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OUTER-CLIQUE ROMAN DOMINATING FUNCTION IN GRAPHS

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Abstract. This paper introduces a new restricted variant of a Roman dominating function in graphs called the outer-clique Roman dominating function and discusses some graph-theoretic properties.

Keywords. Outer-clique domination, Roman domination, outer-clique Roman domination
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1. Introduction and preliminaries

In graph theory, a dominating set is a fascinating topic that has been a center of interest for many discrete mathematicians. Currently, numerous research papers have been published in the literature regarding domination in graphs, which involve motivating theoretical structures [3, 4, 5, 6, 7, 12, 13, 14]. In 2004, Cockayne et al. [11] pioneered the concept of Roman domination in graphs. Roman domination is one of the concepts that has been a center of mathematics research for several years to this day. The idea of Roman domination is founded on the defense strategy of Roman Emperor Constantine the Great sometime in the fourth century A.D. Apparently, there are now a number of restricted variations of Roman domination in the literature, and some of those can be found in [1, 2, 8, 10, 16]. In the year 1990, dominating cliques in graphs was introduced by Cozzens and Kelleher [13], and in 2018, Ravina et al. [17] initiated the concept of outer-clique domination in graphs. Motivated by outer-clique domination and Roman domination, the author combined the two concepts and introduced a new parameter called the outer-clique Roman dominating function in graphs. This new parameter of Roman domination is a stronger version than the usual in the sense that the undefended (labeled 0) regions are connected to each other. The possible application of this new parameter in computer science is designing a network security system and a complete connection of digital infrastructures. The concepts and terminologies used in this paper are found in [9, 15, 17].

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Let $G=(V(G),E(G))$ be a connected graph where $V(G)$ is the collection of vertices and $E(G)$ is the collection of edges. The order of G is the cardinality of $V(G)$, denoted by $|V(G)|$, and the size of G is the cardinality of $E(G)$, denoted by $|E(G)|$. Let $u \in V(G)$. The *open neighborhood* of u in graph G is the set $N_G(u) = \{y \in V(G) : uy \in E(G)\}$ and the *closed neighborhood* of a vertex $u \in V(G)$ is the set $N_G[u] = N_G(u) \cup \{u\}$. Let $C \subseteq V(G)$. The set $N_G(C) = N(C) = \bigcup_{v \in C} N_G(v)$ is called the *open neighborhood* of C and the set $N_G[C] = N[C] = N(C) \cup C$ is called the *closed neighborhood*. Let u and v be two distinct vertices in the graph G. Then the distance between the two u and v, denoted by $d_G(u,v)$, is the length of the shortest walk between u and v. If there is no such walk, then we can define the distance as $d_G(u,v) = \infty$. A *path* is a walk denoted by P_n with order n and size $n-1$. If there exists one path in G that connects every two vertices $u, v \in V(G)$, then G is a *connected graph*; otherwise, G is called *disconnected graph*. Now, a $u-v$ path with length $d_G(u,v)$ is called $u-v$ geodesic. The set $I_G[u,v]$ is a closed interval that consists of all vertices that lie on a $u-v$ geodesic in graph G. Let $P \subset V(G)$. Then the union of all sets $I_G[u,v]$ where $u, v \in P$ is denoted by $I_G[P]$, that is, $\cup_{u,v \in P} I_G[u,v] = I_G[P]$. In that case, $x \in I_G[P]$ if and only if x is in some $u-v$ geodesic for any $u, v \in P$. Hence, a set P is convex if $I_G[P] = P$. Consequently, $V(G)$ is convex if G is connected. A complete graph denoted by K_n is defined as the graph with every pair of distinct vertices joined by an edge, that is, for every $x, y \in K_n$, $xy \in E(K_n)$. Let S be a non-empty subset of $V(G)$. Then S is a clique on G if the subgraph $\langle S \rangle$ is complete. A *maximal clique* on G is a clique that is not a subset of any larger clique on G. Let $D \subseteq V(G)$. Then D is a dominating set of graph G if for every $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$ [15]. Consequently, we get $N[D] = V(G)$. The *domination number* denoted by $\gamma(G)$ is the minimum cardinality of a dominating set D in G. If D is a dominating set with $|D| = \gamma(G)$, then D is called γ-set in G. Let $\varnothing \neq S \subseteq V(G)$. Then S is a *clique dominating set* on G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$ and S is a clique on G [13], [17]. The minimum cardinality of S is called *clique domination number* and is denoted by $\gamma_{cl}(G)$. A clique dominating set S that satisfies $|S| = \gamma_{cl}(G)$ is called γ_{cl}-set of G. A subset O of $V(G)$ is called a *outer-clique dominating set* on G if for every $v \in V(G) \setminus O$, there exists $u \in O$ such that $uv \in E(G)$ and $V(G) \setminus O$ is a clique on G [17]. The minimum cardinality of O is called *outer-clique domination number* and is denoted by $\widetilde{\gamma}_{cl}(G)$. An outer-clique dominating set O that satisfies $|O| = \widetilde{\gamma}_{cl}(G)$ is called $\widetilde{\gamma}_{cl}$-set of G.

