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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A COUPLED GENERALIZED CAMASSA–HOLM SYSTEM

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Abstract. In this paper, we investigate the existence and multiplicity of solutions for a coupled system of generalized Camassa–Holm equations, which arise in the modeling of nonlinear wave propagation in dispersive media. By developing a new integral formulation of the problem and applying fixed-point techniques adapted to the coupled structure, we establish the existence of multiple physically relevant nonnegative classical solutions. In particular, we prove the existence of at least one, two, and three nonnegative solutions under suitable assumptions on the system parameters and initial data. Our results extend those obtained in [1].

Keywords. Nonlinear equations, generalized Camassa–Holm equations, coupled system, iterative methods, classical solutions.
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1. Introduction

The Camassa–Holm (CH) equation was introduced by Camassa and Holm [2, 3] as a model for the unidirectional propagation of shallow water waves. In contrast to classical dispersive models such as the Korteweg–de Vries equation, the CH equation possesses remarkable mathematical properties, including complete integrability, a bi-Hamiltonian structure, and the existence of peaked soliton solutions known as peakons. These properties have stimulated extensive analytical and numerical research over the past decades; see, for example, [4, 5, 6].
Subsequent studies have focused on the well-posedness theory of the CH equation and its various generalizations. Local and global existence results for strong and weak solutions, together with precise blow-up criteria and wave-breaking phenomena, have been obtained in different functional settings; see, for instance, [7, 8]. These works reveal the delicate interplay between nonlinearity and dispersion inherent in CH-type dynamics.
Motivated by both physical considerations and mathematical generalization, modified and generalized Camassa–Holm equations have been proposed, including the

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so-called b-family of equations, where the nonlinear transport structure depends on a parameter $b \geq 1$. For this class, the stability of solitary waves, the global existence of dissipative solutions, and the loss of regularity have been widely investigated; see, for example, [9, 10, 11]. These studies highlight the crucial role of nonlinear transport coefficients in determining the long-time behavior of solutions.

In recent years, increasing attention has been paid to multi-component and coupled Camassa–Holm systems, which model the interaction of multiple wave fields. Such systems exhibit rich dynamics, including coupled peakon solutions, integrable structures, and complex blow-up mechanisms; see, for instance, [12, 13]. Analytical results include local well-posedness, global existence under suitable sign conditions, and detailed descriptions of blow-up scenarios.

In this paper, for $y = y(t,x)$ and $z = z(t,x)$ with $t \geq 0$ and $x \in \mathbb{R}$, we study the following initial value problem:

$$\begin{aligned} m_t + y^{b-1} z m_x + (b+1) y_x z^{b-1} m + \sigma m = 0, \\ n_t + z^{b-1} y n_x + (b+1) z_x y^{b-1} n + \sigma n = 0, \\ y(0,x) = y_0(x), \quad z(0,x) = z_0(x), \end{aligned} \tag{1.1}$$

where

$$m = (1 - \partial_x^2) y, \qquad n = (1 - \partial_x^2) z.$$

Despite the significant progress in the theory of Camassa–Holm–type equations, relatively few works address the existence and multiplicity of nonnegative classical solutions for coupled generalized Camassa–Holm systems using fixed-point or topological methods. In particular, multiplicity results for systems involving nonlinear coupling and damping effects remain scarce.

Beyond their mathematical interest, results of this type may also be relevant to several areas of computer science where nonlinear wave-like or nonlocal interactions are used in modeling and computation. For example, coupled evolution systems can be employed in signal and image processing to describe multiscale propagation and interaction of features, while well-posedness and multiplicity results provide a rigorous foundation for the stability of the underlying algorithms. Similar analytical frameworks also arise in scientific computing, data-driven surrogate modeling, and neural differential equations, where coupled nonlinear dynamics are used to represent interacting information channels. In these contexts, the existence of nonnegative classical solutions helps justify the consistency of computational models and may support the design of reliable numerical schemes for simulation, control, and learning tasks.

The purpose of this paper is to fill this gap. Under suitable assumptions, we develop a new integral representation of the system and construct appropriate operators whose fixed points correspond to solutions of (1.1). This approach allows us to establish the existence of at least one solution, at least two nonnegative solutions, and at least three nonnegative solutions.

The novelty of the present work lies in the development of a new operator framework for coupled generalized Camassa–Holm systems. In contrast to previous

studies, our approach allows us to establish multiple nonnegative classical solutions by combining expansive mappings with compact perturbations in an appropriate Banach space setting.
The paper is organized as follows. Section 2 collects auxiliary definitions and fixed-point results. In Section 3, we prove the existence of at least one classical solution of (1.1). Section 4 is devoted to the existence of at least two nonnegative classical solutions, while Section 5 establishes the existence of at least three nonnegative classical solutions. Finally, in Section 6, we present an explicit example illustrating the applicability of our results.

2. Preliminaries

In what follows, X denotes a real Banach space. We begin by recalling the definition of a completely continuous operator in X, see [14, 15, 16].

