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ON THE EIGENVALUE PROBLEM INVOLVING THE ROBIN $p(x)$-LAPLACIAN

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Abstract. In this study, we investigate a nonlinear eigenvalue problem on a bounded domain in $\mathbb{R}^d, d \geq 2$. The existence of a nondecreasing sequence of nonnegative eigenvalues is established using the Ljusternik-Schnirelmann principle. Using the variational principle, we also prove that there exists a principal eigenvalue which is the smallest of all possible eigenvalues and that the set of eigenvalues is not closed.

1. Introduction

In recent years, nonlinear analysis has played a crucial role in Mathematical applications to physical problems. As a result nonlinear differential operators such as the $p(x)$-Laplacian have emerged. In particular, eigenvalue problems involving operators with variable growth conditions have attracted attention in the recent decades (see e.g., [3, 5, 6, 7, 8] and the references therein). In this paper, we investigate the spectral properties of a nonlinear eigenvalue problem involving the $p(x)$-Laplacian of the form

$$\begin{cases} -\Delta_{p(x)} u = \lambda |u|^{q(x)-2} u \quad \text{in } \Omega, \\ \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \gamma} + \beta(x)|u|^{p(x)-2} u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 2$ with Lipschitz boundary $\partial\Omega$, $\Delta_{p(x)} = div(|\nabla u|^{p(x)-2}\nabla u)$, $\frac{\partial u}{\partial \gamma}$ is the outer normal derivative of u with respect to $\partial\Omega$, $\lambda \in \mathbb{R}$ is a spectral parameter, $\beta : \partial\Omega \longrightarrow (1, \infty)$ is a bounded function, $p, q : \overline{\Omega} \longrightarrow [1, \infty)$ are Lipschitz functions with $\inf_{x\in\Omega} p(x) > 1$, $q \in L^\infty(\Omega)$ and $1 \leq q(x) \leq q(x) + \varepsilon < p^*(x)$, $\varepsilon > 0$ for a.e. $x \in \Omega$, where

$$p^*(x) = \begin{cases} \frac{dp(x)}{d-p(x)} & \text{if } p(x)<d, \\ \infty & \text{if } p(x) \geq d. \end{cases}$$

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The case $p(x) \equiv q(x) \equiv$ constant in problem (1)was investigated and for the Dirichlet, Neumann and Robin boundary conditions and the existence of a non-decreasing sequence of nonnegative eigenvalues was proved. Existence of a principal eigenvalue and closeness of a set of eigenvalues were also proved (see, e.g.,[1, 13, 15]).
For $p(x) \equiv q(x) \not\equiv$ constant,existence of infinitely many eigenvalue sequences has been proved, under some assumptions on $p(x)$ the infimum of all eigenvalues is zero and so, for some variable exponent $p(x)$ there is no principal eigenvalue and the set of eigenvalues is not closed [6, 7, 9]. For $p(x) \not\equiv q(x)$ with Dirichlet boundary conditions, the problem has been investigated in [8].
In section 3.1, we prove the existence of a non decreasing sequence of non negative eigenvalues. We also prove that the first eigenvalue is a principal eigenvalue in section 3.2 and that the set of all possible eigenvalues is not closed in section 3.3.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary $\partial\Omega$ and let $p(x)$ be a continuous positive function (that is, $p \in C(\bar{\Omega}, \mathbb{R})$) with $p(x){>}1$. Denote by $p_-(\Omega) := \inf_{x\in\Omega} p(x)$, $p_+(\Omega) := \sup_{x\in\Omega} p(x)$. Then $p_-(\Omega){>}1$ and $p_+(\Omega){<}+\infty$. In the sequel, we write p_- and p_+ for $p_-(\Omega)$ and $p_+(\Omega)$ respectively.

