

RATIONALIZED TOEPLITZ HANKEL OPERATORS VIA OPERATOR EQUATIONS

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ABSTRACT. In this paper, we compute and study the behaviour of the solutions of the equation $\lambda^{k_2} M_{z^{k_2}} X = X M_{z^{k_1}}$. Precisely, for a general complex number λ and non-zero integers k_1 and k_2 , we introduce λ -Rationalized Toeplitz Hankel operators X on the space $L^2(\lambda - RTHO(L^2))$ as the solution of the equation $\lambda^{k_2} M_{z^{k_2}} X = X M_{z^{k_1}}$. We discuss some properties of solution of these operator equations.

1. INTRODUCTION

The study of Toeplitz and Hankel operators have been a subject of investigations by many researchers along the globe. Along with these operators Slant Toeplitz and Slant Hankel operators have been studied by different mathematicians. Multiple papers have been published that extend and generalize the study made in [4] by Brown and Halmos. In [1], further the notion of Slant Toeplitz operators were generalized to k^{th} -order Slant Toeplitz operators. The class of λ -Toeplitz operators and λ -Slant Toeplitz operators were also introduced.

Motivated by all these we [3] have generalized all kinds of such Toeplitz, Hankel, Slant Toeplitz and Slant Hankel operators and introduce the notion of Rationalized Toeplitz Hankel operators on the space L^2 . The set of all Rationalized Toeplitz Hankel operators is denoted by $RTHO(L^2)$. It is proved [3] that a bounded linear operator R on the space L^2 is a Rationalized Toeplitz Hankel operator of order (k_1, k_2) where k_1 and k_2 are relatively prime integers, if and only if $M_{z^{k_2}} R = R M_{z^{k_1}}$.

Further if k_1 and k_2 are not relatively prime then if $d = g.c.d(k_1, k_2)$ ie. let $k_1 = dn$ and $k_2 = dm$ then a bounded operator R on L^2 is Rationalized Toeplitz Hankel operator if and only if

$$R|_{\tilde{N}_i} = W_m M_{\tilde{\varphi}_i} W_n|_{\tilde{N}_i}$$

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for some $\tilde{\varphi}_i$ in L^∞ and for $i = 1, 2, \dots, d-1$

$$L^2 = \tilde{N}_0 \oplus \tilde{N}_1 \oplus \dots \oplus \tilde{N}_{d-1}$$

where

$$\tilde{N}_0 = N_0 \oplus N_1 \oplus \dots \oplus N_{m-1}$$

$$\tilde{N}_1 = N_m \oplus N_{m+1} \oplus \dots \oplus N_{2m-1}$$

$$\vdots \qquad \qquad \qquad \vdots$$

$$\tilde{N}_{d-1} = N_{(d-1)m} \oplus N_{(d-1)m+1} \oplus \dots \oplus N_{dm-1}$$

N_i = The closed linear span of $\{z^{k_1 t+1} : t \in \mathbb{Z}\}$

In this paper, we introduce the notion of λ -Rationalized Toeplitz Hankel operators. Precisely, we are inspired to solve the equations

$$\lambda^{k_2} M_{z^{k_2}} R = R M_{z^{k_1}}$$

for non zero integers k_1 and k_2 . We begin with the following

2. λ - RATIONALIZED TOEPLITZ HANKEL OPERATORS

Definition 2.1. If k_1 and k_2 are non zero integers and λ be a fixed complex number, an operator X on the space L^2 is said to be a λ -Rationalized Toeplitz Hankel operator (λ -RTHO(L^2)) of order (k_1, k_2) if it is the solution of the equation

$$\lambda^{k_2} M_{z^{k_2}} X = X M_{z^{k_1}}$$

we can observe here, that if $k_1 = k_2 = 1$, then it becomes λ -Toeplitz operator and becomes λ -Slant Toeplitz for $k_1 = 2$ and $k_2 = 1$. We have the following result

Theorem 2.2. If X is the solution of the equation $\lambda^{k_2} M_{z^{k_2}} X = X M_{z^{k_1}}$, $|\lambda| \neq 1$ and k_1 and k_2 are given as pair of non zero integers then X is the zero operator.

Proof: Suppose X is a solution of the equation $\lambda^{k_2} M_{z^{k_2}} X = X M_{z^{k_1}}$, $|\lambda| \neq 1$. If $|\lambda| < 1$, then define an operator Γ on $B(L^2)$ as $\Gamma(X) = \lambda^{k_2} M_{z^{k_2}} X M_{\bar{z}^{k_1}}$. Then $(I - \Gamma)$ is invertible.

