

LATTICES AND THE FORMAL CONTEXT OBTAINED BY SUBSTITUTION SUM

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ABSTRACT. In Formal Context Analysis (FCA), various lattices can be constructed by using different context constructions, one of them being the substitution sum. In this paper, two types of lattices are obtained by substitution sum and their properties have been explored.

1. Let $CS(L_1)$ be the set of all Convex Sublattices of the lattice L_1 which are nonempty. It is known that $CS(L_1)$ can be ordered in such a way that the lattice $CS(L_1)$ becomes a sublattice of $L_1 \times L_1$. The properties of $CS(L_1)$ are already known. We have proved that, the formal context of $CS(L_1)$, that is, $\mathbb{K}_1(CS(L_1))$ is obtained by the substitution sum $\frac{L_1}{L_1} \mid \frac{L_1}{X}$. A characterization of $CS(L_1)$ is given in this paper which helps to construct the lattice $CS(L_1)$.
2. Let L_1 be a lattice with more than one atom. We have constructed a new lattice $TS(L_1)$ and proved that, the formal context of $TS(L_1)$ that is $\mathbb{K}_2(TS(L_1))$ is obtained by the substitution sum $\frac{L_1}{\emptyset} \mid \frac{L_1}{L_1}$. Further, some properties of $TS(L_1)$ have been explored.

1. INTRODUCTION AND PRELIMINARIES

Formal Concept Analysis (FCA) is an application in which the details is arranged as a formal context, consisting of the binary connection between the set of objects and attributes. It is known that every formal context $\mathbb{K}$ is isomorphic to $\langle J(L_1), M(L_1), I \rangle$. Construction of the lattices of various formal contexts and their applications are discussed by B. Ganter and R.Wille[1], one of them is Substitution Sum, in which a context can be placed into an empty cell of another context (For example, see [1], page 151). It is noted that this construction has generalized many of the context sum, disjoint union, direct semi product, direct product etc. In fact, the formal context $\frac{L_1}{X} \mid \frac{X}{L_2}$ corresponds to the lattice of the direct product $L_1 \times L_2$ has already known [1].

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J.M.Rodriguez et.al have given the characterization theorem which gives an isomorphism between the lattices arising from negation-enhanced attributes and mixed concept lattices[2]. This characterization has found wide applications in negative information data mining.

The Convex Sublattices $CS(L_1)$ of a lattice L_1 was first studied by K.M. Koh [3],[4]. He investigated the interdependence of the lattice L_1 and $CS(L_1)$, connection between $CS(L_1)$ and ideal lattice $I(L_1)$. Convex Subnearlattices of a medial nearlattice is studied by A.S.A. Noor and S.N. Begum[5]. S. Lavanya et.al [6] have defined a partial order $\leq$ on $CS(L_1)$ and they proved that the lattice $(CS(L_1), \leq)$ and L_1 are in the same equational class. On this new structure of $CS(L_1)$, S. Parameshwara Bhatta et.al [7] defined a congruence relation Ψ_Θ on $CS(L_1)$ corresponding to a given congruence relation Θ of L_1 and showed that $CS(L_1/\Theta)$ is isomorphic to $CS(L_1)/\Psi_\Theta$. Further, it is shown that L_1/Θ and $CS(L_1)/\Psi_\Theta$ are in the same equational class and if Θ is representable in L_1 , then so is Ψ_Θ in $CS(L_1)$. The bounds for number of elements in $CS(L_1)$ in terms of join irreducible elements have been given[8].

Various lattice constructions have been discussed by G.Gratzer [9], one of them includes chopped lattices. Doubling construction was introduced by Alan Day [10], and further, a class of lattices by doubling construction of a convex set in a given lattice was characterized by W.Geyer [11]. Using the construction concepts, the authors have constructed a lattice $\langle TS(L_1), \wedge, \vee \rangle$ by chopping the minimum element of a lattice L_1 , maximum element of another copy of the same lattice L_1 , and taking the union of $L_1 - \{1\}$ and $L_1 - \{0\}$ by adjoining the corresponding elements other than the minimum and maximum elements.

The paper proceeds as follows: Section 2 introduces the reader to the notations and the definitions used in this paper. In the section 3, considering the fact that a lattice can be identified by its context with objects and attributes consisting of join and meet irreducible elements respectively, the characterization for the irreducible elements of $CS(L_1)$ is given and thereby the lattice $CS(L_1)$ constructed through its formal context table $\mathbb{K}_1(CS(L_1))$ obtained by the substitution sum

$$\frac{L_1 \mid L_1}{L_1 \mid X}.$$

In the section 4, the definition of the lattice $TS(L_1)$ and its elementary properties are given. Further, with the aid of double construction of convex sets, in a lattice, we have constructed the lattice $TS(L_1)$ isomorphic to the concept lattice of formal context $\mathbb{K}_2(TS(L_1)) = \frac{L_1 \mid L_1}{\emptyset \mid L_1}$. Some important properties such as

$TS(L_1)$ cannot be not modular is discussed in section 5.