Definition 2.1. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \Big\{ u \,|\, u : \Omega \to \mathbb{R} \text{ is measurable and } \textstyle\int_\Omega |u(x)|^{p(x)} dx \ < +\infty$$

with the norm given by

$$||u||_{L^{p(x)}(\Omega)} = \inf\left\{ \tau{>}0 : \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \leq 1 \right\}. \tag{2.1}$$

Definition 2.2. The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \}$$

with the norm

$$||u||_{W^{1,p(x)}(\Omega)} = \inf\left\{ \tau{>}0 : \int_\Omega \left(\left| \frac{\nabla u}{\tau} \right|^{p(x)} + \left| \frac{u}{\tau} \right|^{p(x)} \right) dx {<}1 \right\}. \tag{2.2}$$

For $p(x) \geq 1$, $\forall x \in \Omega$, the embedding $W^{1,p(x)} \hookrightarrow C(\bar{\Omega})$ is compact and continuous. Denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ that is, $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. Then for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, the Holder inequality

$$\int_\Omega |uv| \mathrm{dx} \leq \left(\frac{1}{p(x)} + \frac{1}{q(x)} \right) ||u||_{p(x)} ||v||_{q(x)} \tag{2.3}$$

holds. For more information regarding variable exponent Lebesgue spaces see for example [5, 6, 11].

Let $X := W^{1,p(x)}(\Omega)$ and consider the following functional

$$H[u] := \int_\Omega \frac{1}{p(x)} |u|^{p(x)} \, dx, \;\; u \in X.$$

Then $H \in C^1(X, \mathbb{R})$ and the $p(x)$-Laplacian is the derivative operator of H in the weak sense, that is, $\Delta_{p(x)}u = H'(u)$.

Theorem 2.3. [10, Theorem 3.1] *Let X^* be the dual space of X. Then*

(i) *$H' : X \to X^*$ is a continuous, bounded and strictly monotone operator.*
(ii) *H' is a mapping of type (S_+), that is, if $u_n \rightharpoonup u$ in X and $\lim_{n\to\infty}\langle (H^1(u_n) - H^1(u)), (u_n - u)\rangle \leq 0$, then $u_n \to u$ in X.*
(iii) *$H' : X \to X^*$ is a homeomorphism.*

3. Eigenvalue problems

Definition 3.1. For any functional $g : X \to R$, u is a critical point of g if $g'(u) = 0$ (otherwise, it is a regular point) and α is a critical value of g if there exists a critical point u with $g(u) = \alpha$ (otherwise, α is a regular value).

The Ljusternik-Schnirelmann (L-S) principle is applied to establish the existence of eigenvalues for the eigenvalue problems in closed subspaces of $W^{1,p(x)}(\Omega)$. It involves finding critical points of a given functional with respect to a given level set. Thus, to find the eigenvalues of problem (1.1), one needs to find the critical points of the functionals f and g defined on the space X by

$$f(u) = \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\mathrm{dx} \tag{3.1}$$

$$g(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{dx} + \int_{\partial\Omega} \frac{1}{p(x)}\beta(x)|u|^{p(x)}\mathrm{dS} \tag{3.2}$$

restricted to the level set $M(\alpha) = \{u \in X : g(u) = \alpha\}$ for any $\alpha>0$
Let X,f, g and $M(\alpha)$ be defined as above. Consider the eigenvalue problem

$$f'(u) = \mu g'(u), \ \ u \in M(\alpha), \ \ \mu \in \mathbb{R} \tag{3.3}$$

Assume that:

(A) $f, g : X \to \mathbb{R}$ are even functionals and $f, g \in C'(X, \mathbb{R})$ with $f(0) = g(0) = 0$.
(B) f' is weakly-strongly continuous (that is $u_n \rightharpoonup u$ in X implies $f'(u_n) \to f'(u)$) and $\langle f'(u), u\rangle = 0, u \in \overline{coM(\alpha)}$ implies $f(0) = 0$, where $coM(\alpha)$ is the smallest convex set that contains $M(\alpha)$.
(C) g' is continuous, bounded and satisfies condition (S_+), that is; as $n \to \infty$, $u_n \rightharpoonup u$, $g'(u_n) \rightharpoonup v$, $\langle g'(u_n), u_n\rangle \to \langle v, u\rangle$ implies $u_n \to u$.
(D) the level set $M(\alpha)$ is bounded and $u \neq 0$ implies

$$\langle g'(u), u\rangle>0, \ \lim_{t\to+\infty}(g(tu)) = +\infty, \ \text{and} \ \inf_{u\in M(\alpha)}\langle g'(u), u\rangle>0.$$

