Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 34 (2026) 1-12 https://doi.org/10.56947/amcs.v34.799

OUTER-INDEPENDENT CONVEX ROMAN DOMINATION IN GRAPHS

LEOMARICH F. CASINILLO*

Abstract. In this paper, we initiate the study of a new restricted parameter of convex Roman domination in graphs, called the outer-independent convex Roman dominating function, and discuss some of its combinatorial properties.

Keywords. Outer-independent domination, Roman domination, convex Roman domination, outer-independent convex Roman domination
2020 Mathematics Subject Classification. 05C69

1. Introduction

Around the fourth century A.D., the Roman Empire was attacked by its enemies, and hence, Emperor Constantine developed a defense strategy to protect the entire region [14]. The strategy used a limited number of legions (a large number of soldiers) that cannot transfer from one place (location) unless at least one legion remains. This strategy ensures that no place is left undefended. In other words, if one place does not have a legion, then it must be adjacent to a place with two legions. Cockayne et al. [10] in 2004, was inspired by the defense strategy of Emperor Constantine, which can be found in the article of Stewart [17] entitled "Defend the Roman Empire!" and developed a new parameter of domination in graphs called the Roman dominating function. Roman dominating function in graphs is a graph-theoretic concept based on the Roman Empire's defense strategies, where legions are positioned at vertices (places) to protect some adjacent unsecured (no legion) places. A function labels 0, 1, or 2 to the vertices of a graph and requires vertices with label 0 to be adjacent to vertices with label 2 (two legions), while vertices with label 1 do not need to. Nowadays, the concept of Roman dominating function in graphs is studied by many graph theorists and computer scientists, and some notable papers can be found in [1, 2, 7, 8, 16]. Recently, in 2023, Fortosa and Canoy [11] has developed a new variation of Roman domination in graphs, which is called the convex Roman dominating function. Moreover, outer-independent domination in graphs is also extensively studied, and several mathematicians have contributed to the

---

Date: Received: Apr 24, 2026; Revised: May 15, 2026; Accepted: May 20, 2026.

* Corresponding author

topic [4, 15]. Given the two interesting topics, namely convex Roman domination and outer-independent domination, the author is inspired to initiate a new study called outer-independent convex Roman dominating function in graphs. The study's new feature is that the dominating sets (labeled 1 or 2) are closer to each other while defending scattered regions (labeled 0). This concept can be integrated in the study of computer science, particularly in network security, which focus on strategic resource allocation, that is, optimizing the placement of resources in independent places.

2. Preliminaries

Let $G = (V, E)$ be a connected graph. The set V is called the vertex set, and the set E is called the edge set of G. The cardinality of V is denoted by $|V|$, and is called the order of G. The cardinality of E is denoted by $|E|$, and is called the size of G. Let $x \in V$. The *open neighborhood* of x in G is the set $N_G(x) = \{y \in V : xy \in E\}$ and the *closed neighborhood* of a vertex x in G is the set $N_G[x] = N_G(x) \cup \{x\}$. Let $O \subseteq V$. The set $N_G(O) = N(O) = \bigcup_{x \in O} N_G(x)$ is called the *open neighborhood* of O and the set $N_G[O] = N[O] = N(O) \cup O$ is called the *closed neighborhood* of O in G. Let $x, y \in V$. Then, the distance between x and y, denoted by $d_G(x, y)$, is the length of the shortest path between x and y in G. If there is no such path between x and y in G, then the distance is defined as $d_G(x, y) = \infty$. An x-y path with distance $d_G(x, y) > 0$ is called *x-y geodesic*. Let $I_G[x, y]$ be a set of closed intervals that consists of all vertices that lie on an x-y geodesic on G, and let $X \subseteq V$. The union of all sets $I_G[x, y]$ where $x, y \in X$ is denoted by $I_G[X]$, that is, $\cup_{x,y \in X} I_G[x, y] = I_G[X]$. Then $z \in I_G[X]$ if and only if z is in some x-y geodesic for any $x, y \in X$. A set X is a *convex set* in G if and only if $I_G[X] = X$ [11]. Additionally, the vertex set V is a convex set if G is a connected graph. Let $I \subseteq V$. Then I is an *independent vertex set* on G if for every $a, b \in I$, $ab \notin E$ [6].

Let $D \subseteq V$. Then D is called a *dominating set* of G provided that for every $x \in V \setminus D$, there exists $y \in D$ such that $d_G(x, y) = 1$ [13]. The *domination number* of G is denoted by $\gamma(G)$, and defined as the minimum cardinality of a dominating set D on G. If D is a dominating set satisfying the condition $|D| = \gamma(G)$, then D is called the *minimum dominating set* of G or a γ-set in G. For studies on domination in graphs can be found in [3, 5, 6, 13, 15]. Let $I_d \subseteq V$. Then I_d is called an *outer-independent dominating set* of G provided that for every $x \in V \setminus I_d$, there exists $y \in I_d$ such that $d_G(x, y) = 1$ and $V \setminus I_d$ is an independent vertex set on G [15]. The smallest cardinality of an outer-independent dominating set I_d in G is called the *outer-independent domination number* of G, denoted by $\widetilde{\gamma}_i(G)$. An independent dominating set I_d satisfying the condition $|I_d| = \widetilde{\gamma}_i(G)$ is a called a $\widetilde{\gamma}_i$-set on G.

