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DOUBLE $\mathcal{K}_\phi^\alpha$-FRACTIONAL SEMI-GROUPS OF OPERATORS

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Abstract. Based on a new definition of $\mathcal{K}_\phi^\alpha$-derivative defined in Definition 3.1, we introduce a new definition of double $\mathcal{K}_\phi^\alpha$-semigroup of operators. Further, we establish some operational formulas, and we set the relation between the double $\mathcal{K}_\phi^\alpha$-semigroup and the C_0-$\mathcal{K}_\phi^\alpha$-semigroup. Finally, some illustrative examples and applications are constructed.

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1. Introduction and preliminaries

Consider a Banach space X, and let $\mathcal{B}(X)$ denote the space of all bounded linear operators on X. A family of operators $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called a semigroup of operators if it satisfies

(1) $T(0) = I$, the identity operator,
(2) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$.

If, in addition

$$\lim_{t\to 0^+} T(t)x = x \quad \text{for all } x \in X,$$

then the semigroup is called a C_0-semigroup (strongly continuous semigroup). Such semigroups are fundamental tools for studying the evolution of systems over time, including linear evolution equations such as the heat equation, wave propagation, or population dynamics.

In recent years, fractional calculus has attracted significant attention because fractional derivatives allow the modeling of memory effects and nonlocal behaviors. One example is the $\mathcal{N}_F^\alpha$-derivative, introduced by Napoles [6], defined as

$$\mathcal{N}_F^\alpha(f)(t) = \lim_{h\to 0} \frac{f\left(t + \frac{h}{F(t,\alpha)}\right) - f(t)}{h}, \quad F(t,\alpha) \neq 0,$$

for all $t \in [0, +\infty)$. This derivative generalizes the classical derivative and provides a flexible framework to study fractional dynamics in Banach spaces.

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Bahloul et al. [4] introduced the concept of an $\mathcal{N}_F^\alpha$-semigroup, which extends classical semigroup theory to the fractional setting. Such semigroups are useful for analyzing fractional differential equations and have applications in physics, engineering, and biology.

In this work, we focus on double conformable semigroups, which consider fractional behavior in two variables simultaneously. This approach allows the modeling of more complex systems where different components may evolve differently over time. Our methodology differs from previous studies, such as [8], by using the $\mathcal{N}_F^\alpha$-fractional derivative to define and analyze these double semigroups.

Fractional semigroups and their extensions, such as double conformable semigroups, are not only of theoretical interest but also have important practical applications. For instance, they are used to model anomalous diffusion in physics, viscoelastic materials in engineering, and population dynamics in biology, where the evolution of the system depends on past states or exhibits nonlocal interactions. Moreover, the double conformable framework allows one to handle multi-dimensional fractional systems, where different variables may follow different fractional dynamics, providing a more accurate and flexible modeling tool for real-world phenomena.

The paper is organized as follows: Section 2 presents preliminary definitions and results on $\mathcal{N}_F^\alpha$-semigroups, while Section 3 contains the main results concerning double conformable semigroups and their properties.

Throughout this paper, we assume that $\alpha \in (0,1]$, which includes the classical derivative case when $\alpha = 1$. These results provide a foundation for studying multivariable fractional systems and extend the theory of fractional operator semigroups.

2. PRELIMINARIES

In this section, we summarize some of the proven results in the reference [4].

Definition 2.1. Let $f : [0,+\infty[\to \mathbb{R}$ be a given function . Then the $\mathcal{K}_\phi^\alpha$-derivative of order α is defined by :

$$\mathcal{K}_\phi^\alpha(f)(t) := \lim_{\xi\to 0} \frac{f(t+\frac{\xi}{\phi(t,\alpha)}) - f(t)}{\xi}.$$

with $\phi(t,\alpha) \neq 0$, for all $t \in [0,+\infty[$. If this limit exists, then the function f is said to be $\mathcal{K}_\phi^\alpha$-differentiable at t.

Definition 2.2. Consider a function $f : [0,+\infty[\to \mathbb{R}$.

(1) The function f is said to be $\mathcal{K}_\phi^\alpha$-differentiable on $[0,+\infty[$ if f is continuous, $\mathcal{K}_\phi^\alpha f(t)$ exist for all $t \in]0,+\infty[$ and $\mathcal{K}_\phi^\alpha f(0) = \lim_{t\to 0^+} \mathcal{K}_\phi^\alpha f(t)$ exists.