The pair (u, μ) is an eigenpair of the equation (3.3), where μ is an eigenvalue and u is the associated nontrivial eigenfunction. It is known that (u, μ) solves equation (3.3) if and only if u is a critical point of f with respect to $M(\alpha)$, see for example [17].
For any positive integer n, denote by

$$\mathbb{A}_n(\alpha) := \big\{A \subset M(\alpha) : A \ \text{ is compact, symetric } (-\text{A=A}) \text{ and } \ \gamma(A) \geq n\big\},$$

where $\gamma(A)$ denotes the genus of A, that is,

$$\gamma(A) := \inf\big\{k \in \mathbb{N} : \text{there exists } \; h : A \to \mathbb{R}^{\mathrm{k}} - \{0\} \mid h \text{ is continuous and odd}\big\}.$$

Define:

$$a_n(\alpha) = \begin{cases} \sup_{A\in\mathbb{A}_n(\alpha)} \; \inf_{u\in A}(f(u)); & \text{if } \; \mathbb{A}_n(\alpha) \neq \emptyset, \\ 0; & \text{if } \; \mathbb{A}_n(\alpha) = \emptyset \end{cases} \tag{3.4}$$

Also let

$$\mathcal{X}(\alpha) = \begin{cases} \sup\{n \in \mathbb{N} : a_{(n}(\alpha)>0\}, & \text{if } \; a_1(\alpha)>0, \\ 0 & \text{if } \; a_1(\alpha) = 0. \end{cases}$$

Proposition 3.2. *(Ljusternik-Schnirelmann principle) Under assumptions (A) to (D), for any given $\alpha>0$, the following assertions hold:*

(1) If $a_n(\alpha)>0$, then (3.3) possesses a pair of eigenfunctions $\pm u_n(\alpha)$ and an eigenvalue $\mu_n(\alpha) \neq 0$; furthermore, $f(u_n(\alpha)) = a_n(\alpha)$.
(2) If $\mathcal{X}(\alpha) = \infty$, (3.3) has infinitely many pairs of eigenfunctions $\pm u$ corresponding to nonzero eigenvalues.
(3) $\infty>a_1(\alpha) \geq a_2(\alpha) \geq \cdots \geq 0$ and $a_n(\alpha) \to 0$ as $n \to \infty$.
(4) If $\mathcal{X}(\alpha) = \infty$ and $f(u) = 0$, $u \in \overline{coM(\alpha)}$ implies $\langle f(u), u\rangle = 0$, then there exists an infinite sequence $\{\mu_n(\alpha)\}$ of distinct eigenvalues of equation (3.3) such that $\mu_n(\alpha) \to 0$ as $n \to \infty$.
(5) Assume that $f(u) = 0$, $u \in \overline{coM(\alpha)}$ implies $u = 0$. Then $\mathcal{X}(\alpha) = \infty$ and there exists a sequence of eigenpairs $(u_n(\alpha), \mu_n(\alpha))$ of equation (3.3) such that $u_n(\alpha) \rightharpoonup 0$, $\mu_n(\alpha) \to 0$ as $n \to \infty$ and $\mu_n(\alpha) \neq 0, \forall n \in \mathbb{N}$.

Proposition 3.3. *(see, e.g [5, 7]) The mapping $f' : X \to X^*$ is sequentially weakly-strongly continuous, that is, $u_n \rightharpoonup u_0$ in X implies $f'(u_n) \to f'(u_0)$ in X^**

Proposition 3.4. *(see, e.g [5, 7]) Let $\Phi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx$. Then,*

(i) *The mapping $\Phi' : \mathbf{X} \to X^*$ is a strictly monotone, bounded homeomorphism and is of type (S_+), that is, if $u_n \rightharpoonup u_0$ in $\mathbf{X}$ and $\limsup_{n\to\infty}\langle(\Phi'(u_n)\Phi'(u_0)), (u_n - u_0)\rangle \leq 0$, then $u_n \to u_0$ in X.*
(ii) *$(\Phi')^{-1}$ is bounded.*

Remark 3.5. A sum of a mapping of type $(\mathbf{S}_+)$ and a weakly-strongly continuous mapping is still of type $(\mathbf{S}_+)$ as seen for example in [10].