Let $\lambda : V \to \{0, 1, 2\}$ be a function on G. Then consider the following sets:

$$\begin{aligned} V_0 &= \{v \in V : \lambda(v) = 0\}; \\ V_1 &= \{v \in V : \lambda(v) = 1\}; \textit{and} \\ V_2 &= \{v \in V : \lambda(v) = 2\}. \end{aligned}$$

Hence, we denote λ by $\lambda = (V_0, V_1, V_2)$. A function λ is a *Roman dominating function* (RDF) provided that for each $x \in V_0$ there exists $y \in V_2$ such that $d_G(x, y) = 1$ [10]. The *weight* of an RDF λ is given by the formula $\omega_G^R(f) = \sum_{z \in V} \lambda(z)$. The *Roman domination number* of G, denoted by $\gamma_R(G)$, is the smallest weight of an RDF λ on G, that is, $\gamma_R(G) = min\{\omega_G^R(\lambda) : \lambda \ is \ an$ RDF $on \ G\}$. All RDF λ on G with $\omega_G^R(\lambda) = \gamma_R(G)$ is called a γ_R-function on G. A function λ is a convex Roman dominating function (CvRDF) on G provided that for every $x \in V_0$, there exists $y \in V_2$ such that $d_G(x, y) = 1$, and $V_1 \cup V_2$ is a convex set [11]. The weight of CvRDF λ denoted by $\omega_G^{CvR}(f)$ is the sum $\omega_G^{CvR}(f) = \sum_{v \in V(G)} \lambda(v)$, that is, $\omega_G^{CvhR}(f) = |V_1| + 2|V_2|$. The convex Roman domination number of G, denoted by $\gamma_{CvhR}(G)$, is the minimum weight of an CvRDF on G, that is, $\gamma_{CvR}(G) = min\{\omega_G^{CvR}(\lambda) : \lambda \ is \ an$ CvRDF $on \ G\}$. All CvRDF λ on G satisfying $\omega_G^{CvR}(\lambda) = \gamma_{CvR}(G)$ is called a γ_{CvR}-function on G.

A function λ is an outer-independent convex Roman dominating function (OiCvRDF) on G if (i) for each $v \in V_0$, there exists $u \in V_2$ such that $d_G(u, v) = 1$, (ii) V_0 is an independent set and (iii) $V_1 \cup V_2$ is a convex set on G. The weight of OiCvRDF λ is denoted by $\widetilde{\omega}_G^{iCvR}(\lambda)$ and is defined as the sum $\widetilde{\omega}_G^{iCvR}(\lambda) = \sum_{x \in V(G)} \lambda(x) = |V_1| + 2|V_2|$. The outer-independent convex Roman domination number of graph G is denoted by $\widetilde{\gamma}_{iCvR}(G)$ and is defined as the smallest weight of an OiCvRDF λ on G, that is, $\widetilde{\gamma}_{iCvR}(G) = min\{\widetilde{\omega}_G^{iCvR}(\lambda) : \lambda \ is \ an$ OiCvRDF $on \ G\}$. All OiCvRDF λ on G that satisfies $\widetilde{\omega}_G^{iCvR}(\lambda) = \widetilde{\gamma}_{iCvR}(G)$ is called a $\widetilde{\gamma}_{iCvR}$-function on G. Consider the graph G of order $n = 8$ in the Figure 1. Let $\lambda = (V_0, V_1, V_2)$ be an RDF on G such that $V_0 = \{v_3, v_2, v_4, v_7, v_8\}$, $V_1 = \{v_1\}$, and $V_2 = \{v_5, v_6\}$. Clearly, λ is a $\widetilde{\gamma}_{iCvR}$-function on G. Hence, we have $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| + 2|V_2| = |\{v_1\}| + 2|\{v_5, v_6\}| = 5$.

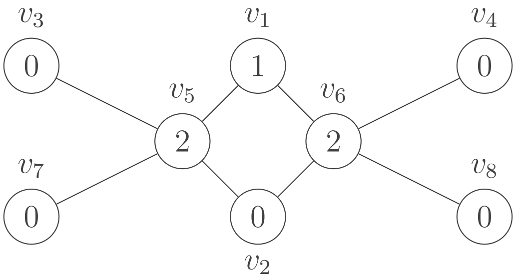

FIGURE 1. A graph G with $|V(G)| = 8$ and $\widetilde{\gamma}_{iCvR}(G) = 5$

The needed concepts and definition of terms in this paper which are not mention here, can be found in the following studies: [9, 12, 13]. In this paper, a new restricted version of convex Roman domination called the outer-independent convex Roman domination in graphs is introduced and initially investigated. The paper explored the graph-theoretic properties of outer-independent convex Roman domination in some graph classes, and obtained some bounds and construct a realization problem.

3. RESULTS

In this section, some combinatorial and graph-theoretic properties of the outer-independent convex Roman dominating function in graphs were investigated. It is worth noting that for a non-connected graph, the definition of OiCvRDF does not hold. Hence, this study only considers connected graphs.

Theorem 3.1. *Let G be a connected graph of order $n \geq 1$ and let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. If $\widetilde{\gamma}_{iCvR}(G) < n$, then $|V_0| \geq 2$, $\langle V_0 \rangle = \overline{K_{|V_0|}}$, and $I_G[V_1 \cup V_2] = V_1 \cup V_2$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. Then λ is an OiCvRDF on G. Assume that $\widetilde{\gamma}_{iCvR}(G) < n$. Then we have that $\widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| + 2|V_2| < |V(G)| = |V_0| + |V_1| + |V_2|$. Thus, $|V_2| < |V_0|$. Since $\widetilde{\gamma}_{iCvR}(G) < n$, $V_0 \neq \varnothing$. Let $x \in V_0$. Since λ is an OiCvRDF on G, there exists $y \in V_2$ such that $d_G(u, v) = 1$, V_0 is an independent set, and $V_1 \cup V_2$ is a convex set on G. In that case, $|V_2| > 0$. Since $|V_2| < |V_0|$, it means that $|V_0| \geq 2$. Additionally, it follows that for every $a, b \in V_0$, $ab \notin E$. Hence, we have $\langle V_0 \rangle = \overline{K_{|V_0|}}$. Moreover, it also implies that for every $u, v \in V_1 \cup V_2$, u-v walk is a u-v geodesic and so, $V_1 \cup V_2 = \cup_{u,v \in V_1 \cup V_2} I_G[u, v] = I_G[V_1 \cup V_2]$. This completes the proof. □

Corollary 3.2 below is quick from Theorem 3.1.