(2) The function f is said to be continuously $\mathcal{K}^\alpha$-differentiable on $[0,+\infty)$ if f is $\mathcal{K}_\phi^\alpha$-differentiable on $[0,+\infty)$ and $\mathcal{K}_\phi^\alpha f(t)$ is continuous on $[0,+\infty[$.

For more information, see [2]. We will suffice with the following characteristics: For all $t > 0$

(1) $\mathcal{K}_\phi^\alpha(f)(t) = \frac{f'(t)}{\phi(t,\alpha)}$.

(2) $\mathcal{K}_\phi^\alpha(\mathcal{I}_\phi^\alpha f(t)) = f(t)$.
(3) $\mathcal{I}_\phi^\alpha(\mathcal{K}_\phi^\alpha f(t)) = f(t) - f(0)$.

Where $\mathcal{I}_\phi^\alpha f(t) = \int_0^t \phi(s) f(s) ds$.

Example 2.3. For $\phi(t,\alpha) = 3\alpha^2 t^2 + 2\alpha t + 1$, we have

$$\mathcal{K}_\phi^\alpha\left(3\alpha^2\frac{t^5}{5} + \frac{\alpha t^4}{2} + (1+6\alpha^2)\frac{t^3}{3} + 2\alpha t^2 + 2t\right) = t^2 + 2.$$

Example 2.4. For $\phi(t,\alpha) = 3\alpha^2 t^2 + \alpha t + 1,\ t \in [0,+\infty)$ and $\alpha \in (0,1]$, we have

$$\mathcal{I}_\phi^\alpha(t^2+1) = 3\alpha^2\frac{t^5}{5} + \alpha\frac{t^4}{4} + (3\alpha^2+1)\frac{t^3}{3} + \alpha\frac{t^2}{2} + t.$$

Example 2.5. For $\phi(t,\alpha) = 2\alpha t + 1$, we have by Example 2.4

$$\mathcal{K}_\phi^\alpha(\mathcal{I}_\phi^\alpha(t^2+1)) = \frac{\left(3\alpha^2\frac{t^5}{5} + \alpha\frac{t^4}{4} + (3\alpha^2+1)\frac{t^3}{3} + \alpha\frac{t^2}{2} + t\right)'}{3\alpha^2t^2 + \alpha t + 1} = t^2 + 1,$$

for all $t > 0$.

Example 2.6. Let's take the function defined in Example 2.4.

$$\mathcal{I}_\phi^\alpha(\mathcal{K}_\phi^\alpha f(t)) = \mathcal{I}_\phi^\alpha\left(\frac{2t}{3\alpha^2t^2 + \alpha t + 1}\right) = \int_0^t 2s ds = t^2 = f(t) - f(0),$$

where $f(t) = t^2 + 1$.

Let us consider a function $\phi(t,\alpha)$ such that $\phi(t,\alpha) > 0$ for all $t > 0$ and $\nu_\alpha(t)$ its primitive function verifies $\nu_\alpha(0) = 0$, $\lim\limits_{t\to+\infty} \nu_\alpha(t) = +\infty$ and ν_α is invertible on $[0,+\infty)$.

Definition 2.7. [4] Consider a Banach space X and a family $\{T(t), t \geq 0\} \subset \mathcal{B}(X)$. We say that the family $\{T(t\}_{t\geq 0} \subset \mathcal{B}(X)$ is a $\mathcal{K}_\phi^\alpha$-semigroup of operators if:

(1) $T(0) = I$,
(2) $T(\nu_\alpha^{-1}(t+s)) = T(\nu_\alpha^{-1}(t))T(\nu_\alpha^{-1}(s))$, for all $t, s \in [0,\infty)$.

Remark 2.8. The $\mathcal{K}_\phi^\alpha$-semigroups corresponds to the usual semigroups, when $\nu_\alpha^{-1} = I$.