Lemma 3.6. [10, Theorem 3.1]

(i) *The mapping $f' : X \to X^*$ is a strictly monotone, bounded homeomorphism and is of type (S_+)*
(ii) *$(f')^{-1}$ is bounded*

Lemma 3.7. *Let g be defined as before, then $B = g'$ satisfies (C) of Ljusternik-Schnirelmann principle. The proof follows directly from that of* [6, Lemma 3.2]

3.1. **Existence of Eigenvalues.** From now onwards, we shall assume that;

(a) $p \in C_0(\bar{\Omega})$ and $p_{-}>1$.
(b) $q \in L^{\infty}(\Omega)$ with $q_{-} \geq 1$ and $q(x) \ll p^*(x)$.

By $q(x) \ll p^*(x)$, it means that there is an $\varepsilon>0$ such that $q(x)+\varepsilon \leq p^*(x)$, $x \in \Omega$ almost everywhere.

Proposition 3.8. [12, Theorem 2.3] *Suppose that $p, q \in C_0(\bar{\Omega})$ and $1 \leq q(x)<p^*(x)$ $\forall\ x \in \bar{\Omega}$. Then,the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.*

Lemma 3.9. *Suppose that $p \in C_0(\bar{\Omega})$ and $q \in L^{\infty}(\Omega)$ with $q_{-} \geq 1$. Then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact provided $q(x) \ll p^*(x)$.*

Proof. If $q \in C_0(\bar{\Omega})$, then $q(x) \ll p^*(x)$ if and only if $q(x)<p^*(x)$ $\forall x \in \Omega$.
Also if $q \in L^{\infty}(\Omega)$, then $q(x) \ll p^*(x)$ if and only if there exists $r \in C_0(\bar{\Omega})$ such that $q(x) \leq r \ll p^*(x)$. Using the above facts , the proof of the Lemma follows by proposition 3.8

□

Let $X = W^{1,p(x)}(\Omega)$ with the norm given by equation (2.2).

Definition 3.10. A pair $(u, \lambda) \in X \times \mathbb{R}$ is called a weak solution of problem (1.1) if

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2} uv \, dS = \lambda \int_{\Omega} |u|^{q(x)-2} uv \, dx,\ \forall v \in X \tag{3.5}$$

If (u,λ) is a solution of (1.1) and $u \in X\backslash\{0\}$, then λ is called an eigenvalue of (1.1) and u is an eigenfunction corresponding to λ.

It follows from the variational principle that

$$\lambda_{\min} = \min_{u \in X, u\neq 0} \frac{\int_{\Omega} |\nabla u|^{p(x)} \, dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)} \, dS}{\int_{\Omega} |u|^{q(x)} \, dx}. \tag{3.6}$$

Theorem 3.11. *Let $\beta(x) > 0$ $\forall x \in \partial\Omega$ and $\beta(x) \in L^{\infty}(\Omega)$. Then $\lambda_{\min} > 0$ and the corresponding eigenfunction $u_{min} > 0$.*

Proof. The proof of this follows directly from the proofs of [14, theorem 3.1] and [14, Theorem 3.2]. □

Let f and g be functionals on X defined by

$$\begin{aligned} f(u) &= \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} \, dx,\ u \in X \\ g(u) &= \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \int_{\partial\Omega} \frac{1}{p(x)} \beta(x)|u|^{p(x)} \, dS,\ u \in X \end{aligned}$$

restricted to the level set $M(\alpha) = \{u \in X : g(u) = \alpha\}$ for $\alpha > 0$. Then, $f, g \in C^1(X, \mathbb{R})$ and

$$\begin{aligned}\langle f'(u), v\rangle &= \int_\Omega |u|^{q(x)-2}uv\,dx,\ \forall v\in X\\ \langle g'(u), v\rangle &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx + \int_{\partial\Omega}\beta(x)|u|^{p(x)-2}uv\,dS,\ \forall v\in X,\end{aligned}$$

where f' and g' are the weak derivatives of f and g respectively. The set $M(\alpha)$ is a C^1 sub-manifold since α is a regular value of g.