Corollary 3.2. *Let G be a connected graph of order $n \geq 1$ and let $\lambda = (V_0, V_1, V_2)$ be $\widetilde{\gamma}_{iCvR}$-function on G. Then $\widetilde{\gamma}_{iCvR}(G) < n$ if and only if $1 \leq |V_2| < |V_0|$.*

Theorem 3.3. *Let G be a connected graph. If $\lambda = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{iCvR}$-function on G, then $V_1 \cup V_2$ is an outer-independent convex dominating set on G.*

Proof. Assume that $\lambda = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{iCvR}$-function on G. Then λ is an OiCvRDF on G. Further, λ is an RDF on G, which implies that $V_1 \cup V_2$ is a dominating set on G. Let $v \in V_0$. Then, by the definition of OiCvHRDF on a graph, it follows that there exists $u \in V_2$ such that $d_G(u, v) = 1$, V_0 is an independent set, and $V_1 \cup V_2$ is a convex set on G. Since $V_0 = V(G) \setminus (V_1 \cup V_2)$ and $V_1 \cup V_2$ is a dominating set on G, it is concluded that $V_1 \cup V_2$ is an outer-independent convex dominating set on G. Which proves the assertion. □

Corollary 3.4. *Let G be a connected graph. If $\lambda = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{iCvR}$-function on G, then $V_1 \cup V_2$ is not necessary a $\widetilde{\gamma}_{iCv}$-set on G.*

To see Corollary 3.4, we consider a counterexample. Let $G = P_4 = [x_1, x_2, x_3, x_4]$ and let $\lambda = (V_0, V_1, V_2)$ be a Roman dominating function on G. Then we let $V_0 = \{x_1\}$, $V_1 = \{x_3, x_4\}$ and $V_2 = \{x_2\}$. In that case, V_0 is a trivial independent set and $V_1 \cup V_2$ is a convex set on G. This implies that $\lambda = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{iCvR}$-function on G. Let D be an outer-independent convex dominating set. Then it is clear that $D = \{x_2, x_3\}$. Thus, $V_1 \cup V_2$ is not a $\widetilde{\gamma}_{iCv}$-set on G. Hence, if we allow $V_1 = \varnothing$, then the following results hold.

Theorem 3.5. *Let G be a connected graph and let $\lambda = (V_0, V_1 = \varnothing, V_2)$ be an OiCvRDF on G. If V_2 is a $\widetilde{\gamma}_{iCv}$-set on G, then λ is $\widetilde{\gamma}_{iCvR}$-function on G.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be an OiCvRDF on G and let $V_1 = \varnothing$. Assume that V_2 is a $\widetilde{\gamma}_{iCv}$-set on G. Suppose, for the contrary that λ is not a $\widetilde{\gamma}_{iCvR}$-function on G. Then there exists $\lambda' = (W_0, W_1 = \varnothing, W_2)$ such that λ' is a $\widetilde{\gamma}_{iCvR}$-function on G. In that case, we get $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda') = |W_1| + 2|W_2| = 2|W_2| < 2|V_2| = |V_1| + 2|V_2| = \widetilde{\omega}_G^{iCvR}(\lambda)$ and so, $|W_2| < |V_2|$, a contradiction. Thus, we have that λ is a $\widetilde{\gamma}_{iCvR}$-function on G. Which proves the assertion. □

The converse of Theorem 3.5 is presented in the next theorem.

Theorem 3.6. *Let G be a connected graph and let $\lambda = (V_0, V_1 = \varnothing, V_2)$ be an OiCvRDF on G. If λ is a $\widetilde{\gamma}_{iCvR}$-function on G, then V_2 is a $\widetilde{\gamma}_{iCv}$-set on G.*

Proof. Assume that $\lambda = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{iCvR}$-function on G for which $V_1 = \varnothing$. By Theorem 3.3, V_2 is an outer-independent convex dominating set on G. Suppose, for a contrary that, V_2 is not a $\widetilde{\gamma}_{iCv}$-set on G. Then there exists a set $V_2' < V_2$ such that it is a $\widetilde{\gamma}_{iCv}$-set on G. Define a function $\lambda' = (U_0, U_1, U_2)$ on G where $U_0 = V \setminus V_2'$, $U_1 = \emptyset$ and $U_2 = V_2'$. Clearly, λ' is an OiCvRDF on G. Hence, $\widetilde{\omega}_G^{iCvR}(\lambda') = |U_1| + 2|U_2| = 2|U_2| < 2|V_2| = |V_1| + 2|V_2| = \widetilde{\omega}_G^{iCvR}(\lambda) = \widetilde{\gamma}_{iCvR}(G)$, a contradiction. Therefore, V_2 is a $\widetilde{\gamma}_{iCv}$-set on G. Which proves the assertion. □

The following corollary is a direct consequence of Theorem 3.6.