Let $\phi : V(G) \to \{0,1,2\}$ be a function on G, and for each $i \in \{0,1,2\}$, let $V_i = \{z \in V(G) : \phi(z) = i\}$. Then we denote ϕ by $\phi = (V_0, V_1, V_2)$ as a function on G. A function $\phi : V(G) \to \{0,1,2\}$ is a *Roman dominating function* (RDF) on G provided that for every vertex $v \in V_0$ is adjacent to at least one vertex $u \in V_2$ [11]. The *weight* of an RDF ϕ is defined by $\omega_G^R(\phi) = \sum_{v \in V(G)} \phi(v) = |V_1| + 2|V_2|$. The *Roman domination number* of G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G, that is, $\gamma_R(G) = min\{\omega_G^R(\phi) : \phi \text{ } is \text{ } an \text{ RDF } on \text{ } G\}$. So, every RDF ϕ on G with $\omega_G^R(\phi) = \gamma_R(G)$ is called a γ_R-function on G. A function ϕ is an outer-clique Roman dominating function (OClRDF) on G provided that for every $v \in V_0$, there exists $u \in V_2$ such that $uv \in E(G)$, and either $V_1 = V(G)$ or V_0 is a clique on G. The *weight* of OClRDF ϕ denoted by $\widetilde{\omega}_G^{clR}(\phi)$ is the sum $\widetilde{\omega}_G^{clR}(\phi) = \sum_{u \in V(G)} \phi(u)$, that is, $\widetilde{\omega}_G^{clR}(\phi) = |V_1| + 2|V_2|$. The *outer-clique Roman domination number* of G denoted by $\widetilde{\gamma}_{clR}(G)$ is the minimum weight of an OClRDF on G, that is, $\widetilde{\gamma}_{clR}(G) = min\{\widetilde{\omega}_G^{clR}(\phi) : \phi \textit{ is an } \text{OClRDF} \textit{ on } G\}$. Any OClRDF ϕ on graph G with $\widetilde{\omega}_G^{clR}(\phi) = \widetilde{\gamma}_{clR}(G)$ is called a $\widetilde{\gamma}_{clR}$-function on G. In this paper, we introduced a new variant of Roman dominating function in graphs, namely outer-clique Roman domination. In addition, the mathematical properties of an outer-clique Roman dominating function were investigated, and characterizations were provided. Moreover, lower and upper bounds were given, and a realization problem was constructed.

2. Main results

In this section, we explore some properties of the outer-clique Roman domination on any connected graph. We start with a very useful remark immediately from the definition of outer-clique Roman domination.

Remark 2.1. Let G be a connected graph of order n and let $\phi = (V_0, V_1, V_2)$ be an OClRDF on G. If $\widetilde{\gamma}_{clR}(G) < n$, then $V_0 \neq \varnothing$ and $\langle V_0 \rangle$ is a complete subgraph of G.

The next results are graph-theoretic properties of OClRDF.