As X being the solution of the equation $\lambda^{k_2} M_{z^{k_2}} X = X M_{z^{k_1}}$, we get $(I - \Gamma)X = 0$. This implies that X is zero operator.

Now if $|\lambda| > 1$, then we define in this case $\Gamma(X) = M_{\bar{z}^{k_2}} X M_{z^{k_1}}$ and we get the invertibility of $(\lambda^{k_2} I - \Gamma)$ which again shows that X is zero operator. This completes the proof.

However for $|\lambda| = 1$, we proceed as follows

Theorem 2.3. For $\lambda \in \mathbb{C}$, with $|\lambda| = 1$, the equation $\lambda^{k_2} M_{z^{k_2}} X = X M_{z^{k_1}}$, admits of a non-zero solution of the form $X = D_{\bar{\lambda}} R$, where R is the Rationalized Toeplitz Hankel operator of order (k_1, k_2) and $D_{\bar{\lambda}}$ is the composition operator on L^2 defined as $D_{\bar{\lambda}} f(z) = f(\lambda z)$ for all f in L^2

Proof: Suppose that X is an operator of the form $D_{\bar{\lambda}} R$ where R is the Rationalized Toeplitz Hankel operator of order (k_1, k_2) on the space $L^2(\mathbb{T})$, $\mathbb{T}$ being the

unit circle. Therefore [3] R must satisfy the equation $M_{z^{k_2}}R = RM_{z^{k_1}}$. Also

$$M_{z^{k_2}}D_{\bar{\lambda}} = \bar{\lambda}^{k_2}D_{\bar{\lambda}}M_{z^{k_2}}$$

Hence

$$\begin{aligned}\lambda^{k_2}M_{z^{k_2}}X &= \lambda^{k_2}M_{z^{k_2}}D_{\bar{\lambda}}R \\ &= \lambda^{k_2}\bar{\lambda}^{k_2}D_{\bar{\lambda}}M_{z^{k_2}}R \\ &= |\lambda|^{2k_2}D_{\bar{\lambda}}RM_{z^{k_1}} \\ &= XM_{z^{k_1}}, \text{ as } |\lambda| = 1\end{aligned}$$

Hence $X = D_{\bar{\lambda}}R$ satisfy the operator equation.

$$\lambda^{k_2}M_{z^{k_2}}X = XM_{z^{k_1}}$$

Conversely, Suppose that X is an operator satisfying the operator equation $\lambda^{k_2}M_{z^{k_2}}X = XM_{z^{k_1}}$. As $M_{z^{k_2}}D_{\bar{\lambda}} = \bar{\lambda}^{k_2}D_{\bar{\lambda}}M_{z^{k_2}}$. So $M_{z^{k_2}}D_{\bar{\lambda}}X = \bar{\lambda}^{k_2}D_{\bar{\lambda}}M_{z^{k_2}}X = D_{\bar{\lambda}}XM_{z^{k_1}}$. This implies that $R = D_{\bar{\lambda}}X$ satisfies the operator equation $M_{z^{k_2}}R = RM_{z^{k_1}}$. So, $R = D_{\bar{\lambda}}X$ is a Rationalized Toeplitz Hankel Operator on L^2 of order (k_1, k_2) . Hence $X = D_{\bar{\lambda}}R$ for some Rationalized Toeplitz Hankel Operator R on L^2 of order (k_1, k_2) . This completes the proof.

Remark 2.4. : If its also defined in [3] that for $k \neq 0$ if $W_k : L^2 \rightarrow L^2$ is defined as

$$W_k(z^i) = \begin{cases} z^{i|k} & \text{if } i \text{ is divisible by } k \\ 0, & \text{otherwise} \end{cases}$$

where $\{z^i; i \in \mathbb{Z}\}$ is orthonormal basis of L^2 . Then the Rationalized Toeplitz Hankel Operator on the space L^2 of order (k_1, k_2) is defined as

$$R_{\varphi} = W_{k_1}M_{\varphi}W_{k_2}^*$$

Hence if $\varphi(z) = \sum a_n z^n$ is the Fourier expansion of φ in L^∞ , then we define λ -Rationalized Toeplitz Hankel operator as $R_{\varphi, \lambda} = D_{\bar{\lambda}}R_{\varphi}$. So for each $j \in \mathbb{Z}$, we have

$$R_{\varphi, \lambda}(z^j) = D_{\bar{\lambda}}R_{\varphi}(z^j) = D_{\bar{\lambda}}\left(\sum_{t=-\infty}^{\infty} a_{k_1 t - k_2 j} z^t\right) = \sum_{t=-\infty}^{\infty} a_{k_1 t - k_2 j} (\lambda z)^t = \lambda^t \sum_{t=-\infty}^{\infty} a_{k_1 t - k_2 j} z^t$$