Corollary 3.7. *Let G be a connected graph and let $\lambda = (V_0, V_1 = \varnothing, V_2)$ be an OiCvRDF on G. Then $\widetilde{\gamma}_{iCvR}(G) = 2\widetilde{\gamma}_{iCv}(G)$.*

Theorem 3.8. *Let G be a connected graph of order $n \geq 1$ and let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. Then $|V_0| = 0$ if and only if $|V_2| = 0$. In that case, $\widetilde{\gamma}_{iCvR}(G) = n$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{iCvR}$-function on G. Assume that $|V_0| = 0$. Suppose, for the contrary, $|V_2| > 0$. Let $x \in V_2$ and let $U_0 = V_0$, $U_1 = V_1 \cup \{x\}$, and $U_2 = V_2 \setminus \{x\}$. This follows that $\lambda' = (U_0, U_1, U_2)$ is an outer-independent convex Roman dominating function on G. Hence, $\widetilde{\omega}_G^{iCvR}(\lambda') = |U_1| + 2|U_2| = (|V_1| + 1) + 2(|V_2| - 1) = |V_1| + 2|V_2| - 1 < \widetilde{\omega}_G^{iCvR}(\lambda) = \widetilde{\gamma}_{iCvR}(G)$, a contradiction. Therefore, $|V_2| = 0$. Conversely, assume that $|V_2| = 0$. Since λ is a OiCvRDF on G, it is clear that $|V_0| = 0$. In that case, $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| = |V(G)| = n$. Which proves the assertion. □

Using the idea of Theorem 3.8, the following result is obtained.

Corollary 3.9. *Let G be a connected graph of order $n \geq 1$. Then $\widetilde{\gamma}_{iCvR}(G) \leq n$.*

Proof. Assume that G is any connected graph of order $n \geq 1$. Clearly, the function $\lambda = (\varnothing, V_1 = V, \varnothing)$ is an outer-independent convex Roman dominating function. In that case, $\widetilde{\gamma}_{iCvR}(G) \leq \widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| + 2|V_2| = |V_1| = |V| = n$. And the conclusion follows. □

Theorem 3.10. *Let G be a connected graph of order $n \geq 1$ and let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. Then $\widetilde{\gamma}_{iCvR}(G) = n$ if and only if $|V_2| = |V_0|$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. Assume that $\widetilde{\gamma}_{iCvR}(G) = n$. Suppose, for the contrary, that $|V_2| \neq |V_0|$. Then $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) =$

$|V_1| + 2|V_2| = n - |V_0| + |V_2|$. Since $|V_2| \neq |V_0|$, it follows that $\widetilde{\gamma}_{iCvR}(G) \neq n$, a contradiction. Therefore, $|V_2| = |V_0|$. As for the converse, we assume that $|V_2| = |V_0|$. If $|V_2| = |V_0| = 0$, then by Theorem 3.8, $\widetilde{\gamma}_{iCvR}(G) = n$. If $|V_2| = |V_0| \geq 1$, then $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| + 2|V_2| = |V_0| + |V_1| + |V_2| = |V| = n$. Which proves the assertion. □

The next corollary is an immediate consequence of Theorem 3.10.

Corollary 3.11. *Let G be a connected graph of order $n \geq 1$ and let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. If $|V_0| = 1$, then $\widetilde{\gamma}_{iCvR}(G) = n$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. Then λ is an OiCvRDF on G. Suppose $|V_0| = 1$. Since λ is an OiCvRDF on G, $V_2 \neq \varnothing$. If $|V_2| > 1$, then $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| + 2|V_2| > |V_0| + |V_1| + |V_2| = |V| = n$, a contradiction to Corollary 3.9. This implies that $|V_2| = 1$. By Theorem 3.10, it is concluded that $\widetilde{\gamma}_{iCvR}(G) = n$. Which proves the assertion. □

Proposition 3.12. *Let $G = K_n$ where $n \geq 1$. Then $\widetilde{\gamma}_{iCvR}(G) = n$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on $G = K_n$. Let $n = 1$. Then it is clear that $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| = |V(K_1)| = 1$. Let $n = 2$. Then it is easy to check that $|V_2| = |V_0| = 1$ and $V_1 = \varnothing$. Hence, $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = 2|V_2| = 2$. Now, let $n \geq 3$. Since for every $x, y \in V(G)$, $xy \in E(G)$, and V_0 must be independent set on G, it follows that $|V_0| = 1$. By Corollary 3.11, it is concluded that $\widetilde{\gamma}_{iCvR}(G) = n$. This completes the proof □

Theorem 3.13. *Let G be a connected non-complete graph of order $n \geq 1$ and let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. Then every cut-vertex $v \in V$, $v \in V_1 \cup V_2$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. Suppose, for the contrary, there exists a cut vertex v of G such that $v \notin V_1 \cup V_2$. Let G' and G'' be the distinct components of the graph $G - \{v\}$. Let $v' \in (V_1 \cup V_2) \cap V(G')$ and $v'' \in (V_1 \cup V_2) \cap V(G'')$ such that for every $x, y \in (V_0 \cap (V(G') \cup V(G'')))$, $xy \notin E(G)$. Since v is a cut vertex, every v'-v'' shortest path must contain v. In that case, $[v', v, v'']$ is a v'-v'' geodesic. Hence, if $v \notin V_1 \cup V_2$, then it follows that $V_1 \cup V_2$ is not a convex set. This is a contradiction. Therefore, it suffices to conclude that every cut-vertex $v \in V(G)$, $v \in V_1 \cup V_2$. Which proves the assertion. □

The following corollary is a direct consequence of Theorem 3.13.

Corollary 3.14. *Let G be a connected non-complete graph of order $n \geq 1$ and let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. Then every pendant vertex $u \in V(G)$, $u \in V_0$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. Suppose there exists a pendant vertex $u \in V$ such that $u \notin V_0$. Then either $\lambda(u) = 1$ or $\lambda(u) = 2$. Without loss of generality, let $\lambda(u) = 1$. Note that $deg_G(u) = 1$. Let $v \in V$ such that $N_G(u) = \{v\}$. By Theorem 3.13, $v \in V_1 \cup V_2$. Hence, we can let $U_0 = V_0 \setminus \{u\}$, $U_1 = V_1 \cup \{u\}$, and $U_2 = V_2$. Then define a function $\lambda' = (U_0, U_1, U_2)$ on G.