Proposition 2.2. *Let G be a connected graph of order n and let $\phi = (V_0, V_1, V_2)$ be an OClRDF on G. Then $|V_0| = |V_2|$ if and only if $\widetilde{\gamma}_{clR}(G) = n$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ is a OClRDF on G. Assume $|V_2| = |V_0|$. Then, it follows that $\widetilde{\gamma}_{clR}(G) = \widetilde{\omega}_G^{clR}(\phi) = |V_1| + 2|V_2| = |V_1| + |V_2| + |V_0| = |V(G)| = n$. Conversely, assume that $\widetilde{\gamma}_{clR}(G) = n$. Seeking a contradiction. Suppose for a moment that $|V_0| \neq |V_2|$. Then either $|V_0| > |V_2|$ or $|V_0| < |V_2|$. In either case, it follows that $\widetilde{\gamma}_{clR}(G) \neq n$. This is a contradiction. Therefore, it is concluded that $|V_0| = |V_2|$. This completes the proof. □

It is worth noting that the possible limitation of the outer-clique Roman dominating function is that for a graph G of order $n \geq 1$ with a small maximal clique, it implies that $\widetilde{\gamma}_{clR}(G) \approx n$. In fact, for triangle-free graphs, $|V_2| = |V_0| \leq 2$, and hence, by Proposition 2.2, we have that $\widetilde{\gamma}_{clR}(G) = n$. Moreover, as a consequence of Proposition 2.2, it is clear that $|V_2| < |V_0|$ if and only if $\widetilde{\gamma}_{clR}(G) < n$. The following corollary is immediate from Proposition 2.2.

Corollary 2.3. *Let n be a positive integer. Then the following holds:*

i.) $\widetilde{\gamma}_{clR}(\overline{K_n}) = n$ *where* $n \geq 1$*;*
ii.) $\widetilde{\gamma}_{clR}(P_n) = n$ *where* $n \geq 1$*;*
iii.) $\widetilde{\gamma}_{clR}(C_n) = n$ *where* $n \geq 3$*; and*
iv.) $\widetilde{\gamma}_{clR}(S_n) = n + 1$ *where* $n \geq 3$.

Proposition 2.4. *Let G be a connected graph. If $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{clR}$-function on G, then $V_1 \cup V_2$ is an outer-clique dominating set on G.*

Proof. Let G be any connected graph. Suppose that $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{clR}$-function on G. Then, it implies that ϕ is an OClRDF on G. In that case, for every $v \in V_0$, there exists $u \in V_2$ such that $uv \in E(G)$, and either $V_1 = V(G)$ or the subgraph $\langle V_0 \rangle$ is complete, that is, V_0 is a clique on G. Note that $V_2 \subseteq V_1 \cup V_2$. Therefore, it follows that $V_1 \cup V_2$ is an outer-clique dominating set on G. This completes the proof. □

Proposition 2.5. *Let G be a connected graph of order n and let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{clR}$-function on G. Then $V_2 = \varnothing$ if and only if $V_0 = \varnothing$. In that case, $\widetilde{\gamma}_{clR}(G) = n$.*

Proof. Let G be any connected graph of order n. Suppose that $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{clR}$-function on G such that $V_2 = \varnothing$. Then ϕ is a Roman dominating function on G. Since $V_2 = \varnothing$, it follows that $V_1 = V(G)$. Hence, we have $V_0 = \varnothing$. Conversely, suppose that $V_0 = \varnothing$. Seeking a contradiction. Assume for a moment that $V_2 \neq \varnothing$. Let $u \in V_2$. Set $X_0 = V_0$, $X_1 = V_1 \cup \{u\}$, and $X_2 = V_2 \setminus \{u\}$. Then, it implies that $\phi' = (X_0, X_1, X_2)$ is an outer-clique Roman dominating function on G. Observe that

$$\begin{aligned}
\widetilde{\omega}_G^{clR}(\phi') &= |X_1| + 2|X_2| \\
&= (|V_1| + 1) + 2(|V_2| - 1) \\
&= |V_1| + 2|V_2| - 1 \\
&< \widetilde{\omega}_G^{clR}(\phi) \\
&= \widetilde{\gamma}_{clR}(G).
\end{aligned}$$

This is a contradiction. Therefore, it is concluded that $V_2 = \varnothing$. In that case, we get $\widetilde{\gamma}_{clR}(G) = |V_1| + 2|V_2| = |V_1| + 2(0) = |V_1| = |V(G)| = n$. This completes the proof. □

Proposition 2.6. *Let G be a connected graph. If $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{clR}$-function on G for which $V_1 = \varnothing$, then V_2 is a $\widetilde{\gamma}_{cl}$-set on G. Moreover, $\widetilde{\gamma}_{clR}(G) = 2\widetilde{\gamma}_{cl}(G)$.*