1.1. Notation and Definitions. To familiarize the reader to formal concept analysis, we present some of the required definitions. For basic concepts, the reader may refer to Formal Concept Analysis by B.Ganter and R. Wille[1]. The formal context $\mathbb{K}_0 := \langle G_0, M_0, I_0 \rangle$ is the relation I_0 between two sets G_0 , the object set and M_0 , the attribute set. $g_0 I_0 m_0$ or $(g_0, m_0) \in I_0$ denotes that the object $g_0 \in G_0$ has an attribute $m_0 \in M_0$.

A Formal Concept is a pair (C, D) with $C \subseteq G_0$, $D \subseteq M_0$ with $C' = D$ and

$D' = C$ where C' denotes those attributes in M_0 common to the objects in C and D' denotes those objects in G_0 common to the attributes in D . For further reading on Clarified contexts and Reduced contexts, see page 27, [1]

Let $\mathbb{K}_1 := \langle G_1, M_1, I_1 \rangle$ and $\mathbb{K}_2 := \langle G_2, M_2, I_2 \rangle$ be the Formal Contexts $(g_1, m_1) \notin I_1$ in $\mathbb{K}_1$. Suppose that $G_2 \neq \emptyset \neq M_2$ and $G_1 \setminus g_1 \cap G_2 = \emptyset = (M_1 \setminus m) \cap M_2$. The Substitution Sum of $\mathbb{K}_1$ with $\mathbb{K}_2$ on (g_1, m_1) is defined to be the Context $\mathbb{K}_1(g_1, m_1)\mathbb{K}_2 := \langle G_0, M_0, I_0 \rangle$ with $G_0 = (G_1 \setminus g_1) \cup G_2$, $M_0 = (M_1 \setminus m_1) \cup M_2$, $I_0 = (h_1, n_1) \in I_1, h_1 \neq g_1, n_1 \neq m_1 \cup G_2 \times g_1^{I_1} \cup m_1^{I_1} \times M_2 \cup I_2$ (For further reading, see page 150, [1]).

We recall some basics in lattice theory, in order to introduce the reader to the notations used in this paper. In this paper, L_1 denotes a finite lattice, maximum element and the minimum element of the lattice L_1 is denoted by 1 and 0 respectively. Note that, in L_1 , every Convex Sublattice is an interval of the form $[a_1, b_1]$ with $a_1 < b_1$.

The one element set $\{m\}$ denotes the interval $[m, m]$.

$[m) = \{x \in L_1 | x \geq m\}$.

$(m] = \{x \in L_1 | x \leq m\}$.

For elements a_1, b_1 of L_1 , a_1 is covered by b_1 is denoted by $a_1 \prec b_1$. An element $m \in L_1$ is *meet-irreducible* if $\exists$ a unique element $m^+ \in L_1$ such that $m \prec m^+$ and the set of all such elements of a lattice L_1 is denoted by $\mathbf{M}(L_1)$. An element $j \in L_1$ is *join-irreducible* if there exists a unique element $j^- \in L_1$ such that $j^- \prec j$ and the set of all such elements are denoted by $\mathbf{J}(L_1)$. For more basic concepts, see [12]. An element a_1 of a lattice L_1 is an atom in L_1 if $0 \prec_{L_1} a_1$ and is denoted by $Atom(L_1)$. Dually, a dual atom of a Lattice L_1 is defined and is denoted by $Coatom(L_1)$. If a_1 and b_1 are non comparable, then we denote it by $a_1 \parallel b_1$.

Given a lattice L_1 , $CS(L_1)$ represents the Convex Sublattices of L_1 . On $CS(L_1)$, define an ordering $\leq$ by: For elements in $CS(L_1)$, $C \leq D$ if and only if for every $c_1 \in C$, $\exists d_1 \in D$ such that $c_1 \leq d_1$ and for every $d_1 \in D$, $\exists c_1 \in C$ such that $d_1 \geq c_1$. Then $\langle CS(L_1); \leq \rangle$ is a lattice which is commonly known as *Lattice of Convex Sublattices* of L_1 (see [6]), denoted by $CS(L_1)$ in this paper.

Let P and Q be posets with the partial order $\leq$ defined on them. Then the Direct Product $P \times Q$ consisting of all ordered pairs (a_1, b_1) and (c_1, d_1) such that $(a_1, b_1) \leq (c_1, d_1)$ with $a_1 \leq c_1$ in P and $b_1 \leq d_1$ in Q [12].

2. MAIN RESULTS

2.1. Formal Context for $CS(L_1)$. In this section, we characterize the irreducible elements of $CS(L_1)$ and study the relation between them. These relations characterize the formal context $\mathbb{K}_1(CS(L_1))$ which aids the construction of the lattice $CS(L_1)$.

In the following theorem, we give a characterization for irreducible elements of $CS(L_1)$.

Theorem 2.1. *Let L_1 be a lattice.*

(α) *If $m \in \mathbf{M}(L_1)$, then $\{m\}$ and $[m) \in \mathbf{M}(CS(L_1))$*

- (β) If $a \in \mathbf{J}(L_1)$, then $\{a\}$ and $[a] \in \mathbf{J}(CS(L_1))$.
 (γ) If $b \in \mathbf{M}(L_1) \cap \mathbf{J}(L_1)$, then $\{b\} \in \mathbf{M}(CS(L_1)) \cap \mathbf{J}(CS(L_1))$, $[b] \in \mathbf{J}(CS(L_1))$. and $[b] \in \mathbf{M}(CS(L_1))$.