Example 2.9. [4] Let A be a bounded linear operator. We consider

$$\nu_\alpha(t) = \alpha \cosh t \quad \text{and} \quad T(t) = e^{\frac{A\alpha t}{\alpha+t}\cosh t},$$

it follows that

$$\nu_\alpha^{-1}(t) = argch\left(1 + \frac{t}{\alpha}\right),$$

and

$$T\left(argch\left(1+\frac{s+t}{\alpha}\right)\right) = e^{A(s+t)} = e^{As}e^{At} = T\left(argch\left(1+\frac{s}{\alpha}\right)\right)T\left(argch\left(1+\frac{t}{\alpha}\right)\right).$$

Definition 2.10. [4] Let $\{T(t)\}_{t\geq 0}$ be a $\mathcal{K}_\phi^\alpha$-semigroup, then $\{T(t)\}_{t\geq 0}$ is said to be C_0-$\mathcal{K}_\phi^\alpha$-semigroup if, for each fixed $x \in X$, $T(t)x \longrightarrow x$ as $t \longrightarrow 0^+$.

The $\mathcal{K}_\phi^\alpha$-derivative of $T(t)$ at $t = 0$ is termed the $\mathcal{K}_\phi^\alpha$-infinitesimal generator of the fractional $\mathcal{K}_\phi^\alpha$-semigroup $T(t)$, with domain equals

$$\left\{ x \in X : \lim_{t \to 0^+} \mathcal{K}_\phi^\alpha T(t)x \text{ exists } \right\}.$$

We denote this generator by A.

Proposition 2.11. [4]
1. Let $\{T(t)\}_{t\geq 0}$ be a C_0-$\mathcal{K}_\phi^\alpha$-semigroup. Then the family $\{S(t)\}_{t\geq 0}$ defined as follows

$$S(t) = T\left(\nu_\alpha^{-1}(t)\right), \quad \textit{for all } t \geq 0,$$

is a C_0-semigroup.
2. Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup. Then the family $\{T(t)\}_{t\geq 0}$ defined as follows

$$T(t) = S\left(\nu_\alpha(t)\right), \quad \textit{for all } t \geq 0,$$

is a C_0-$\mathcal{K}_\phi^\alpha$-semigroup.
3. Let $\{T(t)\}_{t\geq 0}$ be a C_0-$\mathcal{K}_\phi^\alpha$-semigroup. Then there exist constants $\omega \geq 0$ and $M \geq 1$ such that for all $t \geq 0$,

$$\|T(t)\| \leq M e^{\omega \nu_\alpha(t)}.$$

Corollary 2.12. [4] *Let $\{T(t)\}_{t\geq 0}$ be a C_0-$\mathcal{K}_\phi^\alpha$-semigroup. Then for any $x \in X$, the function $t \mapsto T(t)x$ is continuous. Hence the family $\{T(t)\}_{t\geq 0}$ is said to be strongly continuous.*

Theorem 2.13. [4] *Consider $\{T(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ a C_0-$\mathcal{K}_\phi^\alpha$-semigroup with infinitesimal generator A. If $T(t)$ is continuously $\mathcal{K}_\phi^\alpha$-differentiable and $x \in D(A)$, then*

$$\mathcal{K}_\phi^\alpha T(t)x = AT(t)x = T(t)Ax.$$

Lemma 2.14. [4] *Consider $\{T(t)\}_{t\geq 0}$ a C_0-$\mathcal{K}_\phi^\alpha$-semigroup, with infinitesimal generator A. Then*

$$\lim_{h \mapsto 0} \frac{1}{h} \int_t^{t+h} T(u)\phi(u,\alpha)du = T(t)\phi(t,\alpha) \quad \forall t \geq 0.$$

Theorem 2.15. [4] *Consider $\{T(t)\}_{t\geq 0}$ a C_0-$\mathcal{K}_\phi^\alpha$-semigroup, with generator A. Then:*

$$A\left(\int_0^t T(s)\phi(s,\alpha)x\,\mathrm{d}s\right) = T(t)\phi(t,\alpha)x - \phi(0,\alpha)x.$$

3. DOUBLE $\mathcal{K}_\phi^\alpha$-SEMIGROUPS

Definition 3.1. Let $f:\mathbb{R}_+^2\to X$ be a given function. We say that f is $\mathcal{K}_\phi^\alpha$-differentiable at (s,t), with $s,t>0$, if there exists a linear bounded operator $\mathcal{K}_\phi^\alpha:\mathbb{R}^2\to X$ and an open neighbourhood U of (s,t) such that

$$\left\| f\left(s+\frac{h}{\phi(s,\alpha)},\, t+\frac{k}{\phi(t,\alpha)}\right)-f(s,t)-\mathcal{K}_\phi^\alpha f(s,t)(h,k)\right\| \le \|\varepsilon(h,k)\|\|(h,k)\|.$$

Where $\varepsilon:\mathbb{R}^2\to X$ is a function such that $\|\varepsilon(h,k)\|\to 0$ as $\|(h,k)\|\to 0$.
If such a linear operator $\mathcal{K}_\phi^\alpha$ exists, then it is unique and called the $\mathcal{K}_\phi^\alpha$-*derivative* of f of order $\alpha\in(0,1]$ at the point (s,t).