Proof. Let G be any connected graph. Suppose that $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{clR}$-function on G such that $V_1 = \varnothing$. Then, by Proposition 2.4, we have $V_1 \cup V_2 = V_2$ is an outer-clique dominating set on G. Seeking a contradiction. Assume for a moment that V_2 is not a $\widetilde{\gamma}_{cl}$-set on G. Let V_2' be a $\widetilde{\gamma}_{cl}$-set on G. Then, it follows that $|V_2'| < |V_2|$. Now, we define a function $\phi' = (U_0, U_1, U_2)$ on G for which $U_0 = V(G) \setminus V_2'$, $U_1 = \emptyset$ and $U_2 = V_2'$. Thus, it is clear that ϕ' is an OClRDF on G. And so, we get $\widetilde{\omega}_G^{clR}(\phi') = 2|U_2| < 2|V_2| = \widetilde{\gamma}_{clR}(G)$. A contradiction. Therefore, it means V_2 is a $\widetilde{\gamma}_{cl}$-set on G. To this end, we obtain $\widetilde{\gamma}_{clR}(G) = \widetilde{\omega}_G^{clR}(\phi) = |V_1| + 2|V_2| = 2|V_2| = 2\widetilde{\gamma}_{cl}(G)$. This completes the proof. □

Proposition 2.7. *Let G be a connected graph and let $\phi = (V_0, V_1, V_2)$ be an OClRDF on G for which $V_1 = \varnothing$. If V_2 is a $\widetilde{\gamma}_{cl}$-set on G, then ϕ is $\widetilde{\gamma}_{clR}$-function on G.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be an OClRDF on a connected graph G such that $V_1 = \varnothing$. Assume that V_2 is a $\widetilde{\gamma}_{cl}$-set on G. Then $V_0 = V(G) \setminus V_2$ is a clique on G.

Seeking a contradiction. Suppose for a moment that ϕ is not a $\widetilde{\gamma}_{clR}$-function on G. Then it follows that there exists a function $\phi' = (W_0, W_1 = \varnothing, W_2)$ such that ϕ' is a $\widetilde{\gamma}_{clR}$-function on G. Hence, $\widetilde{\gamma}_{clR}(G) = \widetilde{\omega}_G^{clR}(\phi') = |W_1| + 2|W_2| = 2|W_2| < \widetilde{\omega}_G^{clR}(\phi) = |V_1| + 2|V_2| = 2|V_2|$. Thus, we have $|W_2| < |V_2|$. This is a contradiction to our assumption. Therefore, ϕ is a $\widetilde{\gamma}_{clR}$-function on G. This completes the proof. $\square$

The next result is a characterization of an outer-clique Roman dominating function on a complete graph.

Theorem 2.8. *Let G be a connected graph and let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{clR}$-function on G. Then $|V_1| = 0$ and $|V_2| = 1$ if and only if $G = K_n$ where $n \geq 2$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{clR}$-function on a connected graph G. Suppose that $|V_1| = 0$ and $|V_2| = 1$. By Proposition 2.4, it follows that $V_1 \cup V_2 = V_2$ is an outer-clique dominating set on G. Thus, $V(G) \setminus V_2 = V_0$ is a clique on G, and so, the subgraph $\langle V_0 \rangle$ is complete. Hence, it means that V_0 is a convex set, that is, $I_G[V_0] = V_0$. Assume for a moment that there exist distinct vertices $x, y \in V_0$ such that $xy \notin E(G)$. Since V_2 is a dominating set, it implies that $xv, vy \in E(G)$ for some $v \in V_2$. Thus, it follows that $x, v, y \in I_G[x, y] \subseteq I_G[V_0]$. Since $v \notin V_0$, it implies that $I_G[V_0] \neq V_0$, a contradiction to the assumption that V_0 is a clique on G. Hence, it means that for any $x, y \in V_0$, $xy \in E(G)$ and so $\langle V_0 \rangle$ is a complete graph. Since $V_0 \subseteq N_G[v]$, it follows that $G = \langle V_2 \rangle + \langle V_0 \rangle$ and thus, $G = K_n$. Conversely, suppose that $G = K_n$ where $n \geq 2$. If $G = K_2$, then it is easy to check that $|V_1| = 0$ and $|V_2| = 1$. Now, let $n \geq 3$. Let $v \in V(G)$. Set $V_0 = V(G) \setminus \{v\}$, $V_1 = \varnothing$ and $V_2 = \{v\}$. Since $v \in V(G)$ is arbitrary, it follows that $\phi = (V_0, V_1, V_2)$ is an RDF on G. In addition, the subgraph $\langle V_0 \rangle$ is a complete graph, and thus, V_0 is a clique in G. So, it follows that ϕ is an OClRDF on G. Therefore, it is concluded that $|V_1| = 0$ and $|V_2| = 1$. This completes the proof. $\square$