Conversely,

- (α') If $A \in \mathbf{M}(CS(L_1))$, then either $A = \{m\}$ or $A = [m]$ for some $m \in \mathbf{M}(L_1)$.
 (β') If $A \in \mathbf{J}(CS(L_1))$, then either $A = \{m\}$ or $A = [m]$ for some $m \in \mathbf{J}(L_1)$.
 (γ') If $A \in \mathbf{M}(CS(L_1)) \cap \mathbf{J}(CS(L_1))$, then $A = \{b\}$ for some $b \in \mathbf{M}(L_1) \cap \mathbf{J}(L_1)$.

Proof. Let $m \in \mathbf{M}(L_1)$ and $m \prec_{L_1} m^+$. Then, $\{m\} \leq_{CS(L_1)} [m \ m^+]$ and $[m] \leq_{CS(L_1)} [m^+]$. As $m \in \mathbf{M}(L_1)$, clearly $[m \ m^+]$ is a unique cover $\{m\}$ in $CS(L_1)$ and $[m^+]$ is unique cover of $[m]$ in $CS(L_1)$, proving (α).

Now we prove (α'), let $A = [m \ n] \in \mathbf{M}(CS(L_1))$.

Assume the contrary that $m \neq n$ and $n \neq 1$. Then, there exist $p, q \in L_1$ such that $m \prec_{L_1} p \leq_{L_1} n \prec_{L_1} q \leq_{L_1} 1$.

Claim: $[m \ n] \prec_{CS(L_1)} [p \ n]$, $[m \ n] \prec_{CS(L_1)} [m \ q]$ and $[p \ n] || [m \ q]$.

Since, $m \prec_{L_1} p$ and $n \prec_{L_1} q$, clearly, $[m \ n] \prec_{CS(L_1)} [p \ n]$ and $[m \ n] \prec_{CS(L_1)} [m \ q]$.

Since, there exists no $x \in [p \ n]$ such that $x \leq_{L_1} m$, $[p \ n] \not\leq_{CS(L_1)} [m \ q]$. Also, there exists no $y \in [p \ n]$ such that $q \leq_{L_1} y$, $[m \ q] \not\leq_{CS(L_1)} [p \ n]$. This proves that $[p \ n] || [m \ q]$ and hence our claim.

From above claim, we proved that $A \notin \mathbf{MCS}(L_1)$, a contradiction. Therefore, either $A = \{m\}$ or $A = [m]$.

We prove now that, if $A = \{m\}$ and $\{m\} \in \mathbf{M}(CS(L_1))$ then, $m \in \mathbf{M}(L_1)$. Let $\{m\} \in \mathbf{M}(CS(L_1))$. If $m \notin \mathbf{M}(L_1)$ then we can find $m_1, m_2 \in L_1$ such that $m \prec_{L_1} m_1$ and $m \prec_{L_1} m_2$. But then $\{m\} \prec_{CS(L_1)} [m \ m_1]$ and $\{m\} \prec_{CS(L_1)} [m \ m_2]$, a contradiction to the fact that $\{m\} \in \mathbf{M}(CS(L_1))$. We prove now that, if $A = [m]$ and $[m] \in \mathbf{M}(CS(L_1))$, then $m \in \mathbf{M}(L)$.

Let $[m] \in \mathbf{M}(CS(L_1))$. If $m \notin \mathbf{M}(L_1)$ then we can find $m_1, m_2 \in L_1$ such that $m \prec_{L_1} m_1$ and $m \prec_{L_1} m_2$. But then $[m] \prec_{CS(L_1)} [m_1]$ and $[m] \prec_{CS(L_1)} [m_2]$, a contradiction to the fact that $[m] \in \mathbf{M}(CS(L_1))$. This completes proof of (α').

(β) follows by duality of (α), (β') follows by duality of (α').

(γ) follows from (α) and (β), (γ') follows from (α') and (β'). $\square$

From the above theorem it is clear that if $|\mathbf{M}(L_1)| = n_1$ and $|\mathbf{J}(L_1)| = n_2$ then $|\mathbf{M}(CS(L_1))| = 2n_1$ and $|\mathbf{J}(CS(L_1))| = 2n_2$.

The following theorem gives the relation between elements of L_1 and $CS(L_1)$. In particular, the relations hold good for the irreducible elements of L_1 .

Theorem 2.2. *Let $a, m \in L_1$. Then*

$$(\alpha) [a]_{CS(L_1)} \leq [m].$$

$$(\beta) a \leq_{L_1} m \iff \{a\}_{CS(L_1)} \leq \{m\} \iff [a]_{CS(L_1)} \leq [m].$$

Proof. For any $x \in [a]$ there is $1 \in [m]$ and $x \leq_{L_1} 1$ holds. On the other hand, for any $y \in [m]$ there is $0 \in [a]$ and $y \geq_{L_1} 0$ holds, proving (α) . Proof of (β) is trivial. $\square$

Remark 2.3. Note that the object and attributes of the formal context are represented by the join and meet irreducible elements of $CS(L_1)$ respectively. If a and m are join and meet irreducible elements respectively in L_1 , the Theorem 2.2 describes the corresponding relations between the elements in $CS(L_1)$.