Definition 2.1. Let $K : M \subset X \to X$ be a mapping. The operator K is said to be compact if $K(M)$ is contained in a compact subset of X. It is called completely continuous if it is continuous and maps every bounded subset of X into a relatively compact subset of X.

The concept of a k-set contraction is based on the Kuratowski measure of noncompactness; for the reader's convenience, we recall this notion below; see [17].

Definition 2.2. Let Γ_X be the class of all bounded sets of X. The Kuratowski measure of noncompactness $\alpha : \Gamma_X \to [0, \infty)$ is defined by

$$\alpha(Y) = \inf\left\{\delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad \text{and} \quad \operatorname{diam}(Y_j) \leq \delta, \quad j \in \{1, \dots, m\}\right\},$$

where $\operatorname{diam}(Y_j) = \sup\{\|x-y\|_X : x, y \in Y_j\}$ is the diameter of Y_j, $j \in \{1, \dots, m\}$.

Definition 2.3. [18] A mapping $K : X \to X$ is said to be a k-set contraction if there exists a constant $k \geq 0$ such that

$$\alpha(K(Y)) \leq k\alpha(Y)$$

for any bounded set $Y \subset X$.

Clearly, if $K : X \to X$ is a completely continuous, then K is a 0-set contraction(see [19]).

Theorem 2.4. [20] *Let E be a Banach space, Y a closed, convex subset of E, Ω be any open subset of Y with $0 \in \Omega$. Consider two operators T and Λ, where*

$$Ty = \varepsilon\, y, \ y \in \overline{\Omega},$$

for $\varepsilon > 1$ and $\Lambda : \overline{\Omega} \to E$ be such that

- **(i):** *$I - \Lambda : \overline{\Omega} \to Y$ continuous, compact and*
- **(ii):** *$\{y \in \partial\Omega : y = \sigma(I - \Lambda)y\} = \emptyset, \ \forall \sigma \in \left(0, \frac{1}{\varepsilon}\right)$.*

Then there exists $y^* \in \overline{\Omega}$ such that

$$Ty^* + \Lambda y^* = y^*.$$

Definition 2.5. Let X and Y be real Banach spaces. A map $K : X \to Y$ is called expansive if there exists a constant $h > 1$ for which the following inequality

$$\|Kz - Ky\|_Y \geq h\|z - y\|_X$$

for any $z, y \in X$.

In what follows, we recall the definition of a cone in a Banach space.

Definition 2.6. A closed, convex set $\mathcal{R}$ in X is said to be a cone if

(1) $\alpha y \in \mathcal{R}$ $\forall \alpha \geq 0$ and $\forall y \in \mathcal{R}$,
(2) $y, -y \in \mathcal{R} \implies y = 0$.

Denote $\mathcal{R}^* = \mathcal{R} \backslash \{0\}$. The following fixed point theorem will be a key tool in our analysis and will be employed to prove the existence of at least two nonnegative global classical solutions.

Theorem 2.7. [21] *Let $\mathcal{R}$ be a cone of a Banach space E; Γ a subset of $\mathcal{R}$ and Ω_1, Ω_2 and Ω_3, three open bounded subsets of $\mathcal{R}$ such that $\overline{\Omega}_1 \subset \overline{\Omega}_2 \subset \Omega_3$ and $0 \in \Omega_1$. Assume that $T : \Gamma \to \mathcal{R}$ is an expansive mapping, $\Lambda : \overline{\Omega}_3 \to E$ is a completely continuous map, and $\Lambda(\overline{\Omega}_3) \subset (I - T)(\Gamma)$. Suppose that $(\Omega_2 \setminus \overline{\Omega}_1) \cap \Gamma \neq \emptyset$, $(\Omega_3 \setminus \overline{\Omega}_2) \cap \Gamma \neq \emptyset$, and there exists $\varpi_0 \in \mathcal{R}^*$ such that the following conditions hold:*

(i): $\Lambda x \neq (I - T)(x - \sigma\varpi_0)$, *for all $\sigma > 0$ and $x \in \partial\Omega_1 \cap (\Gamma + \sigma\varpi_0)$,*
(ii): *There exists $\varepsilon \geq 0$ such that $\Lambda x \neq (I - T)(\sigma x)$, for all $\sigma \geq 1 + \varepsilon$, $x \in \partial\Omega_2$ and $\sigma x \in \Gamma$,*
(iii): $\Lambda x \neq (I - T)(x - \sigma\varpi_0)$, *for all $\sigma > 0$ and $x \in \partial\Omega_3 \cap (\Gamma + \sigma\varpi_0)$.*

Then $T + \Lambda$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{R}$ such that

$$x_1 \in \partial\Omega_2 \cap \Gamma \text{ and } x_2 \in (\overline{\Omega}_3 \setminus \overline{\Omega}_2) \cap \Gamma$$

or

$$x_1 \in (\Omega_2 \setminus \Omega_1) \cap \Gamma \text{ and } x_2 \in (\overline{\Omega}_3 \setminus \overline{\Omega}_2) \cap \Gamma.$$

The following result is a key tool in our analysis and will be employed to prove the existence of three nonnegative solutions to the problem under consideration.

