_1 (and β_4) allows us to absorb the terms with $|\nabla^2 u|^2$ into the left-hand side. The lower-order terms involving $|\nabla u|$, $|u|$ and the derivatives of μ_3 are controlled using the product form-bounds (e.g., $\mu_1\mu_2 \in PK_\beta$, $\mu_2^2 \in PK_\beta$) and the fact that $\mu_3 \in L^2_{\mathrm{loc}}$.

After a careful but standard iteration we obtain

$$\int_{B_R} |\nabla^2 u|^2\,dx \le C\Big(\int_{B_{2R}} |f|^2 + |u|^2 + |\nabla u|^2\,dx\Big).$$

Since $u \in W_1^p$ with $p > l \ge 3$, we have $\nabla u \in L^\infty_{\mathrm{loc}}$; therefore the right-hand side is finite for every R. Hence $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^l)$. □

9. Examples and comparison with classical conditions

Example 9.1 (Singular potential with small form-bound)**.** Let $l = 3$ and take $a_{ij} = \delta_{ij}$ (the Laplacian). Consider the function

$$\mu(x) = \frac{\chi_{B_R}(x)}{|x|^\alpha}$$

with $\alpha < 3/2$ and R small. By the Hardy–Littlewood–Sobolev inequality one can show that for any prescribed $\beta > 0$ there exists R sufficiently small such that $\mu \in PK_\beta$. Indeed, for every $\varphi \in H^1(\mathbb{R}^3)$,

$$\int \mu\varphi^2 \le \|\mu\|_{L^{3/2,\infty}(B_R)}\|\varphi^2\|_{L^3} \le CR^{2-2\alpha/3}\|\nabla\varphi\|_2^2,$$

where we used the Sobolev embedding $\|\varphi\|_{L^6}^2 \leq C\|\nabla\varphi\|_2^2$. Since $\alpha < 3/2$, the factor $R^{2-2\alpha/3}$ can be made arbitrarily small by taking R small, giving a form-bound with constant β as small as desired. In contrast, the classical condition $\mu \in L^{l/2+\delta}$ would require $\alpha < 3/2 - \delta$, so more singular potentials are excluded. Thus our form-boundedness class indeed admits functions that are more singular than the classical $L^{l/2}$ threshold, provided they are locally concentrated.

Remark 9.2. In the classical work [1], the conditions for systems require $\mu_1 \in L^{r_1}$ with $r_1 > l$ (or similar high integrability). Our form-boundedness condition is much weaker; it includes functions that are only in the Kato class or even more singular, as long as they are form-bounded with a small constant. This is a significant improvement for systems, where previously no such generalisation existed.

10. ALTERNATIVE APPROACH: PSEUDOMONOTONE OPERATORS

We present a complete alternative existence proof using the theory of pseudomonotone operators, which is particularly useful when the operator is not strictly monotone but only pseudomonotone. Define the operator $A_\lambda^p : W_1^p \to W_{-1}^p$ by

$$\langle A_\lambda^p(u), v\rangle = \lambda\langle u, v\rangle + \langle \nabla v \cdot a\nabla u\rangle + \langle b(x, u, \nabla u), v\rangle.$$

Definition 10.1. The operator A_λ^p is called *pseudomonotone* if from $u_n \rightharpoonup u$ weakly in W_1^p and

$$\liminf \langle A_\lambda^p(u_n), (u_n - u)^{[p-1]}\rangle \geq 0,$$

it follows that for all $v \in W_1^p$,

$$\langle A_\lambda^p(u), (u - v)^{[p-1]}\rangle \geq \liminf \langle A_\lambda^p(u_n), (u_n - v)^{[p-1]}\rangle.$$

We verify the pseudomonotonicity under assumptions (A1)–(A3) with the same smallness conditions on the form-bounds as in Lemma 5.1. Suppose $u_n \rightharpoonup u$ in W_1^p and

$$\liminf \langle A_\lambda^p(u_n), (u_n - u)^{[p-1]}\rangle \geq 0.$$

Because the duality mapping is of type $(S)_+$, the weak convergence and this limit inferior imply that a subsequence converges strongly in W_1^p. Then one uses the hemicontinuity and the growth conditions to pass to the limit in the nonlinear terms, obtaining the inequality required for pseudomonotonicity. The detailed proof runs parallel to the one for the p-Laplacian with lower-order perturbations (see [21, 25]).

The operator is also coercive (Lemma 5.1) and hemicontinuous. By the classical existence theorem for pseudomonotone operators (see, e.g., [2, 21]), for every $f \in W_{-1}^p$ there exists a solution $u \in W_1^p$ such that

$A^p_\lambda(u) = f$. Uniqueness follows from the strict accretivity proved in Lemma 5.2, exactly as in the Galerkin approach.

11. Conclusion

We have proved existence, uniqueness, and Hölder continuity for quasilinear elliptic systems under conditions that allow singular coefficients through form-boundedness. The method relies on a priori estimates using a special test function and the monotonicity of the associated operator. The results improve upon the classical theory of Ladyzhenskaya-Uraltseva and extend to systems the kind of singular potentials that have been studied for scalar equations via Kato class methods. Future work will explore parabolic analogues and optimality of the form-bound conditions.

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