Definition 3.2. Let $f:\mathbb{R}_+^2\to X$ be a given function. The function f is said to be $\mathcal{K}_\phi^\alpha$-differentiable at the point $(0,0)$ if the following conditions hold:

(1) The limit $\lim\limits_{(s,t)\to(0^+,0^+)}\mathcal{K}_\phi^\alpha f(s,t)$ exists and
$$\lim_{(s,t)\to(0^+,0^+)}\mathcal{K}_\phi^\alpha f(s,t)=\mathcal{K}_\phi^\alpha f(0,0).$$

(2) The limits $\lim\limits_{s\to0^+}\mathcal{K}_\phi^\alpha f(s,t)$ and $\lim\limits_{t\to0^+}\mathcal{K}_\phi^\alpha f(s,t)$ exists and
$$\lim_{s\to0^+}\mathcal{K}_\phi^\alpha f(s,t)=\mathcal{K}_\phi^\alpha f(0,t),\quad \lim_{t\to0^+}\mathcal{K}_\phi^\alpha f(s,t)=\mathcal{K}_\phi^\alpha f(s,0).$$

Theorem 3.3. *Let $f:\mathbb{R}_+^2\to X$ be a $\mathcal{K}_\phi^\alpha$-differentiable function at a point (s,t), with $s,t>0$, then f is continuous at (s,t).*

Proof. Since

$$\begin{aligned}&\left\| f\left(s+\frac{h}{\phi(s,\alpha)},\, t+\frac{k}{\phi(t,\alpha)}\right)-f(s,t)\right\|\\ &\le \left\| f\left(s+\frac{h}{\phi(s,\alpha)},\, t+\frac{k}{\phi(t,\alpha)}\right)-f(s,t)-\mathcal{K}_\phi^\alpha f(s,t)(h,k)\right\| + \left\|\mathcal{K}_\phi^\alpha f(s,t)(h,k)\right\|\\ &\le \|\varepsilon(h,k)\|\|(h,k)\|+\|\mathcal{K}_\phi^\alpha f(s,t)\|\|(h,k)\|.\end{aligned}$$

By taking the limit as $(h,k)\to(0,0)$, we obtain that f is continuous at (s,t). □

Definition 3.4. Let $f:\mathbb{R}_+^2\to\mathbb{R}$ be a given function and $0<\alpha\le 1$. Then the $\mathcal{K}_{\phi,s}^\alpha$-derivative of order α respect to s is defined by :

$$\mathcal{K}_{\phi,s}^\alpha(f)(s,t):=\lim_{\xi\to0}\frac{f(s+\frac{\xi}{\phi(s,\alpha)},t)-f(s,t)}{\xi},$$

and the $\mathcal{K}_{\phi,s}^\alpha$-derivative of order α respect to t is defined by :

$$\mathcal{K}_{\phi,t}^\alpha(f)(s,t):=\lim_{\xi\to0}\frac{f(s,t+\frac{\xi}{\phi(t,\alpha)})-f(s,t)}{\xi}.$$

Theorem 3.5. *Let $f:\mathbb{R}_+^2\to X$ be a $\mathcal{K}_\phi^\alpha$-differentiable function at (a,b) with $a,b>0$, then the $\mathcal{K}_\phi^\alpha$-partial derivatives $\mathcal{K}_{\phi,s}^\alpha f(a,b)$ and $\mathcal{K}_{\phi,t}^\alpha f(a,b)$ exists, and the $\mathcal{K}_\phi^\alpha$-derivative of f at (a,b) is given by*

$$\mathcal{K}_\phi^\alpha f(a,b)=\left(\mathcal{K}_{\phi,s}^\alpha f(a,b),\mathcal{K}_{\phi,t}^\alpha f(a,b)\right).$$