The following propositions are immediate from Theorem 2.8.

Proposition 2.9. *Let G be a connected graph. Then $\widetilde{\gamma}_{clR}(G) = 1$ if and only if $G = K_1$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{clR}$-function on G. Suppose that $\widetilde{\gamma}_{clR}(G) = 1$. Then, it follows that $|V_1| + 2|V_2| = 1$. This implies that $V_2 = \varnothing$. By Proposition 2.5, it follows that $V_0 = \varnothing$. Hence, $\widetilde{\gamma}_{clR}(G) = \widetilde{\omega}_G^{clR}(\phi) = |V_1| + 2|V_2| = |V_1| = 1 = |V(G)|$. Therefore, it suffices to conclude that $G = K_1$. The converse is clear. This completes the proof. $\square$

Proposition 2.10. *Let G be a connected graph. Then $\widetilde{\gamma}_{clR}(G) = 2$ if and only if $G = K_n$ where $n \geq 2$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{clR}$-function on a connected graph G. Assume that $\widetilde{\gamma}_{clR}(G) = 2$. Then it follows that $\widetilde{\omega}_G^{clR}(\phi) = |V_1| + 2|V_2| = 2$. Thus, $|V_2| \leq 1$. Consider that $|V_2| = 0$. By Proposition 2.5, it implies that $V_0 = \varnothing$. Hence, we have $\widetilde{\gamma}_{clR}(G) = \widetilde{\omega}_G^{clR}(\phi) = |V_1| + 2|V_2| = |V_1| = 2 = |V(G)|$. Since G is connected, we obtain $G = K_2$. Now, suppose $|V_2| = 1$. Then it follows that $|V_1| = 0$. Since

ϕ is a $\widetilde{\gamma}_{clR}$-function on G, by Theorem 2.8, we have that $G = K_n$ where $n \geq 2$. Conversely, assume that $G = K_n$ where $n \geq 2$. Since ϕ is a $\widetilde{\gamma}_{clR}$-function on G, by Theorem 2.8, it follows that $|V_1| = 0$ and $|V_2| = 1$. Therefore, we obtain $\widetilde{\gamma}_{clR}(G) = \widetilde{\omega}_G^{clR}(\phi) = |V_1| + 2|V_2| = 2(1) = 2$. This completes the proof. □

The next corollary is a direct consequence of Proposition 2.10.

Corollary 2.11. *Let G and H be two complete graphs. Then $\widetilde{\gamma}_{clR}(G + H) = 2$.*