Assume for a moment that λ' is also a $\widetilde{\gamma}_{iCvR}$-function on G. Observe that

$$\begin{aligned}\widetilde{\omega}_G^{iCvR}(\lambda') &= |U_1| + 2|U_2| \\ &= |V_1 \cup \{u\}| + 2|V_2| \\ &= |V_1| + 1 + 2|V_2| \\ &> |V_1| + 2|V_2| \\ &= \widetilde{\omega}_G^{iCvR}(\lambda) \\ &= \widetilde{\gamma}_{iCvR}(G).\end{aligned}$$

This is a contradiction. Therefore, every pendant vertex $u \in V(G)$, $u \in V_0$. This completes the proof. □

Theorem 3.15. *Let G be a connected non-complete graph of order $n \geq 1$ and let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. If for every $x, y \in V_1 \cup V_2$, $d_G(x, y) > 1$, then every v in x-y geodesic, $v \in V_1 \cup V_2$. Moreover, $N_G(x) \cap N_G(y) \subset V_1 \cup V_2$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. Assume that for every $x, y \in V_1 \cup V_2$, $d_G(x, y) > 1$. Suppose, for the contrary, there exists v in x-y geodesic such that $v \notin V_1 \cup V_2$. This follows that $v \in V_0$ and so $d_{\langle V_1 \cup V_2 \rangle}(x, y) = \infty$. This is a contradiction to the convexity of the set $V_1 \cup V_2$. Hence, for every v in x-y geodesic, $v \in V_1 \cup V_2$. Moreover, if $d_G(x, y) > 2$, then $N_G(x) \cap N_G(y) = \varnothing \subset V_1 \cup V_2$. Now, suppose that $d_G(x, y) = 2$. Then we let $z \in N_G(x) \cap N_G(y)$. Since $V_1 \cup V_2$ is a convex set, it follows that $z \in V_1 \cup V_2$. Therefore, it suffices to conclude that $N_G(x) \cap N_G(y) \subset V_1 \cup V_2$. This completes the proof. □

Using the concepts of Theorems 3.13 and 3.15, and Corollary 3.14, the following propositions are immediate.

Proposition 3.16. *If $G = P_n$ where $n \geq 1$, then*

$$\widetilde{\gamma}_{iCvR}(G) = \begin{cases} 1, & \text{if } n = 1, \\ 2, & \text{if } n \in \{2, 3\}, \\ n, & \text{if } n \geq 4. \end{cases}$$

Proposition 3.17. *If $G = C_n$ where $n \geq 3$, then $\widetilde{\gamma}_{iCvR}(G) = n$.*

The following results are characterizations of outer-independent convex Roman domination number involving small values.

Theorem 3.18. *Let G be a connected graph. Then*

- *(i)* $\widetilde{\gamma}_{iCvR}(G) = 1$ *if and only if* $G = K_1$.
- *(ii)* $\widetilde{\gamma}_{iCvR}(G) = 2$ *if and only if* $G \in \{K_2, P_3, S_n\}$ *where* $n \geq 3$.

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}(G)$-function on G. Assume that $\widetilde{\gamma}_{iCvR}(G) = 1$. Suppose, for the contrary, that $G \neq K_1$. Then $|V| \geq 2$. On one hand, let $|V_0| \neq 0$. By Theorem 3.8, $|V_2| \neq 0$. Hence, we have that $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| + 2|V_2| > 1$. This is a contradiction. On the other hand, let $|V_0| = 0$. By Theorem 3.8, $|V_2| = 0$. Thus, we get $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| + 2|V_2| = |V_1| = |V| > 1$. This is again a contradiction. Therefore, we have $G = K_1$. As for the converse, we assume that $G = K_1$. Since λ is a $\widetilde{\gamma}_{iCvR}(G)$-function on G, it is clear that

$|V_0| = 0 = |V_2|$. Therefore, $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| = |V(K_1)| = 1$. And (i) is proven. Assume that $\widetilde{\gamma}_{iCvR}(G) = 2$. Then $\widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| + 2|V_2| = 2$. In that case, we get $|V_2| \leq 1$. Consider $|V_2| = 0$. By Theorem 3.8, $|V_0| = 0$. So, we have $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| + 2|V_2| = |V_1| = |V| = 2$. Hence, $G = K_2$. Moreover, consider $|V_2| = 1$. Then $|V_1| = 0$. Let $x \in V_2$. Then V_2 is a convex set on G. Set $G = K_1 + \langle V \setminus \{x\} \rangle$ where $V(K_1) = \{x\}$. Let $V_0 = V \setminus \{x\}$ be an independent set. If $|V \setminus \{x\}| = 1$, then $G = K_1 + K_1 = K_2$. If $|V \setminus \{x\}| = 2$, then $G = K_1 + \overline{K_2} = P_3$. If $|V \setminus \{x\}| \geq 3$, then $G = K_1 + \overline{K_{n\geq 3}} = S_n$. Therefore, $\widetilde{\gamma}_{iCvR}(G) = 2$ if and only if $G \in \{K_2, P_3, S_n\}$ where $n \geq 3$. Conversely, we assume that $G \in \{K_2, P_3, S_n\}$ where $n \geq 3$. Then $G = K_1 + H$ where $H = \{K_1, \overline{K_2}, \overline{K_{n\geq 3}}\}$. In that case, we let $V_0 = H$, $V_1 = \varnothing$, and $V_2 = V(K_1)$. By construction, $\lambda = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{iCvR}$-function on G. Therefore, $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| + 2|V_2| = 2|V(K_1)| = 2$. And (ii) is proven. □

Theorem 3.19. *Let G be a non-complete connected graph. Then $\widetilde{\gamma}_{iCvR}(G) \neq 3$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. Suppose, for the contrary, that there exists a connected graph G such that $\widetilde{\gamma}_{iCvR}(G) = 3$. Then $\widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| + 2|V_2| = 3$. Hence, $|V_2| \leq 1$. Consider $|V_2| = 0$. By Theorem 3.8, $|V_0| = 0$. So, we get $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| = |V| = 3$. Since G is a non-complete connected graph, it implies that $G = P_3$. However, by Proposition 3.16, $\widetilde{\gamma}_{iCvR}(P_3) = 2$, a contradiction. Now, consider $|V_2| = 1$. Then $|V_1| = 1$. Let $V_1 = \{a\}$ and $V_2 = \{b\}$. Then $d_G(a, b) = 1$ since $V_1 \cup V_2$ is a convex set on G. In that case, $V_0 \cup V_1 \subseteq N_G(b)$, and so V_2 is a dominating set on G. This forced that $|V_1| = 0$. Thus, we have $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = 2|V_2| = 2$, a contradiction. Therefore, we end up with $\widetilde{\gamma}_{iCvR}(G) \neq 3$. This completes the proof. □

The corollary below is quick from Theorem 3.19.