Proposition 2.4 ([1]). *For any finite lattice L_1 there is, upto isomorphism, a unique reduced context $\mathbb{K}(L_1)$ with $L_1 \cong \mathbb{B}(\mathbb{K}(L_1))$ that is $\mathbb{K}(L_1) := (\mathbf{J}(L_1), \mathbf{M}(L_1), \leq)$*

Theorem 2.5. *Let L_1 be a lattice. Define*

$$G_1 = \{A \in CS(L_1) | A = \{a\} \text{ for some } a \in \mathbf{J}(L_1)\}$$

$$G_2 = \{A \in CS(L_1) | A = [a] \text{ for some } a \in \mathbf{J}(L_1)\}$$

$$M_1 = \{A \in CS(L_1) | A = \{m\} \text{ for some } m \in \mathbf{M}(L_1)\}$$

$$M_2 = \{A \in CS(L_1) | A = [a] \text{ for some } a \in \mathbf{M}(L_1)\}$$

Define a context $\mathbb{K}_1(CS(L_1)) = (G_1 \cup G_2, M_1 \cup M_2, I_1 \cup I_2 \cup I_3 \cup I_4)$ as follows: for $\{g_1\} \in G_1$, $(g_2) \in G_2$, $\{m_1\} \in M_1$, $[m_2] \in M_2$,

$$(i) \{g_1\} I_1 \{m_1\} \text{ if } g_1 \leq m_1 \text{ in } L_1,$$

$$(ii) \{g_1\} I_2 [m_2] \text{ if } g_1 \leq m_2 \text{ in } L_1,$$

$$(iii) (g_2) I_3 \{m_1\} \text{ if } g_2 \leq m_1 \text{ in } L_1,$$

$$(iv) (g_2) I_4 [m_2] \text{ for all } (g_2) \in G_2, [m_2] \in M_2$$

Then $\mathbb{K}_1(CS(L_1))$ is the formal context of $CS(L_1)$.

Proof. From the theorem 2.1, it is clear that $\mathbf{J}(CS(L_1)) = G_1 \cup G_2$ and $\mathbf{M}(CS(L_1)) = M_1 \cup M_2$. Also, using theorem 2.2, the object- attribute relation in the first, second, third and fourth cells of the formal context of $CS(L_1)$ are defined by (i),(ii),(iii),(iv) respectively. From Proposition 2.4, $\mathbb{K}_1(CS(L_1))$ is the formal context of $CS(L_1)$. $\square$

Hence, given any lattice L_1 , the substitution sum $\mathbb{K}_1(CS(L_1)) = \frac{L_1}{L_1} \Big| \frac{L_1}{X}$ gives $CS(L_1)$.

Example 2.6. The lattice L_1 and $CS(L_1)$ is given in Figure 1, Figure 2 respectively, and the corresponding formal context of L_1 and $CS(L_1)$ is given in Figure 3 and Figure 4 respectively.

2.2. Formal Context for $TS(L_1)$. In this section, we discuss the structural properties of the lattice $TS(L_1)$ obtained by substitution sum $\mathbb{K}_2(TS(L_1)) =$

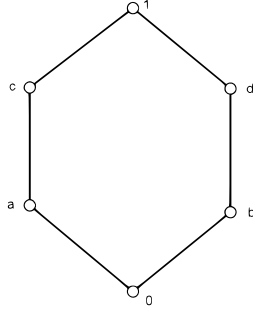


FIGURE
1. Lattice L_1

	1	2	3	4
a	X		X	
b	X			
c		X		X
d		X		

FIGURE 3. Formal Context of L_1

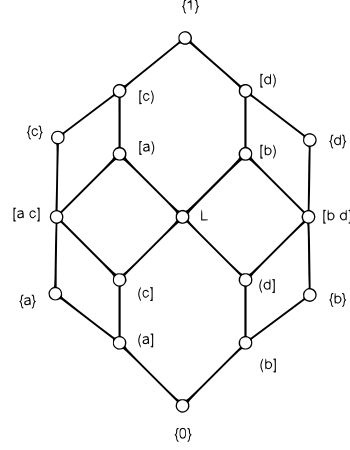


FIGURE 2. Lattice $CS(L_1)$

	{1}	{2}	{3}	{4}	[1]	[2]	[3]	[4]
{a}	X		X		X		X	
{b}	X				X			
{c}		X		X		X		X
{d}		X				X		
[a]	X		X		X	X	X	X
[b]	X				X	X	X	X
[c]		X		X	X	X	X	X
[d]		X			X	X	X	X

FIGURE 4. Formal Context of $CS(L_1)$

$\frac{L_1 \mid L_1}{\emptyset \mid L_1}$. Further, we discuss the characterization for the irreducible elements of the lattice $TS(L_1)$ and its formal context $\mathbb{K}_2(TS(L_1))$ which was generally discussed by Geyer [11].