Remark 2.5. As we have seen that for $|\lambda| \neq 1$, the only solution of the equation $\lambda^{k_2}M_{z^{k_2}}X = XM_{z^{k_1}}$ is the zero operator. Therefore, hence onwards $R_{\varphi, \lambda}, \varphi \in L^\infty$. is further used in the reference to the solution of the equation $\lambda^{k_2}M_{z^{k_2}}X = XM_{z^{k_1}}$ where $|\lambda| = 1$.

If $R_{\varphi, \lambda}^*$ is the adjoint of $R_{\varphi, \lambda}$ than we have for all $i, j \in \mathbb{Z}$

$$\begin{aligned}\langle R_{\varphi, \lambda}^*, z^j, z^i \rangle &= \langle z^j, R_{\varphi, \lambda} z^i \rangle \\ &= \langle z^j, \sum_{t=-\infty}^{\infty} \lambda^t a_{k_1 t - k_2 i} z^t \rangle \\ &= \bar{\lambda}^j \bar{a}_{k_1 j - k_2 i}\end{aligned}$$

Therefore, we have the following.

Theorem 2.6. *Adjoint of a non-zero λ -Rationalized Toeplitz Hankel operator is not a λ -Rationalized Toeplitz Hankel operator.*

Proof Let if possible $R_{\varphi,\lambda}^*$, be a non zero λ - Rationalized Toeplitz Hankel operator of order (k_1, k_2) on L^2 . Then for each $i, j \in \mathbb{Z}$.

$$\lambda^{k_2} M_{z^{k_2}} R_{\varphi,\lambda}^* = R_{\varphi,\lambda}^* M_{z^{k_1}} \quad (2.1)$$

$$\Rightarrow \lambda^{k_2} \langle R_{\varphi,\lambda}^* z^j, z^i \rangle = \langle R_{\varphi,\lambda}^* z^{j+k_1}, z^{i+k_2} \rangle \quad (2.2)$$

$$\Rightarrow \lambda^{k_2} \bar{\lambda}^j \bar{a}_{k_1 j - k_2 i} = \bar{\lambda}^{(j+k_1)} \bar{a}_{k_1(j+k_1) - k_2(i+k_2)} \quad (2.3)$$

$$\Rightarrow \bar{a}_{k_1 j - k_2 i} = \bar{\lambda}^{(k_1+k_2)} \bar{a}_{k_1(j+k_1) - k_2(i+k_2)} \quad (2.4)$$

Put $j = 0$, we get $\bar{a}_{-k_2 i} = \bar{\lambda}^{(k_1+k_2)} \bar{a}_{k_1 - k_2 - k_2 i}$. Since $a_n \rightarrow 0$ as $n \rightarrow \infty$, we get $\bar{a}_t = 0$ for each $t \in \mathbb{Z}$. So $\varphi = 0$ which is a contradiction to the fact that $R_{\varphi,\lambda}^*$ is a non zero operator. This completes the proof.

3. λ -RATIONALIZED TOEPLITZ HANKEL MATRIX

If $\varphi(z) = \sum a_n z^n$ in L^∞ , the matrix representation of $R_{\varphi,\lambda}$, ($|\lambda| = 1$) with respect to the standard orthonormal basis $\{e_n(z) = z^n\}_{n \in \mathbb{N}}$ of L^2 given by