Corollary 3.20. *$\widetilde{\gamma}_{iCvR}(G) = 3$ if and only if $G = K_3$.*

Theorem 3.21. *Let $G \neq S_n$ be a connected graph where $n \geq 3$ and let $\lambda = (V_0, V_1 = \varnothing, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. Then $\widetilde{\gamma}_{iCvR}(G) = 4$ if and only if there exists a dominating set $D = \{u, v\}$ such that D is a convex set and $V \setminus \{u, v\}$ is an independent vertex set on G.*

Proof. Let $\lambda = (V_0, V_1 = \varnothing, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on $G \neq S_n$ where $n \geq 3$. Assume that $\widetilde{\gamma}_{iCvR}(G) = 4$. Then $\widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| + 2|V_2| = 4$. Since $|V_1| = 0$, $|V_2| = 2$. By Theorem 3.6, V_2 is a $\widetilde{\gamma}_{iCv}$-set on G. Hence, V_2 is an outer-independent convex dominating set on G. Let $V_2 = \{u, v\}$. Then $D = \{u, v\}$ is a dominating set on G and D is a convex set on G. Thus, $V_0 = V \setminus \{u, v\}$. In that case, $V \setminus \{u, v\}$ is an independent vertex set on G. Conversely, assume that there exists a dominating set $D = \{u, v\}$ such that D is a convex set and $V \setminus \{u, v\}$ is an independent vertex set on G. Then D is an outer-independent convex dominating set on G. Since $\lambda = (V_0, V_1 = \varnothing, V_2)$ is a $\widetilde{\gamma}_{iCvR}$-function on G, by Theorem 3.6, V_2 is a $\widetilde{\gamma}_{iCv}$-set on G. Hence, we have that $2 = |D| \geq |V_2|$. Suppose that $|V_2| = 0$. Then $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = 2|V_2| = 0$, a contradiction. Next, suppose that $|V_2| = 1$. By Theorem 3.18(ii), $G = S_n$ where $n \geq 3$, a contradiction to our assumption. Consequently, we get $|V_2| \geq 2$. Thus, $|V_2| = 2$. Therefore, we end up with $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = 2|V_2| = 4$. Which proves the assertion. □

Theorem 3.22. *Let G be a connected graph of order $n \geq 1$ and let $\lambda = (V_0, V_1, V_2)$ be an OiCvRDF on G for which $|V_1| = 0$. Then $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\gamma}_{iCv}(G) + 1$ if and only if there exists $v \in V$ such that $deg_G(v) = n - \widetilde{\gamma}_{iCv}(G)$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be an OiCvRDF on G for which $|V_1| = 0$. Assume that $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\gamma}_{iCv}(G) + 1$. Then $\widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| + 2|V_2| = \widetilde{\gamma}_{iCv}(G) + 1$. Since $|V_1| = 0$, by Theorem 3.6, V_2 is a $\widetilde{\gamma}_{iCv}$-set on G. In that case, $\widetilde{\omega}_G^{iCv}(G) = |V_2|$. Thus, we have that $|V_2| + 1 = 2|V_2|$. So, $|V_2| = 1$. Let $V_2 = \{v\}$. Since V_2 is a $\widetilde{\omega}_G^{iCv}$-set on G, $V = N_G[v]$ and $V \setminus \{v\}$ is an independent vertex set on G. Hence, we have $deg_G(v) = |N_G(v)| = |V \setminus \{v\}| = |V| - |\{v\}| = n - |V_2| = n - \widetilde{\omega}_G^{iCv}(G)$. Conversely, assume that there exists $v \in V$ such that $deg_G(v) = n - \widetilde{\gamma}_{iCv}(G)$. Then let $V_2 = \{v\}$ and so, V_2 is a convex set. Since $|V_1| = 0$ and V_2 is a dominating set, it is clear that V_2 is a $\widetilde{\gamma}_{iCv}$-set on G. This follows that $|V_2| = \widetilde{\gamma}_{iCv}(G) = n - deg_G(v)$. Now, since V_2 is a $\widetilde{\gamma}_{iCv}$-set on G, it means that V_2 is an outer-independent convex dominating set on G. This implies that $V \setminus V_2$ is an independent vertex set on G. Hence, we let $V_0 = V \setminus V_2$. Since V_2 is a $\widetilde{\gamma}_{iCv}$-set on G and $V_1 = \varnothing$, by Theorem 3.5, $\lambda = (V_0, V_1 = \varnothing, V_2)$ is a $\widetilde{\gamma}_{iCvR}$-function on G. Therefore, we end up with $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = 2|V_2| = |V_2| + |V_2| = \widetilde{\gamma}_{iCv}(G) + 1$. This completes the proof. □

The following theorem presents a realization problem for OiCvRDF and CvRDF on a connected graph.