Definition 2.7. Let $\langle L_1, \wedge, \vee \rangle$ be a lattice, having more than one atom and coatom. We construct a lattice $TS(L_1)$ by chopping the minimum element of L_1 , maximum element of another copy of the lattice L_1 , and taking the union of $L_1 - \{1\}$ and $L_1 - \{0\}$ by adjoining the the corresponding elements other than the minimum and maximum elements. For all $a \in L_1$, we denote the elements of $L_1 - \{1\}$ as $(a, \perp)$ and those of $L_1 - \{0\}$ as $(a, \top)$. Let $TS(L_1) = \{(a, \perp) \cup (a, \top) \mid a \in L_1\}$. On $TS(L_1)$, we define $\wedge$ and $\vee$ as follows:

For all $(a, \perp), (b, \perp), (a, \top), (b, \top) \in TS(L_1)$

1. $(a, \perp) \wedge (b, \perp) = (a \wedge b, \perp)$
2. $(a, \perp) \wedge (b, \top) = (a \wedge b, \perp)$
3. $(a, \perp) \vee (b, \perp) = (a \vee b, \perp)$
4. $(a, \perp) \vee (b, \top) = (a \vee b, \top)$
5. $(a, \top) \wedge (b, \top) = (a \wedge b, \top)$
6. $(a, \perp) \vee (b, \perp) = (1, \top)$ if $a \vee b = 1$ in L_1

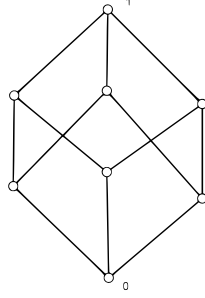


FIGURE 5. Lattice L_1

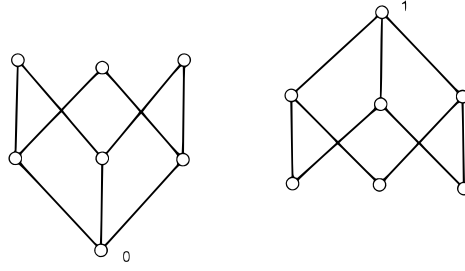


FIGURE 6. $L_1 - \{1\}$ and $L_1 - \{0\}$

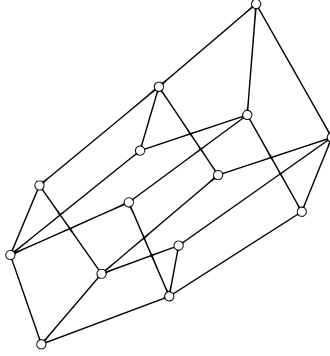


FIGURE 7. Lattice $TS(L_1) = L_1 - \{1\} \cup L_1 - \{0\}$

7. $(a, \top) \wedge (b, \top) = (0, \perp)$ if $a \wedge b = 0$ in L_1

Then, one can easily verify that $\langle TS(L_1), \wedge, \vee \rangle$ is a lattice.

A simple illustration is given in figures 5, 6 and 7.

Remark 2.8. The set theoretic definition for $TS(L_1)$ is given as follows: The binary relation $\leq$ on $TS(L_1)$ is defined by, for all $a \in L_1, a \neq 0, a \neq 1, (a, \perp) \leq (a, \top), (a, \perp) \leq (b, \perp)$ and $(a, \top) \leq (b, \top)$, if $a \leq b \in L_1$ is a lattice ordering on $TS(L_1)$. $(0, \perp)$ and $(1, \top)$ is the minimum and the maximum respectively of $TS(L_1)$.

Remark 2.9. Note that the lattice $TS(L_1)$ can also be constructed by taking the direct product of the two element chain C_2 and the poset $L_1 - \{0, 1\}$ and adjoining the minimum $(0, \perp)$ and the maximum $(1, \top)$ to it.

We shall see some elementary properties of the lattice of $TS(L_1)$.

Lemma 2.10. *Let L_1 be a lattice and $a \in L_1, a \neq 0$ and $a \neq 1$. Let $(a, \perp), (a, \top) \in TS(L_1)$. Then $(a, \perp) \prec_{TS(L_1)} (a, \top)$ for all $a \in L_1$.*

Proof. By definition of $TS(L_1)$, we have, $\forall a, a \neq 0, a \neq 1$ in $L_1, (a, \perp) \wedge (a, \top) = (a, \perp)$ and $(a, \perp) \vee (a, \top) = (a, \top)$.

Therefore, $(a, \perp) \leq_{TS(L_1)} (a, \top)$.

Suppose that $(a, \perp) \not\leq_{TS(L_1)} (a, \top)$. Then there exists some $A \in TS(L_1)$ such that $(a, \perp) \leq_{TS(L_1)} A \leq_{TS(L_1)} (a, \top)$. Then, A is either $(x, \perp)$ or $(x, \top)$. In the former case, we have $x = a$. In the latter case, $(a, \perp) \leq_{TS(L_1)} (x, \top)$ implies $a \leq_{L_1} x$. Also, $(x, \top) \leq_{TS(L_1)} (a, \top)$ implies $x \leq_{L_1} a$. Therefore, we have $x = a$.