Theorem 2.8. [21] *Let $\mathcal{R}$ be a cone in a Banach space E; Γ a subset of $\mathcal{R}$ and Ω_1, Ω_2 and Ω_3 three open bounded subsets of $\mathcal{R}$ such that $\overline{\Omega}_1 \subset \overline{\Omega}_2 \subset \Omega_3$ and $0 \in \Omega_1$. Assume that $T : \Gamma \to E$ is an expansive mapping, $\Lambda : \overline{\Omega}_3 \to E$ is a completely continuous one, and $\Lambda(\overline{\Omega}_3) \subset (I - T)(\Gamma)$. Suppose that $(\Omega_2 \setminus \overline{\Omega}_1) \cap \Gamma \neq \emptyset$, $(\Omega_3 \setminus \overline{\Omega}_2) \cap \Gamma \neq \emptyset$, and there exist $w_0 \in \mathcal{R}^*$ and $\varepsilon > 0$ small enough such that the following conditions hold:*

(i): $\Lambda x \neq (I - T)(\sigma x)$, *for all $\sigma \geq 1 + \varepsilon$, $x \in \partial\Omega_1$ and $\sigma x \in \Gamma$,*
(ii): $\Lambda x \neq (I - T)(x - \sigma w_0)$, *for all $\sigma \geq 0$ and $x \in \partial\Omega_2 \cap (\Gamma + \sigma w_0)$,*
(iii): $\Lambda x \neq (I - T)(\sigma x)$, *for all $\sigma \geq 1 + \varepsilon$, $x \in \partial\Omega_3$ and $\sigma x \in \Gamma$.*

Then $T + \Lambda$ has at least three non trivial fixed points $x_1, x_2, x_3 \in \mathcal{R}$ such that

$$x_1 \in \overline{\Omega}_1 \cap \Gamma \text{ and } x_2 \in (\Omega_2 \setminus \overline{\Omega}_1) \cap \Gamma \text{ and } x_3 \in (\overline{\Omega}_3 \setminus \overline{\Omega}_2) \cap \Gamma.$$

For $w \in \mathcal{C}^1([0,\infty),\mathcal{C}^3(\mathbb{R}))$, define

$$\|w\|_1 = \max\Bigg\{ \sup_{0\le t,\ x\in\mathbb{R}} |w|, \quad \sup_{0\le t,\ x\in\mathbb{R}} |w_t|, \quad \sup_{0\le t,\ x\in\mathbb{R}} |w_{txx}|, \quad \sup_{0\le t,\ x\in\mathbb{R}} |w_x|, \quad \sup_{0\le t,\ x\in\mathbb{R}} |w_{xx}|, \sup_{0\le t,\ x\in\mathbb{R}} |w_{xxx}| \Bigg\},$$

provided it exists. Let $X = \left(\mathcal{C}^1\left([0,\infty),\mathcal{C}^3(\mathbb{R})\right)\right)^4$ be endowed with the norm

$$\|\varpi\| = \max_{j\in\{1,\ldots,4\}} \|\varpi_j\|_1, \quad \varpi = (\varpi_1,\ldots,\varpi_4).$$

3. Existence of at least one solution

The aim of this section is to prove that problem (1.1) has at least one solution. Let

$$p = (1-\partial_x^2)\varpi_1, \qquad q = (1-\partial_x^2)\varpi_2.$$

For $\varpi \in X$, define the operators

$$\begin{aligned}
\Lambda_1^1(\varpi) &= p_t + (\varpi_1)^{b-1}\varpi_2 p_x + (b+1)\varpi_{1x}(\varpi_2)^{b-1}p + \sigma p,\\
\Lambda_1^2(\varpi) &= q_t + (\varpi_2)^{b-1}\varpi_1 q_x + (b+1)\varpi_{2x}(\varpi_1)^{b-1}q + \sigma q,\\
\Lambda_1^3(\varpi) &= \varpi_1(0,x) - y_0(x),\\
\Lambda_1^4(\varpi) &= \varpi_2(0,x) - z_0(x),\\
\Lambda_1(\varpi) &= \left(\Lambda_1^1(\varpi),\ldots,\Lambda_1^4(\varpi)\right),
\end{aligned}$$

where

$$p = (1-\partial_x^2)\varpi_1,$$

$$q = (1-\partial_x^2)\varpi_2.$$

Note that if $\varpi \in X$ and $\varpi = (\varpi_1,\ldots,\varpi_4)$ are solutions to the equation $\Lambda_1(\varpi) = 0$, then ϖ is a solution to the IVP (1.1).
Let

$$B_1 = 2(1+\sigma)B + 2(b+2)B^{b+1}.$$

Lemma 3.1. *Suppose* (A1). *If* $\varpi \in X$, $\|\varpi\| \le B$, *then*

$$|\Lambda_1^j(\varpi)| \le B_1, \quad j \in \{1,\ldots,4\}.$$

Proof. We have

$$\begin{aligned}
|\Lambda_1^1(\varpi) =& |p_t + (\varpi_1)^{b-1}\varpi_2 p_x + (b+1)\varpi_{1x}(\varpi_2)^{b-1}p + \sigma p| \\
\leq& |p_t| + |\varpi_1|^{b-1}|\varpi_2||p_x| + (b+1)|\varpi_{1x}||\varpi_2|^{b-1}|p| + \sigma|p| \\
\leq& 2B + 2B^{b+1} + 2(b+1)B^{b+1} + 2\sigma B \\
=& 2(1+\sigma)B + 2(b+2)B^{b+1} \\
=& B_1,
\end{aligned}$$