Theorem 3.23. *Let p, q and n be positive integers and let $1 \leq p \leq q \leq n$. Then there exists a connected graph G such that $|V| = n$, $\gamma_{CvR}(G) = p$ and $\widetilde{\gamma}_{iCvR}(G) = q$.*

Proof. Let G be a connected graph with $|V| = n$ and let $\xi = (W_0, W_1, W_2)$ is a $\widetilde{\gamma}_{CvR}$-function on G and $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. Then consider the following cases:

Case 1. $1 \leq p \leq q = n$

Let $G = K_1$. By Theorem 3.18(i), we get $1 = \gamma_{CvR}(G) = \widetilde{\gamma}_{iCvR}(G) = |V| = n$. Let $\gamma_{CvR}(G) = p = 1 = q = \widetilde{\gamma}_{iCvR}(G)$. Then $1 = p = q = n$. Moreover, consider the graph $G = K_n$ where $n \geq 3$. Then $G = K_1 + K_{n-1}$ where $n \geq 2$. Here, we let $W_0 = V(K_{n-1})$, $W_1 = \varnothing$, and $W_2 = V(K_1)$. Hence, by construction, we get $\gamma_{CvR}(G) = \omega_G^{CvR}(\xi) = |W_1| + 2|W_2| = 2|V(K_1)| = 2$. By Proposition 3.12, $\widetilde{\gamma}_{iCvR}(G) = n$. Let $\gamma_{CvR}(G) = 2 = p$ and $\widetilde{\gamma}_{iCvR}(G) = n = q$. Since $n \geq 3$, it follows that $1 < p < q = n$. Therefore, Case 1 is satisfied.

Case 2. $1 < p < q < n$

Let $G = F_m$ where $m \geq 3$ and $m \equiv 1 \pmod 2$. Then $|V| = n = m + 1$ and $G = K_1 + P_m$ where $m \geq 3$ and $m \equiv 1 \pmod 2$. In that case, let $W_0 = V(P_m)$, $W_1 = \varnothing$, and $W_2 = V(K_1)$. By construction, we obtain $\gamma_{CvR}(G) = \omega_G^{CvR}(\xi) = |W_1| + 2|W_2| = 2|V(K_1)| = 2$. Now, let $P_m = [p_1, p_2, ..., p_m]$. Since $m \equiv 1 \pmod 2$, we let $A = \{p_1, p_3, ..., p_m\}$ and $B = \{p_2, p_4, ..., p_{m-1}\}$. This means that the set A is an independent vertex set on G. In addition, for every $a, b \in B$, $d_G(a, b) = 2$,

which indicates that $V(K_1) \cup B$ is a convex set on G. Hence, we set $V_0 = A$, $V_1 = B$, and $V_2 = V(K_1)$. By construction, we get $\widetilde{\gamma}_{iCvR}(G) = \widetilde{\omega}_G^{iCvR}(\lambda) = |V_1| + 2|V_2| = \frac{m-1}{2} + 2(1) = \frac{m+3}{2}$. Let $\gamma_{CvR}(G) = 2 = p$ and $\widetilde{\gamma}_{iCvR}(G) = \frac{m+3}{2} = q$. Since $m \geq 3$, it implies that $1 < p < q < n$. Therefore, Case 2 is satisfied.
Case 3. $1 < p = q < n$
Let $G = S_n$ where $n \geq 3$. Then $G = K_1 + \overline{K_n}$ where $n \geq 3$. Let $W_0 = V(\overline{K_n})$, $W_1 = \varnothing$, and $W_2 = V(K_1)$. By construction, we get $\gamma_{CvR}(G) = \omega_G^{CvR}(\xi) = |W_1| + 2|W_2| = 2|V(K_1)| = 2$. By Theorem 3.18$(ii)$, $\widetilde{\gamma}_{iCvR}(G) = 2$. Let $\gamma_{CvR}(G) = 2 = p$ and $\widetilde{\gamma}_{iCvR}(G) = 2 = q$. Consequently, $1 < p = q < n$. Therefore, Case 3 is satisfied.
Which proves the assertion. ☐

The next corollary is a direct consequence of Theorem 3.23.

Corollary 3.24. *Let G be a connected graph of order $n \geq 1$. Then the difference $\widetilde{\gamma}_{iCvR}(G) - \gamma_{CvR}(G)$ can be made arbitrarily large.*

Proof. Let m be a positive integer. By Theorem 3.23, there exists a connected graph G such that $|V| = n$, $\widetilde{\gamma}_{iCvR}(G) = m+2 < n$ and $\gamma_{CvR}(G) = 2$. In that case, we get $\widetilde{\gamma}_{iCvR}(G) - \gamma_{CvR}(G) = m$. Since m is a positive integer, it follows that the difference $\widetilde{\gamma}_{iCvR}(G) - \gamma_{CvR}(G)$ can be made arbitrarily large. Which proves the assertion. ☐

Theorem 3.25. *Let G be a connected graph of order $n \geq 1$. Then*

$$max\{\widetilde{\gamma}_{iCv}(G), \gamma_{CvR}(G)\} \leq \widetilde{\gamma}_{iCvR}(G) \leq min\{n, 2\widetilde{\gamma}_{iCv}(G)\}.$$