Therefore, $(a, \perp) \preceq_{TS(L_1)} (a, \top)$ for all $a \in L_1$. $\square$

Lemma 2.11. *Let L_1 be a lattice. If $a, b \in L_1$ such that $a < b$, then $(b, \perp) \parallel (a, \top)$*

Proof. Let $a, b \in L_1$. By Lemma 2.10, $(a, \perp) \preceq_{TS(L_1)} (a, \top)$ and $(b, \perp) \preceq_{TS(L_1)} (b, \top)$. Also, by the definition of $TS(L_1)$, $(b, \perp) \wedge (a, \top) = (b \wedge a, \perp) = (a, \perp)$ and $(b, \perp) \vee (a, \top) = (b \vee a, \top) = (b, \top)$. This implies that $(b, \perp) \parallel (a, \top)$. $\square$

Lemma 2.12. *Let $a, b \in L_1$. If $a \leq_{L_1} b$ in L_1 , then the following holds:*

- (α) $(a, \perp) \preceq_{TS(L_1)} (b, \perp)$
- (β) $(a, \top) \preceq_{TS(L_1)} (b, \top)$

Proof. Let $a, b \in L_1$ and $a \leq_{L_1} b$. Since $a \leq_{L_1} b$, we have $a \wedge b = a$ and $a \vee b = b$. Then, by definition 2.7, $(a, \perp) \wedge (b, \perp) = (a, \perp)$ and $(a, \perp) \vee (b, \perp) = (b, \perp)$. This implies that $(a, \perp) \leq_{TS(L_1)} (b, \perp)$.

Suppose that there exists $(x, \perp) \in TS(L_1)$ such that $(a, \perp) \leq_{TS(L_1)} (x, \perp) \leq_{TS(L_1)} (b, \perp)$. Since $a \leq_{L_1} b$, x must be either same as a or b , proving (α).

The proof of (β) follows similarly as (α). $\square$

The following theorem gives a characterization for the irreducible elements of $TS(L_1)$.

Theorem 2.13. *Let L_1 be a lattice.*

- (α) *If $a \in \mathbf{J}(L_1)$, then $(a, \perp) \in \mathbf{J}(TS(L_1))$.*
- (β) *If $a \in \text{Atom}(L_1)$, then $(a, \top) \in \mathbf{J}(TS(L_1))$.*
- (γ) *If $m \in \mathbf{M}(L_1)$ then $(m, \top) \in \mathbf{M}(TS(L_1))$*
- (δ) *If $m \in \text{Coatom}(L_1)$, then $(m, \perp) \in \mathbf{M}(TS(L_1))$.*

Conversely, if $A \in \mathbf{J}(TS(L_1))$, then, one of the following holds:

- (α') $A = (a, \perp)$ for some $a \in \mathbf{J}(L_1)$.
- (β') $A = (a, \top)$ for some $a \in \mathbf{J}(L_1) \cap \text{Atom}(L_1)$.

Also, if $A \in \mathbf{M}(TS(L_1))$, then, one of the following holds:

- (γ') $A = (m, \top)$ for some $m \in \mathbf{M}(L_1)$.
- (δ') $A = (m, \perp)$ for some $m \in \mathbf{M}(L_1) \cap \text{Coatom}(L_1)$.

Proof. Let $a \in \mathbf{J}(L_1)$. Then there exists a unique $a^- \in L_1$ such that $a^- \prec_{L_1} a$. Then by Lemma 2.12, $(a^-, \perp) \prec_{TS(L_1)} (a, \perp)$. Clearly $(a^-, \perp)$ is unique and this proves (α) .

Let $a \in \mathbf{J}(L_1)$ such that a is an atom in L_1 . Clearly, $0 \prec_{L_1} a$ and 0 is the unique lower neighbor for a . Since, there exists no $b \in L - \{0\}$ such that $b < a$ in L_1 , there exists no $(b, \top) < (a, \top)$. Also, $(a, \perp) \prec_{TS(L_1)} (a, \top)$, thus being the unique lower neighbor of $(a, \top)$, thus proving (β) .

(γ) and (δ) follows by duality of (α) and (β) respectively.

To Prove (α') : Let $A = (a, \perp) \in \mathbf{J}(TS(L_1))$. Clearly $(a, \perp) \neq \perp$ and $(a, \perp) \neq \top$. Since $(a, \perp) \in \mathbf{J}(TS(L_1))$, there exists $(b, \perp) \in TS(L_1)$ such that $(b, \perp) \prec_{TS(L_1)} (a, \perp)$. From the Lemma 2.12, $b \prec_{L_1} a$.

Now, suppose that, there exists $c \in L$ such that $c \prec_{L_1} a$. Then, $(c, \perp) \prec_{TS(L_1)} (a, \perp)$ implies $(a, \perp) \notin \mathbf{J}(TS(L_1))$, thus proving (α') .

To prove (β') : Let $A = (a, \top) \in \mathbf{J}(TS(L_1))$.

We know that $\forall a \in L_1, (a, \perp) \prec_{TS(L_1)} (a, \top)$. Since $(a, \top) \in \mathbf{J}(TS(L_1))$, $(a, \perp)$ is the unique lower neighbor of $(a, \top)$.

Therefore, $a \in L_1$ must be an atom, for if not, there exists $b \in L_1$ such that $b < L_1 < a$. By Lemma 2.11, $(b, \top) \parallel (a, \perp)$, which implies $(a, \top) \notin \mathbf{J}(TS(L_1))$.