Theorem 2.12. *Let G be a connected graph of order n and let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{clR}$-function on G such that $\widetilde{\gamma}_{clR}(G) < n$. Then $\widetilde{\gamma}_{clR}(G) = \widetilde{\gamma}_{cl}(G) + 1$ if and only if there is only one vertex $v \in V_2$ such that $V_0 \subseteq N_G[v]$ and V_0 is a clique on G.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{clR}$-function on a connected graph G with $|V(G)| = n$ and $\widetilde{\gamma}_{clR}(G) < n$. Suppose that $\widetilde{\gamma}_{clR}(G) = \widetilde{\gamma}_{cl}(G) + 1$. Then it follows that $|V_0| \geq 1$ and by Proposition 2.5, $V_2 \neq \varnothing$. Seeking a contradiction. Assume for a moment that $|V_2| \geq 2$. Then let $v \in V_2$ such that $V_0 \subseteq N_G[v]$ and the subgraph $\langle V_0 \rangle$ is complete. Let $C \subseteq V(G)$ be a $\widetilde{\gamma}_{cl}$-set on G. Then C is an outer-clique dominating set on G. Invoking Proposition 2.4, we can let $C = V_1 \cup V_2$. This implies that $\widetilde{\gamma}_{cl}(G) = |C| = |V_1| + |V_2| = |V_1| + 2|V_2| - |V_2| = \widetilde{\omega}_G^{clR}(\phi) - |V_2| = \widetilde{\gamma}_{clR}(G) - |V_2|$. So, we have $\widetilde{\gamma}_{cl}(G) = \widetilde{\gamma}_{clR}(G) - |V_2|$ where $|V_2| \geq 2$, a contradiction to our assumption. Hence, $V_2 = \{v\}$ and $V_0 \subseteq N_G[v]$ for which the subgraph $\langle V_0 \rangle$ is complete, that is, V_0 is a clique on G. Suppose that there is only one vertex in V_2 such that $V_0 \subseteq N_G[V_2]$ and V_0 is a clique on G, that is, $V_2 = \{v\}$. Then it implies that $V_0 \neq \varnothing$. Now, let $D \subset V(G)$ be a $\widetilde{\gamma}_{cl}$-set on G. Then D is an outer-clique dominating set on G. In view of Proposition 2.4, we let $D = V_1 \cup V_2$. Then $\widetilde{\gamma}_{cl}(G) = |D| = |V_1| + |V_2|$. Now, since $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{clR}$-function on G, it follows that $\widetilde{\gamma}_{clR}(G) = \widetilde{\omega}_G^{clR}(\phi) = |V_1| + 2|V_2| = |V_1| + |V_2| + |V_2| = |D| + |V_2| = \widetilde{\gamma}_{cl}(G) + |V_2|$. Therefore, we end up with $\widetilde{\gamma}_{clR}(G) = \widetilde{\gamma}_{cl}(G) + 1$. This completes the proof. □

Theorem 2.13. *Let G be a non-complete connected graph of order $n \geq 4$ and let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{clR}$-function on G. Then $V_1 = \varnothing$ and $\widetilde{\gamma}_{clR}(G) = 4$ if and only if there exists a dominating set $\{x, y\}$ such that $V(G) \setminus \{x, y\} = V_0$ is a clique on G.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{clR}$-function on a non-complete connected graph G with $|V(G)| = n \geq 4$. Assume that $V_1 = \varnothing$ and $\widetilde{\gamma}_{clR}(G) = 4$. Then, it follows that $\widetilde{\gamma}_{clR}(G) = |V_1| + 2|V_2| = 2|V_2| = 4$ and so, $|V_2| = 2$. Now, let $V_2 = \{x, y\}$. By Proposition 2.4, it implies that $V_1 \cup V_2 = V_2 = \{x, y\}$ is an outer-clique dominating set on G. So, it means that $\{x, y\}$ is a dominating set on G for which $V(G) \setminus \{x, y\} = V_0$ and the subgraph $\langle V_0 \rangle$ is complete, that is, V_0 is a clique on G. Conversely, assume that there exists a dominating set $\{x, y\}$ such that $V(G) \setminus \{x, y\} = V_0$ is a clique on G. Then, it implies that $\{x, y\}$ is an outer-clique dominating set on G. Thus, we have $V_1 = \varnothing$ and $V_2 = \{x, y\}$. It follows that $\widetilde{\gamma}_{clR}(G) \leq |V_1| + 2|V_2| = 2|V_2| = 2(2) = 4$. Since G is a non-complete graph, by Theorem 2.8, it follows that $|V_2| \geq 2$. This means that $\widetilde{\gamma}_{clR}(G) = \widetilde{\omega}_G^{clR}(\phi) = |V_1| + 2|V_2| = 2|V_2| \geq 4$. Therefore, it suffices to conclude that $\widetilde{\gamma}_{clR}(G) = 4$. This completes the proof. □

The next theorem exposed the lower and upper bounds of an outer-clique Roman domination on any connected graph G.