Let $\Sigma_n(\alpha) = \{A\subset X : A\,\text{is compact, symmetric and}\ \gamma(A)\geq n\}$,
$n = 1, 2, ...$
where $\gamma(A)$ is the genus of A defined by

$$\gamma(A) := \inf\{k\in\mathbb{N} : \text{there is}\, h : A\to\mathbb{R}^k - \{0\}\mid h \text{ is continuous and odd}\}.$$

Also let

$$a_n(\alpha) = \sup_{A\in\Sigma_n(\alpha)}\inf_{u\in A}(f(u)),\ n = 1, 2, \ldots.$$

Then, its clear that $a_n(\alpha)>0$ and

$$a_1(\alpha)\geq a_2(\alpha)\geq ...\geq a_n(\alpha)\geq a_{n+1}(\alpha)\geq\ldots. \tag{3.7}$$

Proposition 3.12. [8, Theorem 2.2] *Let $p\in C_0(\bar\Omega)$ and $p_->1$, $q\in L^\infty(\Omega)$ and $q(x)\ll p^*(x)$. Then, for every $n = 1, 2, ...$, $a_n(\alpha)$ is a critical value of f_α on $M(\alpha)$ and $a_n(\alpha)\to 0$ as $n\to\infty$.*

For any $\alpha>0$, and $n = 1, 2, ...$, let

$$\Gamma_n(\alpha) = \{u\in M(\alpha) : \text{u is a critical point of } f_\alpha\,\text{and}\, f_\alpha(u) = a_n(\alpha)\},$$

$$\begin{aligned}\Lambda_n(\alpha) &= \{\lambda(u) : u\in\Gamma_n(\alpha)\}\,\text{and}\\ \mu_n(\alpha) &= \frac{\alpha}{a_n(\alpha)}.\end{aligned} \tag{3.8}$$

Theorem 3.13. *Let $p\in C_0(\bar\Omega)$ and $p_->1$, $q\in L^\infty(\Omega)$ and $q(x)\ll p^*(x)$. Then for each $\alpha>0$ and $n = 1, 2, ...$, $\Lambda_n\to\infty$ as $n\to\infty$.*

Proof. Let $p\in C_0(\bar\Omega)$ and $p_->1$, $q\in L^\infty(\Omega)$ and $q(x)\ll p^*(x)$. Then, it follows from Proposition 3.12 that for every $\alpha>0$ and $n = 1, 2, ...$, $\Gamma_n(\alpha)\neq\emptyset$ and $\mu_n\to\infty$ as $n\to\infty$.
Thus, with $\alpha>0$ and $\lambda\in\Lambda_n(\alpha)$, there is a $u\in\Gamma_n(\alpha)$ such that

$$\lambda = \frac{\int_\Omega|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\beta(x)|u|^{p(x)}\,dS}{\int_\Omega|u|^{q(x)}\,dx}. \tag{3.9}$$

But $\forall x\in\Omega$, $p(x)\leq p_+$ implying that $\frac{p_+}{p(x)}\geq 1$ and $q_-\leq q(x)$ implying that $\frac{q_-}{q(x)}\leq 1$. Thus,

$$\begin{aligned}
\lambda &= \frac{\int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\beta(x)|u|^{p(x)}\,dS}{\int_\Omega |u|^{q(x)}\,dx} \\
&\leq \frac{p_+\left\{\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\frac{1}{p(x)}\beta(x)|u|^{p(x)}\,dS\right\}}{q_-\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx} \\
\therefore \lambda(u) &\leq \frac{p_+}{q_-}\mu_n(\alpha). \qquad (3.10)
\end{aligned}$$