(γ') and (δ') follows by duality of (α') and (β') and respectively. $\square$

2.3. Properties of the lattice $TS(L_1)$.

Theorem 2.14. *If $a \in J(L_1), a \notin Atom(L_1)$, then $(a, \top) \notin J(TS(L_1))$.*

Proof. Let $a \in \mathbf{J}(L_1), a \notin Atom(L_1)$. Then, there exists $a^- \in L_1$ such that $0 \prec_{L_1} a^- \leq a$. By Lemma 2.11, $(a, \perp) \parallel (a^-, \top)$ in $TS(L_1)$. Also, $(a, \perp) \prec_{TS(L_1)} (a, \top)$. Therefore $(a, \top) \notin J(TS(L_1))$. $\square$

Remark 2.15. If a and m are the join and meet irreducible elements of $TS(L_1)$, Theorem 2.13 gives the join and the meet irreducible elements of $TS(L_1)$ and the Lemmas 2.10, 2.11 and 2.12 give the relations between join and the meet irreducible elements of $TS(L_1)$.

Theorem 2.16. *Let L_1 be a lattice. Define*

$$G_1 = \{A \in TS(L_1) | A = (a, \perp) \text{ for some } a \in \mathbf{J}(L_1)\}$$

$$G_2 = \{A \in TS(L_1) | A = (a, \top) \text{ for some } a \in Atom(L_1)\}$$

$$M_1 = \{A \in TS(L_1) | A = (m, \perp) \text{ for some } m \in \mathbf{M}(L_1)\}$$

$$M_2 = \{A \in TS(L_1) | A = (m, \top) \text{ for some } m \in Coatom(L_1)\}$$

Define a context $\mathbb{K}_2(TS(L_1)) = (G_1 \cup G_2, M_1 \cup M_2, I_1 \cup I_2 \cup I_3)$ as follows:

For $(g_1, \perp) \in G_1, (g_2, \top) \in G_2, (m_1, \perp) \in M_1, (m_2, \top) \in M_2$,

- (i) $(g_1, \perp) I_1 (m_1, \perp)$ if $g_1 \leq m_1$ in L_1 .
- (ii) $(g_1, \perp) I_2 (m_2, \top)$ if $g_1 \leq m_2$ in L_1 .
- (iii) $(g_2, \top) I_3 (m_2, \top)$ if $g_2 \leq m_2$ in L_1 .

Then $\mathbb{K}_2(TS(L_1))$ is the formal context of $TS(L_1)$.

Theorem 2.17. *The formal context $\mathbb{K}_2(TS(L_1)) = \frac{L_1 \mid L_1}{\emptyset \mid L_1}$ is reduced if and only if L_1 is atomistic, dually atomistic and its formal context is reduced.*

Proof. Let $\mathbb{K}_2(TS(L_1))$ be reduced. We know that every compatible subcontext of a reduced context is reduced ([1] proposition 38, page 102). Since L_1 is a compatible subcontext of a reduced context $\mathbb{K}_2(TS(L_1))$, we have L_1 is reduced. Further, since the object extents of $\mathbb{K}_2(TS(L_1))$ are independent, we have L_1 is atomistic. Similarly, since the attribute intents of $\mathbb{K}_2(TS(L_1))$ are independent of each other, we have L_1 is dually atomistic.

Conversely, let L_1 be atomistic, dually atomistic and reduced. We observe that the formal context of the form $\frac{L_1 \mid L_1}{\emptyset \mid L_1}$ generates the same concept lattice as

that of $\mathbb{K}_2(TS(L_1))$. From the theorem 2.16, it is clear that its object set is $G_1 \cup G_2$ corresponds to $J(TS(L_1))$ and the attribute set is $M_1 \cup M_2$.

Suppose that, the formal context $\mathbb{K}_2(TS(L_1))$ is not reduced, then there exists an extent A of the forms $(a, \perp)'$ or $(a, \top)'$ which is the intersection of other two extents. Since $G_2 = L_1$ is reduced, we have $(a, \top)'$ is independent of the other object extents, hence cannot be expressed as the intersection of $(b, \top)'$ and $(c, \top)'$. Similarly, $(a, \perp)'$ cannot be the intersection of $(b, \perp)'$ and $(c, \perp)'$. Suppose that, $(a, \top)' = (a, \perp)' \cap (b, \top)'$, then by Lemma 2.11, this corresponds to the lower covers of $(a, \top)'$ which are non comparable. This implies that $b < a$ in L_1 . Hence, a is not an atom in L_1 . Therefore, $\mathbb{K}_2(TS(L_1))$ must be reduced. □

Remark 2.18. For a lattice L_1 , the concept lattice obtained by $\frac{L_1 \mid L_1}{\emptyset \mid L_1}$ is same as that obtained by $\mathbb{K}_2(TS(L_1))$, even though the formal context is not reduced.

Example 2.19. The Lattice L_1 and $TS(L_1)$ is given in Figure 8 and Figure 9 respectively. The corresponding formal context of L_1 and $TS(L_1)$ is given in the Figure 10 and Figure 11 respectively.