Theorem 2.14. *Let G be a connected graph with $|V(G)| = n$. Then,*

$$max\{\gamma_R(G), \widetilde{\gamma}_{cl}(G)\} \leq \widetilde{\gamma}_{clR}(G) \leq min\{2\widetilde{\gamma}_{cl}(G), n\}.$$

Proof. Let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{clR}$-function on a connected graph G of order n. In view of Proposition 2.4, it implies that $V_1 \cup V_2$ is an outer-clique dominating set on G. Observe that $\widetilde{\gamma}_{cl}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \widetilde{\omega}_G^{clR}(\phi) = \widetilde{\gamma}_{clR}(G)$. Thus, we get $\widetilde{\gamma}_{cl}(G) \leq \widetilde{\gamma}_{clR}(G)$. In addition, since every outer-clique Roman dominating function on G is a Roman dominating function on G, it suffices to say that $\gamma_R(G) \leq \widetilde{\gamma}_{clR}(G)$. Consequently, we obtain $max\{\widetilde{\gamma}_{clR}(G), \gamma_R(G)\} \leq \widetilde{\gamma}_{clR}(G)$. On the other hand, we set $V_0 = \varnothing$. Then, we get $V_2 = \varnothing$. On the face of it, $\phi = (\varnothing, V_1 = V(G), \varnothing)$ is an outer-clique Roman dominating function on G. So, we have $\widetilde{\gamma}_{clR}(G) \leq |V_1| + 2|V_2| = |V_1| + 2(0) = |V_1| = |V(G)| = n$. Now, suppose that $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{clR}$-function on G for which $V_1 = \varnothing$. Let C be a $\widetilde{\gamma}_{cl}$-set on G. Then C is an outer-clique dominating set on G. In view of Proposition 2.4, we let $V_1 \cup V_2 = V_2 = C$. Thus, it follows that $\widetilde{\gamma}_{clR}(G) \leq |V_1| + 2|V_2| = 0 + 2|V_2| = 2|C| = 2\widetilde{\gamma}_{cl}(G)$. To this end, we get $\widetilde{\gamma}_{clR}(G) \leq min\{n, 2\widetilde{\gamma}_{cl}(G)\}$. Therefore, we end up with $max\{\gamma_R(G), \widetilde{\gamma}_{cl}(G)\} \leq \widetilde{\gamma}_{clR}(G) \leq min\{2\widetilde{\gamma}_{cl}(G), n\}$. This completes the proof. □

The next result is a direct consequence of Theorem 2.14

Corollary 2.15. *Let G be a connected graph. If $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{clR}$-function on G, then $0 \leq |V_2| \leq |V_0|$.*

Proof. Assume that $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{clR}$-function on G. Invoking Theorem 2.14, we obtain $\widetilde{\gamma}_{clR}(G) \leq n$. First, we suppose $\widetilde{\gamma}_{clR}(G) = n$. In view of Proposition 2.2, it implies that $0 \leq |V_2| = |V_0|$. Secondly, we suppose $\widetilde{\gamma}_{clR}(G) < n$. As a consequence of Proposition 2.2, we have that $|V_2| < |V_0|$. Therefore, $0 \leq |V_2| \leq |V_0|$. This completes the proof. □

The next result is a realization problem.

Theorem 2.16. *Let a, b and n be positive integers for which $1 \leq a \leq b \leq n$. Then there exists a connected graph G such that $|V(G)| = n$, $\gamma_R(G) = a$ and $\widetilde{\gamma}_{clR}(G) = b$.*

Proof. Assume that G is a connected graph. Let $f = (W_0, W_1, W_2)$ be a γ_R-function on G and let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{clR}$-function on G. Then, consider the following cases:
Case 1. Let $1 \leq a \leq b = n$.
Clearly, if $G = K_1$, then $\gamma_R(G) = \widetilde{\gamma}_{clR}(G) = 1$. Let $G = K_1 + \overline{K_{n-1}}$ where $n \geq 2$. Then it is clear that $W_0 = V(\overline{K_{n-1}})$, $W_1 = \varnothing$, and $W_2 = V(K_1)$. So, it follows that $\gamma_R(G) = \omega_G^R(f) = |W_1| + 2|W_2| = 2|W_2| = 2(1) = 2$. Now, let $u \in V(\overline{K_{n-1}})$. Then it is easy to see that $V_0 = \{u\}$, $V_1 = V(\overline{K_{n-1}}) \setminus \{u\}$ and $V_2 = V(K_1) = \{v\}$ such that $uv \in E(G)$. Thus, it follows that $\widetilde{\gamma}_{clR}(G) = \widetilde{\omega}_G^{clR}(\phi) = |V_1| + 2|V_2| = (n-2) + 2(1) = n$, and the conclusion holds.
Case 2. Let $1 < a < b < n$.