$$\left(\begin{array}{c|ccc} \vdots & \vdots & \vdots & \vdots \\ \dots \bar{\lambda} a_{-k_1+k_2} & \bar{\lambda} a_{-k_1} & \bar{\lambda} a_{-k_1-k_2} & \bar{\lambda} a_{-k_1-2k_2} \dots \\ \hline \dots a_{k_2} & a_0 & a_{-k_2} & a_{-2k_2} \dots \\ \dots \lambda a_{k_1+k_2} & \lambda a_{k_1} & \lambda a_{k_1-k_2} & \lambda a_{k_1-2k_2} \dots \\ \dots \lambda^2 a_{2k_1+k_2} & \lambda^2 a_{2k_1} & \lambda^2 a_{2k_1-k_2} & \lambda^2 a_{2k_1-2k_2} \dots \\ \vdots & \vdots & \vdots & \vdots \\ \dots \lambda^{k_2} a_{k_1 k_2 + k_2} & \lambda^{k_2} a_{k_1 k_2} & \lambda^{k_2} a_{k_1 k_2 - k_2} & \lambda^{k_2} a_{k_1 k_2 - 2k_2} \dots \\ \vdots & \vdots & \vdots & \vdots \end{array} \right)$$

Since all kinds of Toeplitz and Hankel operators are characterized in terms of their respective matrices, therefore it motivates us to define the following matrix.

Definition 3.1. Given non zero integers k_1 and k_2 , a λ -Rationalized Toeplitz Hankel matrix is a two-way infinite matrix (α_{ij}) such that $\lambda^{k_2} \alpha_{ij} = \alpha_{i+k_2, j+k_1}$

We have the following.

Theorem 3.2. *A necessary and sufficient condition for an operator R on L^2 to be a λ -Rationalized Toeplitz Hankel operator on L^2 for $|\lambda| = 1$ is that its matrix with respect to the orthonormal basis $\{z^i : i \in \mathbb{Z}\}$ is a λ - Rationalized Toeplitz-Hankel matrix.*

Proof If $R_{\varphi,\lambda}$ is a λ -Rationalized Toeplitz Hankel operator then as above we have seen that its matrix (α_{ij}) is a λ Rationalized Toeplitz Hankel matrix. Conversely,

Let the matrix of a bounded operator R on L^2 is a λ -Rationalized Toeplitz Hankel matrix. Then for all $i, j \in \mathbb{Z}$, we have $\lambda^{k_2} \alpha_{ij} = \alpha_{i+k_2, j+k_1}$

$$\begin{aligned} \Rightarrow \lambda^{k_2} \langle R z^j, z^i \rangle &= \langle R z^{j+k_1}, z^{i+k_2} \rangle \\ &= \langle R M_{z^{k_1}} z^j, M_{z^{k_2}} z^i \rangle \\ &= \langle M_{\bar{z}^{k_2}} R M_{z^{k_1}} z^j, z^i \rangle \end{aligned}$$

Thus

$$\begin{aligned} \lambda^{k_2} R z^j &= M_{\bar{z}^{k_2}} R M_{z^{k_1}} z^j \quad \forall j \in \mathbb{Z} \\ \lambda^{k_2} M_{z^{k_2}} R &= R M_{z^{k_1}} \end{aligned}$$

Hence R satisfy the operator equation $\lambda^{k_2} M_{z^{k_2}} X = X M_{z^{k_1}}$ and therefore is a λ -Rationalized Toeplitz Hankel operator of order (k_1, k_2) .

With these observations and results, one can prove the following trivial results with little efforts.

Proposition 3.3. Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

- i) Sum of two λ -Rationalized Toeplitz Hankel operators is a λ -Rationalized Toeplitz Hankel operator.
- ii) The mapping $\varphi \rightarrow R_{\varphi, \lambda}$ from L^∞ into $B(L^2)$ is linear and one-one.
- iii) The set of all λ -Rationalized Toeplitz Hankel operators (λ -RTHO) on the space L^2 is weakly closed and hence strongly closed.
- iv) For an unimodular complex number μ , $D_{\bar{\mu}\lambda} R_{\varphi, \lambda}$ is a μ -Rationalized Toeplitz Hankel operator.

Further, we can show that if M_φ is the multiplication operator on L^2 and $R_{\psi, \lambda}$ is a λ -Rationalized Toeplitz Hankel operator on L^2 of order (k_1, k_2) then $\lambda^{k_2} M_{z^{k_2}} (M_\varphi R_{\psi, \lambda}) = (M_\varphi R_{\psi, \lambda}) M_{z^{k_1}}$ and $\lambda^{k_2} M_{z^{k_2}} (R_{\psi, \lambda} M_\varphi) = (R_{\psi, \lambda} M_\varphi) M_{z^{k_1}}$ for $\psi, \varphi \in L^\infty$. As a consequence the product is always a λ -Rationalized Toeplitz Hankel operator. Therefore we have the following.