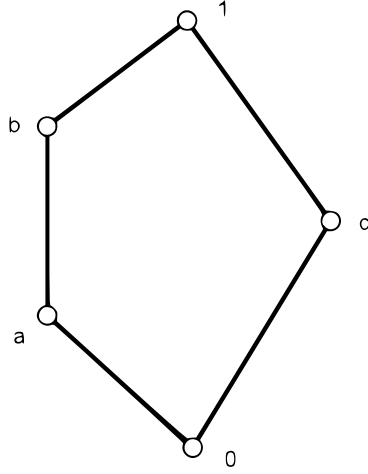
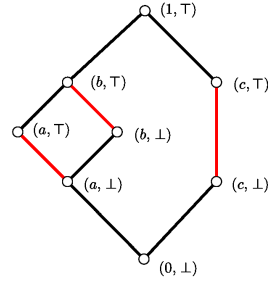
We explore some of the properties of the lattice $TS(L_1)$. The following proposition is easy to observe.

Theorem 2.20. *L_1 can be embedded in $TS(L_1)$ in two ways .*

Proof. Define the maps $h : L_1 \rightarrow TS(L_1)$ and $g : L_1 \rightarrow TS(L_1)$ by

- (i) $h(a) = (a, \perp)$ for $a \neq 1$
 $h(1) = (1, \top)$.
- (ii) $g(a) = (a, \top)$ for $a \neq 0$
 $g(0) = (0, \perp)$

We prove that the map h is well defined. Let $a, b \in L_1$ with $a \neq 0, a \neq 1, a \neq b$. Clearly $h(a) \neq h(b)$. Therefore, h is well defined.

FIGURE 8. Lattice L_1 FIGURE 9. Lattice $TS(L_1)$

	1	2	3
a	X		X
b	X		
c		X	

FIGURE 10. Formal Context of L_1

	(1, ⊥)	(2, ⊥)	(3, ⊥)	(1, T)	(2, T)	(3, T)
(a, ⊥)	X		X	X		X
(b, ⊥)	X			X		
(c, ⊥)		X			X	
(a, T)				X		X
(b, T)				X		
(c, T)					X	

FIGURE 11. Formal Context of $TS(L_1)$

Let $h(a) = h(b)$ in $TS(L_1)$. Then $(a, \perp) = (b, \perp)$. This hold if and only if $a = b$. Therefore h is one to one.

$$h(a) \wedge h(b) = (a, \perp) \wedge (b, \perp) = (a \wedge b, \perp) = h(a \wedge b)$$

$$h(0) \wedge h(a) = (0, \perp) \wedge (a, \perp) = (0 \wedge a, \perp) = h(0 \wedge a).$$

Therefore, h is meet homomorphism.

$$h(a) \vee h(b) = (a, \perp) \vee (b, \perp) = (a \vee b, \perp) = h(a \vee b)$$

$$\therefore h(a \vee b) = h(a) \vee h(b)$$

$$h(1) \vee h(a) = (1, T) \vee (a, \wedge) = (1, T)$$

$$\begin{aligned} h(1 \vee a) &= h(1) = (1, \top) \\ \therefore h(1 \vee a) &= h(1) \vee h(a). \end{aligned}$$

Therefore, h is join homomorphism. Hence h is an embedding. Similarly, it can be proved that g is an embedding. $\square$

Remark 2.21. Define a map $f : TS(L_1) \rightarrow L_1 \times L_1$ by $f((a, \perp)) = (a, 0)$ and $f((1, \top)) = (1, 1)$ where 1 is the maximum of L_1 . This is well defined, one to one map which is order preserving. The map is not a homomorphism in general.

We shall now discuss the Modular(distributive) property of the lattice $TS(L_1)$.

Theorem 2.22. *Let L_1 be a lattice with more than one atom. Then, the lattice $TS(L_1)$ is not modular.*

Proof. Let L_1 be a lattice. Let $a, b \in \text{Atom}(L_1)$. Clearly, $a \parallel b$. Then $(a, \perp)$ and $(b, \perp)$ are atoms in $TS(L_1)$ such that $(a, \perp) \parallel (b, \perp)$ and $(a, \top) \parallel (b, \top)$. By Lemma 2.10, $(a, \perp) < (a, \top)$ and $(b, \perp) < (b, \top)$.

Let $(a, \top), (a, \perp), (b, \top) \in TS(L_1)$ with $(a, \perp) < (a, \top)$, Consider,

$$(a, \top) \wedge [(b, \top) \vee (a, \perp)] = (a, \top) \wedge (b \vee a, \top) = (a, \top) \quad (2.1)$$

Consider

$$[(a, \top) \wedge (a, \perp)] \vee [(a, \top) \wedge (b, \top)] = (a, \perp) \vee (0, \perp) = (a, \perp) \quad (2.2)$$

Therefore, $TS(L_1)$ is not modular. $\square$

CONCLUSION

For a lattice L_1 , we have shown that the formal context $\mathbb{K}_1(CS(L_1))$ obtained from the substitution sum $\frac{L_1}{L_1} \Big| \frac{L_1}{X}$ gives the lattice $CS(L_1)$. Also, if L_1 is a lattice with more than one atom and dual atom, we have shown that the formal context $\mathbb{K}_2(TS(L_1))$ obtained from the substitution sum $\frac{L_1}{\emptyset} \Big| \frac{L_1}{L_1}$ gives the lattice $TS(L_1)$. However, the restrictions on the lattice L_1 can be avoided with small modifications in the results of $TS(L_1)$.

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