Again, $\forall x \in \Omega$, $p_- \leq p(x)$ such that $\frac{p_-}{p(x)} \leq 1$ and $q_+ \geq q(x)$ such that $\frac{q_+}{q(x)} \geq 1$. Thus,

$$\begin{aligned}
\lambda &= \frac{\int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\beta(x)|u|^{p(x)}\,dS}{\int_\Omega |u|^{q(x)}\,dx} \\
&\geq \frac{p_-\left\{\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\frac{1}{p(x)}\beta(x)|u|^{p(x)}\,dS\right\}}{q_+\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx} \\
\therefore \frac{p_-}{q_+}\mu_n(\alpha) &\leq \lambda(u). \qquad (3.11)
\end{aligned}$$

Consequently from inequalities (3.10) and (3.11), one has

$$\frac{p_-}{q_+}\mu_n(\alpha) \leq \Lambda_n(\alpha) \leq \frac{p_+}{q_-}\mu_n(\alpha). \qquad (3.12)$$

Since $\mu_n(\alpha) \to \infty$ as $n \to \infty$, then the squeeze Theorem for limits of sequences implies that $\Lambda_n(\alpha) \to \infty$ as $n \to \infty$ for each $\alpha>0$.

□

3.2. **The First Eigenvalue.** In this section, we prove whether the first eigenvalue, λ_1, is a principal eigenvalue of (1.1). Let

$$\Lambda = \{\lambda : \lambda \text{ is an eigenvalue of the problem (1.1) }\}. \qquad (3.13)$$

Definition 3.14. $\lambda \in \Lambda$ is a principal eigenvalue of (1.1) if there is a $u>0$ such that (u, λ) is a solution of (1.1).

Suppose that $p \in C_0(\bar{\Omega})$ with $p_->1$, $q \in L^\infty(\Omega)$ with $q_- \geq 1$ and $q(x) \ll p^*(x)$. Then, it follows from (3.7) that for $\alpha>0$,

$$a_1(\alpha) = \sup_{u \in M(\alpha)} f(u).$$

Now, define $\Lambda^+ = \{\lambda \in \mathbb{R} : \lambda \text{ is a principal eigenvalue of (1.1)}\}$. Then, we have the following theorem.

Theorem 3.15. *For every $\alpha>0$, $\Lambda_1(\alpha) \subset \Lambda^+$.*

Proof. Suppose that $\alpha>0$ and $\lambda\in\Lambda_1(\alpha)$. Then there exists a $u\in M(\alpha)$ such that $f(u)=a_1(\alpha)=\sup_{u\in M(\alpha)}f(u)$ with

$$\lambda=\frac{\int_\Omega|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}\beta(x)|u|^{p(x)}\,dS}{\int_\Omega|u|^{q(x)}\,dx}$$

Suppose also that $v(x)=|u(x)|\quad\forall x\in\Omega$. Then, $g(v)=g(u)=\alpha$ and $f(v)=f(u)=a_1(\alpha)$, $\int_\Omega|\nabla v|^{p(x)}\,dx+\int_{\partial\Omega}\beta(x)|v|^{p(x)}\,dS=\int_\Omega|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}\beta(x)|u|^{p(x)}\,dS$ and $\int_\Omega|v|^{q(x)}\,dx=\int_\Omega|u|^{q(x)}\,dx$. Therefore, $v>0$, $v\in M(\alpha)$, $v\in\Gamma_1(\alpha)$ gives

$$\lambda=\frac{\int_\Omega|\nabla v|^{p(x)}\,dx+\int_{\partial\Omega}\beta(x)|v|^{p(x)}\,dS}{\int_\Omega|v|^{q(x)}\,dx}.$$

Thus, it follows that (v,λ) is a solution of (1.1) and $\lambda\in\Lambda^+$ and the proof is complete. Therefore, λ_1 is a principal eigenvalue since $\Lambda_1(\alpha)\subset\Lambda^+$ for every $\alpha>0$.