and

$$\begin{aligned}
|\Lambda_1^2(\varpi) =& |q_t + (\varpi_2)^{b-1}\varpi_1 q_x + (b+1)\varpi_{2x}(\varpi_1)^{b-1}q + \sigma q| \\
\leq& |q_t| + |\varpi_2|^{b-1}|\varpi_1||q_x| + (b+1)|\varpi_{2x}||\varpi_1|^{b-1}|q| + \sigma|q| \\
\leq& 2B + 2B^{b+1} + 2(b+1)B^{b+1} + 2\sigma B \\
=& 2(1+\sigma)B + 2(b+2)B^{b+1} \\
=& B_1,
\end{aligned}$$

and

$$\begin{aligned}
|\Lambda_1^3(\varpi)| =& |\varpi_1(0,x) - y_0(x)| \\
\leq& |\varpi_1(0,x)| + y_0(x) \\
\leq& 2B \\
\leq& B_1,
\end{aligned}$$

and

$$\begin{aligned}
|\Lambda_1^4(\varpi)| =& |\varpi_2(0,x) - z_0(x)| \\
\leq& |\varpi_2(0,x)| + |z_0(x)| \\
\leq& 2B \\
\leq& B_1.
\end{aligned}$$

This completes the proof. □

(A2): Suppose that $g \in \mathcal{C}([0,\infty)\times\mathbb{R})$ is a nonnegative function such that $g > 0$ on $(0,\infty)\times(\mathbb{R}\backslash\{0\})$,

$$24(1+t)(1+|x|+x^2+|x|^3)\left|\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\right| \le A,$$

for some nonnegative constant A.

In the last section we will give an example for a function g and a constant A that satisfy $(A2)$. For $\varpi \in X$, define the operators

$$\begin{aligned}\Lambda_2^j(\varpi) &= \int_0^t\int_0^x (t-t_1)(x-x_1)^3\Lambda_1^j(\varpi)(t_1,x_1)dx_1dt_1, \quad j\in\{1,\dots,4\},\\ \Lambda_2(\varpi) &= (\Lambda_2^1(\varpi),\dots,\Lambda_2^4(\varpi)).\end{aligned}$$

Lemma 3.2. *Let $\varpi \in X$ be a solution to the equation*

$$\Lambda_2(\varpi) = D,$$

for some constant D. Then u is a solution to the IVP (1.1).

Proof. We have

$$\int_0^t\int_0^x (t-t_1)(x-x_1)^3\Lambda_1(\varpi)(t_1,x_1)dx_1dt_1 = D,$$

which we differentiate two times in t and four times in x and we obtain

$$g\Lambda_1(\varpi) = 0.$$

Therefore

$$\Lambda_1(\varpi) = 0, x \in (\mathbb{R}\backslash\{0\}).$$

Since $\Lambda_1(\varpi)(\cdot,\cdot)$ is continuous with respect to t and x, we get

$$\begin{aligned}0 =& \lim_{t\to 0}\Lambda_1(\varpi)\\ =&\Lambda_1(\varpi)(0,x)\\ =& \lim_{x\to 0}\Lambda_1(\varpi)\\ =&\Lambda_1(\varpi)(t,0).\end{aligned}$$

Consequently

$$\Lambda_1(\varpi) = 0.$$

This completes the proof. □

Lemma 3.3. *Suppose* $(A1)$ *and* $(A2)$. *If $\varpi \in X$ and $\|\varpi\| \le B$, then*

$$\|\Lambda_2\varpi\| \le AB_1.$$

Proof. We have

$$
\begin{aligned}
|\Lambda_2^j(\varpi)| =& \left| \int_0^t \int_0^x (t-t_1)(x-x_1)^3 g(t_1,x_1)\Lambda_1(\varpi)(t_1,x_1)dx_1dt_1 \right| \\
\leq & \left| \int_0^t \int_0^x (t-t_1)|x-x_1|^3 g(t_1,x_1)|\Lambda_1(\varpi)(t_1,x_1)|dx_1dt_1 \right| \\
\leq & 2^3(1+t)B_1 \left| \int_0^t \int_0^x (|x|^3+|x_1|^3) g(t_1,x_1)dx_1dt_1 \right| \\
\leq & 24(1+t)(1+|x|+x^2+|x|^3)B_1 \left| \int_0^t \int_0^x g(t_1,x_1)dx_1dt_1 \right| \\
\leq & AB_1, \quad j\in\{1,\ldots,4\},
\end{aligned}
$$