Theorem 3.4. If for ψ, φ in L^∞ , M_φ is the multiplication operator on L^2 , then $M_\varphi R_{\psi, \lambda}$ and $R_{\psi, \lambda} M_\varphi$ are always λ -Rationalized Toeplitz Hankel operator. Further $M_\varphi R_{\psi, \lambda} = R_{\psi, \lambda} M_\varphi$ if and only if $\varphi(\bar{\lambda} z^{k_1}) \psi(z) = \varphi(z) \psi(z^{k_2})$.

Proof The first part is trivial. However for the later part we proceed as follows

$$\begin{aligned} M_{\varphi(z)} R_{\psi, \lambda} &= M_{\varphi(z)} D_{\bar{\lambda}} R_{\psi(z)} \\ &= M_{\varphi(z)} D_{\bar{\lambda}} W_{k_1} M_{\psi(z)} W_{k_2}^* \\ &= D_{\bar{\lambda}} M_{\varphi(\bar{\lambda} z)} W_{k_1} M_{\psi(z)} W_{k_2}^* \\ &= D_{\bar{\lambda}} W_{k_1} M_{\varphi(\bar{\lambda} z^{k_1})} M_{\psi(z)} W_{k_2}^* \\ &= D_{\bar{\lambda}} W_{k_1} M_{\varphi(\bar{\lambda} z^{k_1}) \psi(z)} W_{k_2}^* \\ &= R_{\varphi(\bar{\lambda} z^{k_1}) \psi(z), \lambda} \end{aligned}$$

On the other hand,

$$\begin{aligned}
 R_{\psi(z),\lambda} M_{\varphi(z)} &= D_{\bar{\lambda}} W_{k_1} M_{\psi(z)} W_{k_2}^* M_{\varphi(z)} \\
 &= D_{\bar{\lambda}} W_{k_1} M_{\psi(z)} M_{\varphi(z^{k_2})} W_{k_2}^* \\
 &= D_{\bar{\lambda}} W_{k_1} M_{\psi(z)\varphi(z^{k_2})} W_{k_2}^* \\
 &= D_{\bar{\lambda}} R_{\psi(z)\varphi(z^{k_2})} \\
 &= R_{\psi(z)\varphi(z^{k_2}),\lambda}
 \end{aligned}$$

As the mapping $\varphi \rightarrow R_{\varphi,\lambda}$ is linear and one-one we get the desired result.

So if $\lambda = 1$ and $k_1 = k_2 = 1$ we get from the previous result that the product of 1-Rationalized Toeplitz Hankel operator of order (1,1) and λ -Rationalized Toeplitz Hankel operator of order (k_1, k_2) is λ -Rationalized Toeplitz Hankel operator. This leads us to think about the product of two λ -Rationalized Toeplitz Hankel operator. To answer this, we can prove that the product of two λ -Rationalized Toeplitz Hankel operator is a λ -Rationalized Toeplitz Hankel operator if and only if the product is zero.

Lemma 3.5. *For $\varphi \in L^\infty$, $D_{\bar{\lambda}} W_{k_1} R_{\varphi,\lambda}$ is a λ -Rationalized Toeplitz Hankel operator iff $\varphi = 0$*

Proof Suppose $D_{\bar{\lambda}} W_{k_1} R_{\varphi,\lambda}$ is a λ -Rationalized Toeplitz Hankel Operator. Then for integers i, j we have

$$\lambda^{k_2} \langle D_{\bar{\lambda}} W_{k_1} R_{\varphi,\lambda} z^j, z^i \rangle = \langle D_{\bar{\lambda}} W_{k_1} R_{\varphi,\lambda} z^{j+k_1}, z^{i+k_2} \rangle$$

This further gives us that

$$a_{k_2 j - k_2} = \lambda^{k_1 k_2} a_{k_1(k_2 + i) - k_2(j + k_1)}$$

which further shows that $a_t = 0 \forall t \in \mathbb{Z}$. Hence $\varphi = 0$. The converse is obvious.

Thus with the help of this lemma and little calculations, we can prove the following result for product.

Theorem 3.6. *The product of two λ -Rationalized Toeplitz Hankel Operator is a λ -Rationalized Toeplitz Hankel Operator if and only if the product is zero.*

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