□

3.3. **Closeness of the Set of Eigenvalues.** In this section, we check whether or not the set of eigenvalues of (1.1) is closed. To achieve this, we need to show that the infimum of the set of eigenvalues of (1.1) is not zero, since zero is not an eigenvalue.
Let $\lambda_*=\inf\Lambda$ where Λ is as in (3.13). Then,

Theorem 3.16. *Suppose that there exists an open set $U\subset\Omega$ and a compact set $E\subset U$ with a positive Lebesgue measure such that $q_+(E)<p_-(\partial U)$, $x\in\partial U$. Suppose also that there are $x_0\in\Omega$ and an open neighbourhood G of x_0 such that $q\in C_0(G)$ and $q(x_0)<p(x_0)$. Then, as $\alpha\to0$, $\mu_1(\alpha)\to0$ and $\Lambda_1(\alpha)\to0$. Thus,* $\inf\Lambda=0$, *that is,*$\lambda_*=0$.

Proof. Let $U_\varepsilon=\{x\in\Omega: dist(x,\partial U)<\varepsilon\}$, for $\varepsilon>0$. Then, there is an $\varepsilon>0$ small enough such that $E\cap U_\varepsilon=\emptyset$ and $q_+(x_0)<p_-(x)$. Thus if $\varepsilon_0=p_-(x)-q_+(x_0)$, then $\varepsilon_0>0$. Take a function $u_0\in C^\infty(\Omega)$ such that for any $x\in\Omega$,

$$u_0(x)=\begin{cases}0, & \text{for}\quad x\in\Omega\backslash U\\ 1, & \text{for}\quad x\in U\backslash U_\varepsilon\end{cases}$$

Given any $\alpha>0$, there exists a $t>0$ such that $tu_0\in M(\alpha)$, that is, $g(tu_0)=\alpha$. Moreover, $t\to0$ as $\alpha\to0$. Indeed,

$$\begin{aligned}g(tu_0)&=\int_\Omega\frac{1}{p(x)}|\nabla tu_0|^{p(x)}\mathrm{dx}+\int_{\partial\Omega}\frac{1}{p(x)}\beta(x)|tu_0|^{p(x)}\mathrm{dS}=\alpha\\&=\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{dx}+\int_{\partial\Omega}\frac{1}{p(x)}\beta(x)|u|^{p(x)}\mathrm{dS}.\end{aligned}$$

Thus,

$$\begin{aligned}&t^{p(x)}\left[\int_\Omega\frac{1}{p(x)}|\nabla u_0|^{p(x)}\mathrm{dx}+\int_{\partial\Omega}\frac{1}{p(x)}\beta(x)|u_0|^{p(x)}\mathrm{dS}\right]\\&=\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{dx}+\int_{\partial\Omega}\frac{1}{p(x)}\beta(x)|u|^{p(x)}\mathrm{dS}\end{aligned}$$

giving

$$\begin{aligned} t^{p(x)} &= \frac{\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{dx} + \int_{\partial\Omega}\frac{1}{p(x)}\beta(x)|u|^{p(x)}\mathrm{dS}}{\int_\Omega \frac{1}{p(x)}|\nabla u_0|^{p(x)}\mathrm{dx} + \int_{\partial\Omega}\frac{1}{p(x)}\beta(x)|u_0|^{p(x)}\mathrm{dS}} \\ &= \frac{\alpha}{g(u_0)}. \end{aligned}$$

Thus, one gets

$$t = \left\{\frac{\alpha}{g(u_0)}\right\}^{\frac{1}{p(x)}}$$

and

$$\lim_{\alpha\to 0} t = \lim_{\alpha\to 0}\left\{\frac{\alpha}{g(u_0)}\right\}^{\frac{1}{p(x)}} = 0.$$