and

$$
\begin{aligned}
|\Lambda_{2t}^j(\varpi)| =& \left| \int_0^t \int_0^x (x-x_1)^3 g(t_1,x_1)\Lambda_1(\varpi)(t_1,x_1)dx_1dt_1 \right| \\
\leq & \left| \int_0^t \int_0^x |x-x_1|^3 g(t_1,x_1)|\Lambda_1(\varpi)(t_1,x_1)|dx_1dt_1 \right| \\
\leq & 2^3 B_1 \left| \int_0^t \int_0^x (|x|^3+|x_1|^3) g(t_1,x_1)dx_1dt_1 \right| \\
\leq & 24(1+t)(1+|x|+x^2+|x|^3)B_1 \left| \int_0^t \int_0^x g(t_1,x_1)dx_1dt_1 \right| \\
\leq & AB_1, \quad j\in\{1,\ldots,4\},
\end{aligned}
$$

and

$$
\begin{aligned}
|\Lambda_{2x}^{j}(\varpi)| =& 3\left|\int_0^t\int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1)\Lambda_1(\varpi)(t_1,x_1)dx_1dt_1\right| \\
\leq& 3\left|\int_0^t\int_0^x (t-t_1)|x-x_1|^2 g(t_1,x_1)|\Lambda_1(\varpi)(t_1,x_1)|dx_1dt_1\right| \\
\leq& 3\cdot 2^2 B_1(1+t)\left|\int_0^t\int_0^x (x^2+x_1^2)g(t_1,x_1)dx_1dt_1\right| \\
\leq& 24(1+t)(1+|x|+x^2+|x|^3)B_1\left|\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\right| \\
\leq& AB_1, \quad j\in\{1,\ldots,4\},
\end{aligned}
$$

and

$$
\begin{aligned}
|\Lambda_{2xx}^{j}(\varpi)| =& 6\left|\int_0^t\int_0^x (t-t_1)(x-x_1) g(t_1,x_1)\Lambda_1(\varpi)(t_1,x_1)dx_1dt_1\right| \\
\leq& 6\left|\int_0^t\int_0^x (t-t_1)|x-x_1| g(t_1,x_1)|\Lambda_1(\varpi)(t_1,x_1)|dx_1dt_1\right| \\
\leq& 6B_1(1+t)\left|\int_0^t\int_0^x (|x|+|x_1|)g(t_1,x_1)dx_1dt_1\right| \\
\leq& 24(1+t)(1+|x|+x^2+|x|^3)B_1\left|\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\right| \\
\leq& AB_1, \quad j\in\{1,\ldots,4\},
\end{aligned}
$$

and

$$
\begin{aligned}
|\Lambda^j_{2xxx}(\varpi)| =& 6\left|\int_0^t \int_0^x (t-t_1)g(t_1,x_1)\Lambda_1(\varpi)(t_1,x_1)dx_1dt_1\right| \\
\leq& 6\left|\int_0^t \int_0^x (t-t_1)g(t_1,x_1)|\Lambda_1(\varpi)(t_1,x_1)|dx_1dt_1\right| \\
\leq& 6B_1(1+t)\left|\int_0^t \int_0^x g(t_1,x_1)dx_1dt_1\right| \\
\leq& 24(1+t)(1+|x|+x^2+|x|^3)B_1\left|\int_0^t \int_0^x g(t_1,x_1)dx_1dt_1\right| \\
\leq& AB_1, \quad j\in\{1,\dots,4\},
\end{aligned}
$$

and

$$
\begin{aligned}
|\Lambda^j_{2txx}(\varpi)| =& 6\left|\int_0^t \int_0^x (x-x_1)g(t_1,x_1)\Lambda_1(\varpi)(t_1,x_1)dx_1dt_1\right| \\
\leq& 6\left|\int_0^t \int_0^x |x-x_1|g(t_1,x_1)|\Lambda_1(\varpi)(t_1,x_1)|dx_1dt_1\right| \\
\leq& 6B_1\left|\int_0^t \int_0^x (|x|+|x_1|)g(t_1,x_1)dx_1dt_1\right| \\
\leq& 24(1+t)(1+|x|+x^2+|x|^3)B_1\left|\int_0^t \int_0^x g(t_1,x_1)dx_1dt_1\right| \\
\leq& AB_1, \quad j\in\{1,\dots,4\},
\end{aligned}
$$

whereupon we get the desired result. This completes the proof. □

Under condition

(A3): $\varepsilon > 1$.

We have the main result reads as follow.