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iCvR}$-function on G. By Theorem 3.3, $V_1 \cup V_2$ is an outer-independent convex dominating set on G. Hence, we have that $\widetilde{\gamma}_{iCv}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \widetilde{\omega}_G^{iCvR}(\lambda) = \widetilde{\gamma}_{iCvR}(G)$. Moreover, since every OiCvRDF on G is a CvRDF, it follows that $\gamma_{CvR}(G) \leq \widetilde{\gamma}_{iCvR}(G)$. Thus, $max\{\widetilde{\gamma}_{iCv}(G), \gamma_{CvR}(G)\} \leq \widetilde{\gamma}_{iCvR}(G)$. By Corollary 3.9, we have that $\widetilde{\gamma}_{iCvR}(G) \leq n$. Now, let O be a $\widetilde{\gamma}_{iCv}$-set on G. Then $\widetilde{\gamma}_{iCv}(G) = |O|$. Let $U_0 = V \setminus O$, $U_1 = \varnothing$, and $U_2 = O$. This implies that $\lambda^* = (U_0, U_1, U_2)$ is an OiCvRDF on G, that is, $U_0 = V \setminus O$ is an independent set on G and U_2 is a convex set on G. So, we obtain $\widetilde{\gamma}_{iCvR}(G) \leq \widetilde{\omega}_G^{iCvR}(\lambda^*) = |U_1| + 2|U_2| = 2|O| = 2\widetilde{\gamma}_{iCv}(G)$. Consequently, $\widetilde{\gamma}_{iCvR}(G) \leq min\{n, 2\widetilde{\gamma}_{iCv}(G)\}$. And the conclusion follows. ☐

Corollary 3.26. *Let G be a connected graph of order n. Then $1 \leq \gamma_R(G) \leq \widetilde{\gamma}_{iCvR}(G) \leq min\{n, 2\widetilde{\gamma}_{iCv}(G)\}$.*

4. CONCLUSION

This research paper has introduced a new version of the Roman dominating function, which is called the outer-independent convex Roman dominating function. Some graph-theoretic properties of outer-independent convex Roman dominating function in some graphs were investigated, which include some bounds, the realization problem, characterization of outer-independent convex Roman domination number for small values, and exact values for outer-independent convex Roman domination number in some special graphs. The paper highly recommends that the said new parameter should be characterized under some binary operations of two graphs to strengthen the current findings. Moreover, for future research, one may consider a computational complexity (e.g., proving NP-completeness for general graphs) or polynomial-time algorithms for trees involving OiCvRDF.

5. Acknowledgement

The author would like to thank the reviewers for giving significant comments to improve this paper.

References

1. Ahangar, H. A., Henning, M. A., Samodivkin, V., & Yero, I. G. (2016). Total Roman domination in graphs. *Applicable Analysis and Discrete Mathematics*, 10(2), 501–517. `https://doi.org/10.2298/AADM160802017A`
2. Alqesmah, A., & Gangabylaiah, D. (2025). On the roman domination polynomial of the commuting and non-commuting graphs associated to the dihedral groups. *Annals of Mathematics and Computer Science*, 27, 44–54. `https://doi.org/10.56947/amcs.v27.482`
3. Anand, B. S., Dayap, J. A., Casinillo, L. F., Pepper, R., & Nair, R. S. (2025). On distance k-domination number of graphs under product operations. *Gulf Journal of Mathematics*, 19(1), 117–124. `https://doi.org/10.56947/gjom.v19i1.2032`
4. Cabrera-Martinez, A. (2024). Total outer-independent domination in regular graphs. *Filomat*, 38(18), 6581-6586. `https://www.jstor.org/stable/27387558`
5. Casinillo, L. F. (2018). A note on Fibonacci and Lucas number of domination in path, *Electronic Journal of Graph Theory and Applications*, 6(2), 317–325. `http://dx.doi.org/10.5614/ejgta.2018.6.2.11`
6. Casinillo, L. F. (2020). Odd and even repetition sequences of independent domination number. *Notes on Number Theory and Discrete Mathematics*, 26(1), 8–20. `https://doi.org/10.7546/nntdm.2020.26.1.8-20`
7. Casinillo, L., & Casinillo, E. (2025). Outer-connected fair Roman dominating function in graphs. *Annals of Mathematics and Computer Science*, 31, 32-42. `https://doi.org/10.56947/amcs.v31.686`
8. Casinillo, L. (2026). Outer-clique Roman dominating function in graphs. *Annals of Mathematics and Computer Science*, 33, 1-9. `https://doi.org/10.56947/amcs.v33.753`
9. Chartrand, G., Lesniak, L., & Zhang, P. (2016). *Graphs and Digraphs*. Textbooks in Mathematics. CRC Press, 1(2.4), 3. `https://doi.org/10.1201/b19731`
10. Cockayne, E. J., Dreyer Jr, P. A., Hedetniemi, S. M., & Hedetniemi, S. T. (2004). Roman domination in graphs. *Discrete mathematics*, 278(1-3), 11–22. `https://doi.org/10.1016/j.disc.2003.06.004`
11. Fortosa, R. J., & Canoy, S. (2023). Convex roman dominating function in graphs. *European Journal of Pure and Applied Mathematics*, 16(3), 1705-1716. `https://doi.org/10.29020/nybg.ejpam.v16i3.4828`
12. Gross, J. L., Yellen, J., & Anderson, M. (2018). *Graph theory and its applications*. Chapman and Hall/CRC. `https://doi.org/10.1201/9780429425134`
13. Haynes, T. W., Hedetniemi, S., & Slater, P. (2013). *Fundamentals of domination in graphs*, CRC press. `https://doi.org/10.1201/9781482246582`
14. Henning, M. A., & Hedetniemi, S. T. (2003). Defending the Roman Empire—A new strategy. *Discrete mathematics*, 266(1-3), 239-251. `https://doi.org/10.1016/S0012-365X(02)00811-7`
15. Jafari Rad, N., & Krzywkowski, M. (2015). 2-outer-independent domination in graphs. *National Academy Science Letters*, 38(3), 263–269. `https://link.springer.com/article/10.1007/s40009-015-0389-x`

16. Pushpam, R. L., & Padmapriea, S. (2015). Restrained Roman domination in graphs. *Transactions on combinatorics*, 4(1), 1–17. `https://doi.org/10.22108/toc.2015.4395`
17. Stewart, I. (1999). Defend the Roman empire!. *Scientific American*, 281(6), 136-138. `https://www.jstor.org/stable/26058532`

Department of Mathematics, Visayas State University, Philippines
Email address: `leomarichcasinillo02011990@gmail.com`