Now, suppose that $\alpha>0$ is small enough such that $0<t<1$. Then,

$$\begin{aligned} \mu_1(\alpha) &= \frac{\alpha}{a_1(\alpha)} \\ &= \frac{g(tu_0)}{f(tu_0)} \\ &= \frac{\int_\Omega \frac{1}{p(x)}|\nabla tu_0|^{p(x)}dx + \int_{\partial\Omega}\frac{1}{p(x)}\beta(x)|tu_0|^{p(x)}\mathrm{dS}}{\int_\Omega \frac{1}{q(x)}|tu_0|^{q(x)}\mathrm{dx}} \\ &= \frac{\int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla u_0|^{p(x)}dx + \int_{\partial\Omega}\frac{t^{p(x)}}{p(x)}\beta(x)|u_0|^{p(x)}\mathrm{dS}}{\int_\Omega \frac{t^{q(x)}}{q(x)}|u_0|^{q(x)}\mathrm{dx}} \\ &= \frac{\int_{U\setminus U_\varepsilon} \frac{t^{p(x)}}{p(x)}|\nabla u_0|^{p(x)}dx + \int_{U_\varepsilon}\frac{t^{p(x)}}{p(x)}|\nabla u_0|^{p(x)}dx + \int_{\partial U_\varepsilon}\frac{t^{p(x)}\beta(x)}{p(x)}|u_0|^{p(x)}\mathrm{dS}}{\int_E \frac{t^{q(x)}}{q(x)}|u_0|^{q(x)}\mathrm{dx} + \int_{U\setminus E}\frac{t^{q(x)}}{q(x)}|u_0|^{q(x)}\mathrm{dx}} \\ &\leq \frac{\int_{U_\varepsilon}\frac{t^{p(x)}}{p(x)}|\nabla u_0|^{p(x)}dx + \int_{\partial U_\varepsilon}\frac{t^{p(x)}\beta(x)}{p(x)}|u_0|^{p(x)}\mathrm{dS}}{\int_E \frac{t^{q(x)}}{q(x)}|u_0|^{q(x)}\mathrm{dx}}. \end{aligned}$$

But

$$t^{p(x)} \leq t^{p_-(\bar{U_\varepsilon})},\ \forall x\in U_\varepsilon \text{ and } t^{q_+(E)} \leq t^{q(x)}\ \forall x\in E.$$

Thus,

$$\begin{aligned} \mu_1(\alpha) &\leq \frac{t^{p_-(U_\varepsilon)}\left(\int_{U_\varepsilon}\frac{1}{p(x)}|\nabla u_0|^{p(x)}dx + \int_{\partial U_\varepsilon}\frac{\beta(x)}{p(x)}|u_0|^{p(x)}\mathrm{dS}\right)}{t^{q_+(E)}\int_E \frac{1}{q(x)}|u_0|^{q(x)}\mathrm{dx}} \\ &= t^{p_-(U_\varepsilon)-q_+(E)}\left[\frac{\int_{U_\varepsilon}\frac{1}{p(x)}|\nabla u_0|^{p(x)}dx + \int_{\partial U_\varepsilon}\frac{\beta(x)}{p(x)}|u_0|^{p(x)}\mathrm{dS}}{\int_E \frac{1}{q(x)}|u_0|^{q(x)}\mathrm{dx}}\right] \\ &= t^{\varepsilon_0}\left[\frac{\int_{U_\varepsilon}\frac{1}{p(x)}|\nabla u_0|^{p(x)}dx + \int_{\partial U_\varepsilon}\frac{\beta(x)}{p(x)}|u_0|^{p(x)}\mathrm{dS}}{\int_E \frac{1}{q(x)}|u_0|^{q(x)}\mathrm{dx}}\right], \end{aligned}$$

from which it follows that $\mu_1(\alpha) \to 0$ as $\alpha \to 0$. Thus, by (3.12), $\Lambda_1(\alpha) \to 0$ as $\alpha \to 0$. Hence $\lambda_* = \inf \Lambda = 0$. □

Therefore from the above theorem, one can see that if there exists an open set $U \subset \Omega$ and a compact set $E \subset U$ such that $q_+(E) < p_-(\partial U), \quad x \in \partial U$, or if there exists an open set $U \subset \Omega$ and a compact set $E \subset U$ such that $q_-(E) < p_+(\partial U)$, then $\lambda_* = 0$. Therefore, the infimum of the set of eigenvalues of (1.1) is zero and so, the set of eigenvalues is not closed since zero is not an eigenvalue.

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