Theorem 3.4. *Suppose* (*A*1) - (*A*3)*. Then the equation* (1.1) *has at least one solution in* X*.*

Proof. Let $\widetilde{Y}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also,

$$\widetilde{\widetilde{Y}} = \left\{ \varpi \in \widetilde{Y} : \varpi \geq \frac{\|\varpi\|}{2}, \right\}$$

and $Y = \overline{\widetilde{\widetilde{Y}}}$ be the closure of $\widetilde{Y}$,

$$\Omega \;=\; \{\varpi \in Y : \|\varpi\| < B\}.$$

For $\varpi \in \overline{\Omega}$ and $\varepsilon > 0$, define the operators

$$\begin{aligned} T\varpi &= \varepsilon\varpi, \\ \Lambda\varpi &= \varpi - \varepsilon\varpi - \varepsilon\Lambda_2(\varpi). \end{aligned}$$

For $\varpi \in \overline{\Omega}$, we have

$$\begin{aligned} \|(I-\Lambda)\varpi\| &= \|\varepsilon\varpi + \varepsilon\Lambda_2(\varpi)\| \\ &\leq \varepsilon\|\varpi\| + \varepsilon\|\Lambda_2(\varpi)\| \\ &\leq \varepsilon B + \varepsilon A B_1. \end{aligned}$$

Thus, $\Lambda : \overline{\Omega} \to X$ is continuous and $(I-\Lambda)(\overline{\Omega})$ resides in a compact subset of Y. Now, suppose that there is a $\varpi \in \overline{\Omega}$ so that $\|\varpi\| = B$ and

$$\varpi = \sigma(I-\Lambda)\varpi$$

or

$$\varpi = \sigma\varepsilon\left(\varpi + \Lambda_2(\varpi)\right), \tag{3.1}$$

for some $\sigma \in \left(0, \frac{1}{\varepsilon}\right)$. Then, using that $\|\varpi\| \geq \frac{B}{2}$, we get $\varpi(0,x) > \frac{B}{2}$, $x \in \mathbb{R}$, and

$$\varpi(0,x) = \sigma\varepsilon\varpi(0,x), \quad x \in \mathbb{R},$$

whereupon $\sigma\varepsilon = 1$, which is a contradiction. Consequently

$$\{\varpi \in \overline{\Omega} : \varpi = \sigma_1(I-\Lambda)\varpi,\ \|\varpi\| = B\} = \emptyset$$

for any $\sigma_1 \in \left(0, \frac{1}{\varepsilon}\right)$. Then, from Theorem 2.4, it follows that the operator $T + \Lambda$ has a fixed point $\varpi^* \in Y$. Therefore

$$\begin{aligned} \varpi^* &= T\varpi^* + \Lambda\varpi^* \\ &= \varepsilon\varpi^* + \varpi^* \\ &\quad -\varepsilon\varpi^* - \varepsilon\Lambda_2(\varpi^*), \end{aligned}$$

whereupon

$$\Lambda_2(\varpi^*) = 0.$$

From here, ϖ^* is a solution to the problem (1.1), it follows that ϖ is a solution to the equation (1.1). This completes the proof. $\square$

4. Existence of at Least Two Solutions

Let X be the space used in the previous section. Suppose

(A4): Let m, r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1 \leq B.$$

Our main result in this section is as follows.

Theorem 4.1. *Suppose that (A1)-(A4) holds. Then the equation (1.1) has at least two nonnegative solutions in X.*

Proof. Let

$$\widetilde{P} = \{\varpi \in X : \varpi \geq 0 \quad \text{on} \quad [0, \infty) \times \mathbb{R}\}.$$

With $\mathcal{R}$, we will denote the set of all equi-continuous families in $\widetilde{P}$. For $w \in X$, define the operators

$$\begin{aligned} T_1 w &= (1 + m\varepsilon)w, \\ \Lambda_3 w &= -\varepsilon|\Lambda_2(w)| - m\varepsilon w. \end{aligned}$$

Note that any fixed point $w \in X$ of the operator $T_1 + \Lambda_3$ is a solution to the equation (1.1). Define

$$\begin{aligned} \Gamma &= \mathcal{R}, \\ \Omega_1 &= \mathcal{R}_r = \{w \in \mathcal{R} : \|w\| < r\}, \\ \Omega_2 &= \mathcal{R}_L = \{w \in \mathcal{R} : \|w\| < L\}, \\ \Omega_3 &= \mathcal{R}_{R_1} = \{w \in \mathcal{R} : \|w\| < R_1\}. \end{aligned}$$

(1) For $w_1, w_2 \in \Gamma$, we have

$$\|T_1 w_1 - T_1 w_2\| = (1 + m\varepsilon)\|w_1 - w_2\|,$$

whereupon $T_1 : \Gamma \to X$ is an expansive operator with a constant $h = 1 + m\varepsilon > 1$.

(2) For $w \in \overline{\mathcal{R}}_{R_1}$, we get

$$\begin{aligned} \|\Lambda_3 w\| &\leq \varepsilon\|\Lambda_2(w)\| + m\varepsilon\|w\| \\ &\leq \varepsilon\Big(AB_1 + mR_1\Big). \end{aligned}$$

Therefore, $\Lambda_3(\overline{\mathcal{R}}_{R_1})$ is uniformly bounded. Since $\Lambda_3 : \overline{\mathcal{R}}_{R_1} \to X$ is continuous, we have that $\Lambda_3(\overline{\mathcal{R}}_{R_1})$ is equi-continuous. Consequently, $\Lambda_3 : \overline{\mathcal{R}}_{R_1} \to X$ is a 0-set contraction.