Consider the graph $G = K_p \circ K_2$ where $p \geq 3$. Then it follows that $|V(G)| = n = 3p$ where $p \geq 3$. Let $K_p = [v_1, v_2, ..., v_p]$. So, it implies that $W_0 = \cup_{i=1}^{p} V(K_2^{v_i})$, $W_1 = \varnothing$, and $W_2 = V(K_p)$. In that case, we get $\gamma_R(G) = \omega_G^R(f) = |W_1| + 2|W_2| = 2|W_2| = 2|V(K_p)| = 2p < 3p = n$. Now, let $v \in V(K_p)$. Then it is clear that $V_0 = V(K_p - \{v\})$, $V_1 = \cup_{i=1}^{p} V(K_2^{v_i})$, and $V_2 = \{v\}$. Thus, we obtain $\widetilde{\gamma}_{clR}(G) = \widetilde{\omega}_G^{clR}(\phi) = |V_1| + 2|V_2| = |\cup_{i=1}^{p} V(K_2^{v_i})| + 2(1) = 2p + 2$. Since $p \geq 3$, it implies that $\widetilde{\gamma}_{clR}(G) = 2p + 2 < 3p = n$. Set $a = 2p$ and $b = 2p + 2$, and the result follows.
Case 3. Let $1 < a = b < n$.
Consider $G = K_n$ where $n \geq 3$. Let $v \in V(G)$. Then it is easy to check that $W_0 = V(G) \setminus \{v\}$, $W_1 = \varnothing$, $V_2 = \{v\}$. This implies that $\gamma_R(G) = \omega_G^R(f) = |W_1| + 2|W_2| = 2|W_2| = 2(1) = 2$. Note that $V(G) \setminus \{v\}$ is a clique on G. Hence, we have $V_0 = V(G) \setminus \{v\}$, $V_1 = \varnothing$, and $V_2 = \{v\}$. Thus, we also get $\widetilde{\gamma}_{clR}(G) = \widetilde{\omega}_G^{clR}(\phi) = |V_1| + 2|V_2| = 2|V_2| = 2(1) = 2$. Therefore, we end up with $1 < a = 2 = b < n$ and the assertion follows.
This completes the proof. □

The next corollary is a direct consequence of Theorem 2.16.

Corollary 2.17. *Let G be a non-trivial connected graph with $|V(G)| = n \geq 1$. Then the difference $\widetilde{\gamma}_{clR}(G) - \gamma_R(G)$ can be made arbitrarily large.*

Proof. Let k be a positive integer. By Theorem 2.16, there exists a connected graph G such that $\widetilde{\gamma}_{clR}(G) = k+2$ and $\gamma_R(G) = 2$. Then, it follows that $\widetilde{\gamma}_{clR}(G) - \gamma_R(G) = k$. Increasing k as large as possible, it implies that $\widetilde{\gamma}_{clR}(G) - \gamma_R(G)$ can be made arbitrarily large. This completes the proof. □

3. Conclusion

In this paper, we introduced a new variation of Roman domination in graphs called outer-clique Roman domination. Some theoretical properties were exposed and presented with formal and detailed mathematical proofs. It is shown that for a connected graph G of order n, the lower bound of $\widetilde{\gamma}_{clR}(G)$ is $max\{\gamma_R(G), \widetilde{\gamma}_{cl}(G)\}$ and the upper bound is $min\{2\widetilde{\gamma}_{cl}(G), n\}$. Moreover, this study revealed that given positive integers a, b and n satisfying to the condition $1 \leq a \leq b \leq n$, there exists a connected graph G such that $|V(G)| = n$, $\gamma_R(G) = a$ and $\widetilde{\gamma}_{clR}(G) = b$. For future research, characterization of outer-clique Roman domination in graphs under some binary operations, such as join, corona, Cartesian, and lexicographic products, is recommended for investigation to strengthen the current findings of this study.

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