(3) Let $w_1 \in \overline{\mathcal{R}}_{R_1}$. Set

$$w_2 = w_1 + \frac{1}{m}|\Lambda_2(w_1)|.$$

We have $w_2 \geq 0$ on $[0,\infty) \times \mathbb{R}$. Therefore, $w_2 \in \Gamma$ and

$$-\varepsilon m w_2 = -\varepsilon m w_1 - \varepsilon|\Lambda_2(w_1)|$$

or

$$\begin{aligned}(I - T_1)w_2 &= -\varepsilon m w_2 \\ \\ &= \Lambda_3 w_1.\end{aligned}$$

Consequently, $\Lambda_3(\overline{\mathcal{R}}_{R_1}) \subset (I - T_1)(\Gamma)$.

(4) Assume that for any $w_0 \in \mathcal{R}^*$ there exist $\sigma \geq 0$ and $w \in \partial\mathcal{R}_r \cap (\Gamma + \sigma w_0)$ or $w \in \partial\mathcal{R}_{R_1} \cap (\Gamma + \sigma w_0)$ such that

$$\Lambda_3 w = (I - T_1)(w - \sigma w_0).$$

Then

$$-\varepsilon|\Lambda_2(w)| - m\varepsilon w = -m\varepsilon(w - \sigma w_0)$$

or

$$-|\Lambda_2(w)| = \sigma m w_0.$$

This is a contradiction.

(5) Let $\varepsilon_1 = \frac{AB_1}{mL}$. Suppose that there exists a $w_1 \in \partial\mathcal{R}_L$ and $\sigma_1 \geq 1+\varepsilon_1$ such that

$$\Lambda_3 w_1 = (I - T_1)(\sigma_1 w_1). \tag{4.1}$$

Moreover,

$$-\varepsilon|\Lambda_2(w_1)| - m\varepsilon w_1 = -\sigma_1 m\varepsilon w_1,$$

or

$$|\Lambda_2(w_1)| + m w_1 = \sigma_1 m w_1.$$

From here,

$$\sigma_1 m L = \sigma_1 m\|w_1\| \leq \|\Lambda_2 w_1\| + m\|w_1\| \leq AB_1 + mL$$

and

$$\sigma_1 \leq 1 + \frac{AB_1}{mL},$$

which is a contradiction.

Therefore, all conditions of Theorem 2.7 hold. Hence, the problem (1.1) has at least two solutions ϖ_1 and ϖ_2 so that

$$\|\varpi_1\| = L < \|\varpi_2\| < R_1$$

or

$$r < \|\varpi_1\| < L < \|\varpi_2\| < R_1.$$

□

5. Existence of at least three solutions

Our next main result is as follows.

Theorem 5.1. *Under the hypotheses (A1)-(A4), the problem (1.1) has at least three nonnegative solutions* $\varpi_1, \varpi_2, \varpi_3 \in X$.

Proof. (1) Assume that there are $\sigma_1 \geq 1 + \frac{2AB_1}{mr}$, $\varpi \in \partial\Omega_1$, and $\sigma_1\varpi \in \Gamma$ such that

$$\Lambda_3(\varpi) = (I - T_1)(\sigma_1\varpi).$$

Then

$$-\varepsilon|\Lambda_2(\varpi)| - m\varepsilon\varpi = -m\varepsilon\sigma_1\varpi$$

or

$$|\Lambda_2(\varpi)| + m\varpi = m\sigma_1\varpi$$

Hence,

$$\sigma_1 mr = \sigma_1 m\|\varpi\| \leq \|\Lambda_2(\varpi)\| + m\|\varpi\| \leq AB_1 + \sigma_1 mr,$$

whereupon

$$\sigma_1 \leq 1 + \frac{AB_1}{mr},$$

which is a contradiction. Thus, the condition (**i**) of Theorem 2.8 holds.

(2) Now, assume that there are $\sigma_1 \geq 1 + \frac{2AB_1}{mr}$, $\varpi \in \partial\Omega_3$, and $\sigma_1\varpi \in \Gamma$ such that

$$\Lambda_3(\varpi) = (I - T_1)(\sigma_1\varpi).$$

As above,

$$\sigma_1 mR_1 = \sigma_1 m\|\varpi\| \leq \|\Lambda_2(\varpi)\| + m\|\varpi\| \leq AB_1 + \sigma_1 mR_1,$$

whereupon

$$\sigma_1 \leq 1 + \frac{AB_1}{mR_1} \leq 1 + \frac{AB_1}{mr},$$

which is a contradiction. Hence, the condition (**iii**) of Theorem 2.8 holds.

(3) Assume that for any $\varpi_0 \in \mathcal{R}^*$ there exist $\sigma_1 \geq 0$ and $\varpi \in \partial\mathcal{R}_L \cap (\Gamma + \sigma_1\varpi_0)$ such that

$$\Lambda_3(\varpi) = (I - T_1)(\varpi - \sigma_1\varpi_0).$$

Then

$$-\varepsilon|\Lambda_2(\varpi)| - m\varepsilon\varpi = -m\varepsilon(\varpi - \sigma_1\varpi_0)$$

or

$$-|\Lambda_2(\varpi)| = \sigma_1 m\varpi_0.$$

This is a contradiction. From here, the condition (**ii**) of Theorem 2.8 holds.

Now, by Theorem 2.8, it follows that the problem (1.1) has at least three classical solutions ϖ_1, ϖ_2, and ϖ_3 such that

$$\varpi_1 \in \partial\Omega_1 \cap \Gamma \text{ and } \varpi_2 \in (\Omega_2 \setminus \overline{\Omega}_1) \cap \Gamma \text{ and } \varpi_3 \in (\overline{\Omega}_3 \setminus \overline{\Omega}_2) \cap \Gamma,$$

or

$$\varpi_1 \in \Omega_1 \cap \Gamma \text{ and } \varpi_2 \in (\Omega_2 \setminus \overline{\Omega}_1) \cap \Gamma \text{ and } \varpi_3 \in (\overline{\Omega}_3 \setminus \overline{\Omega}_2) \cap \Gamma.$$

□

6. Example

Let $y_0(x)=\frac{1}{1+x^2}$, $z_0(x)=\frac{1}{1+2x^2+4x^4}$, $b=2$, $\sigma=3$, and

$$R_1=1,\quad L=\frac{1}{4},\quad r=\frac{1}{5},\quad m=10^{50},\quad B=1,\quad p=2\quad A=\frac{1}{10B_1},\quad \varepsilon=4.$$

Then $B_1=16$, $A=\frac{1}{160}$, and $\varepsilon>1$, i.e., $(A3)$ hold. Next,

$$0<r<L<R_1=B.$$

i.e., $(A4)$ holds.
Let us give an example of such a function g.: Take

$$h(s)=\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}},\quad l(s)=\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}},\quad s\in\mathbb{R},\quad s\neq\pm1.$$

Then

$$\begin{aligned} h'(s) &= \frac{22\sqrt{2}s^{10}(1-s^{22})}{(1-s^{11}\sqrt{2}+s^{22})(1+s^{11}\sqrt{2}+s^{22})},\\ l'(s) &= \frac{11\sqrt{2}s^{10}(1+s^{22})}{1+s^{44}},\quad s\in,\quad s\neq\pm1. \end{aligned}$$

Therefore

$$\begin{aligned} -\infty &< \lim_{s\to\pm\infty}(1+s+s^2+s^3)h(s)<\infty,\\ -\infty &< \lim_{s\to\pm\infty}(1+s+s^2+s^3)l(s)<\infty. \end{aligned}$$

Hence, there exists a positive constant C_1 such that

$$(1+s+s^2+s^3)\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}+\frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right)\ \leq\ C_1,$$

$s\in\mathbb{R}$. Note that $\lim\limits_{s\to\pm1}l(s)=\frac{\pi}{2}$ and by [22] (pp. 707, Integral 79) and [23], we have

$$\int\frac{dz}{1+z^4}=\frac{1}{4\sqrt{2}}\log\frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2}+\frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s)=\frac{s^{10}}{(1+s^{44})(1+s+s^2)^2},\quad s\in\mathbb{R},$$

and

$$g_1=Q(t)Q(x).$$

Then there exists a constant $C_2>0$ such that

$$24(1+t)(1+|x|+x^2+|x|^3)\left|\int_0^t\int_0^x g_1(t_1,x_1)dx_1dt_1\right|\leq C_2.$$

Let

$$g=\frac{A}{C_2}g_1.$$

Then

$$24(1+t)(1+|x|+x^2+|x|^3)\left|\int_0^t\int_0^x g(t_1,x_1)dx_1dt_1\right| \leq A.$$

Therefore, all conditions of Theorem 3.4, Theorem 4.1, and Theorem 5.1 are fulfilled.

7. Conclusion

In this paper, we investigate a coupled system of generalized Camassa–Holm equations. The main objective was to establish existence and multiplicity results for nonnegative classical solutions under suitable structural and parameter assumptions. The first novelty of this work lies in the derivation of a new integral representation for the considered system. This formulation allows the original evolution problem to be rewritten as a fixed-point problem in an appropriate Banach space, providing a flexible framework for the application of topological methods to a coupled system.

A second important contribution is the construction of two carefully designed operators whose sum characterizes the solution set of the system. By exploiting their compactness, continuity, and cone-preserving properties, we successfully applied fixed-point theorems in cones to obtain multiplicity results. In particular, we proved the existence of at least one solution, at least two nonnegative solutions, and at least three nonnegative solutions, depending on the imposed assumptions. Such multiplicity results for coupled generalized Camassa–Holm systems appear to be new in the existing literature. Moreover, our analysis does not rely on integrability or variational structures, which are often unavailable in coupled or damped Camassa–Holm-type equations. This highlights the robustness of the proposed approach and its potential applicability to a broader class of nonlinear evolution systems with nonlocal or coupled interactions.

Finally, an explicit example was provided to illustrate the applicability of the theoretical results and to demonstrate the effectiveness of the developed framework. The techniques introduced in this work open several perspectives for future research, including extensions to higher-dimensional settings, more general nonlinear coupling terms, and the study of qualitative properties such as stability, long-time behavior, and decay rates of solutions.

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