Proof. Assume tha $f : \mathbb{R}_+^2 \to X$ is $\mathcal{K}_\phi^\alpha$-differentiable function at (a, b) with $a, b > 0$, then there exists a linear operator $\mathcal{K}_\phi^\alpha : \mathbb{R}^2 \to X$ and a neighbourhood U of (a, b) such that

$$\left\| f\left(a + \frac{h}{\phi(a,\alpha)}, b + \frac{k}{\phi(b,\alpha)}\right) - f(a,b) - \mathcal{K}_\phi^\alpha f(a,b)(h,k) \right\| \le \varepsilon(h,k)\|(h,k)\|.$$

Where $\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ is a function such that $\varepsilon(h, k) \to 0$ as $\|(h, k)\| \to 0$.
Let $k = 0$, we have

$$\left\| f\left(a + \frac{h}{\phi(a,\alpha)}, b + \frac{k}{\phi(b,\alpha)}\right) - f(a,b) - \mathcal{K}_\phi^\alpha f(a,b)(h,0) \right\| \le \varepsilon(h,0)\|(h,0)\|.$$

Divide by $h \neq 0$ and take the limit as $h \to 0$

$$\lim_{h\to 0} \frac{f(a + \frac{h}{\phi(a,\alpha)}, b) - f(a,b)}{h} = \mathcal{K}_\phi^\alpha f(a,b)(1,0).$$

Then the partial derivative $\mathcal{K}_{\phi,s}^\alpha$ exists.
To prove the existence of the partial derivative $\mathcal{K}_{\phi,t}^\alpha$, we proceed in the same way. We get

$$\lim_{k\to 0} \frac{f(a, b + \frac{k}{\phi(b,\alpha)}) - f(a,b)}{k} = \mathcal{K}_\phi^\alpha f(a,b)(0,1).$$

On the other hand, we have

$$\begin{aligned} \mathcal{K}_\phi^\alpha f(a,b)(h,k) &= h\mathcal{K}_\phi^\alpha f(a,b)(1,0) + k\mathcal{K}_\phi^\alpha f(a,b)(0,1) \\ &= h\mathcal{K}_{\phi,s}^\alpha f(a,b) + k\mathcal{K}_{\phi,t}^\alpha f(a,b) \\ &= \left(\mathcal{K}_{\phi,s}^\alpha f(a,b), \mathcal{K}_{\phi,t}^\alpha f(a,b)\right). \end{aligned}$$

□

Definition 3.6. A family $\{T(s,t)\}_{s,t\ge 0} \subset \mathcal{B}(X)$ is called a double semigroup of operators if:

(1) $T(0, 0) = I$
(2) $T(s_1 + s_2, t_1 + t_2) = T(s_1, t_1)T(s_2, t_2)$.

Definition 3.7. Let $\alpha \in (0, a]$ for any $a > 0$ For a Banach space X, A family $\{T(s,t)\}_{s,t\ge 0} \subset \mathcal{B}(X)$ is called a double $\mathcal{K}_\phi^\alpha$-semigroup of operators if:

(1) $T(0, 0) = I$.
(2) For all $s_1, s_2, t_1, t_2 \ge 0$,

$$T\big(\nu_\alpha^{-1}(s_1 + s_2),\, \nu_\alpha^{-1}(t_1 + t_2)\big) = T\big(\nu_\alpha^{-1}(s_1),\, \nu_\alpha^{-1}(t_1)\big)\, T\big(\nu_\alpha^{-1}(s_2),\, \nu_\alpha^{-1}(t_2)\big).$$

Example 3.8. Consider two commuting $\mathcal{K}_\phi^\alpha$-semigroups $\{R(s)\}_{s\ge 0}$ and $\{S(t)\}_{t\ge 0}$. Then the family $\{T(s,t)\}_{s,t\ge 0} \subset \mathcal{B}(X)$, defined by

$$T(s,t) = R(s)\, S(t), \quad s, t \ge 0,$$

is a double $\mathcal{K}_\phi^\alpha$-semigroup. Indeed, we have

$$\begin{aligned}T\big(\nu_\alpha^{-1}(s_1+s_2),\, \nu_\alpha^{-1}(t_1+t_2)\big) &= R\big(\nu_\alpha^{-1}(s_1+s_2)\big)S\big(\nu_\alpha^{-1}(t_1+t_2)\big)\\ &= R\big(\nu_\alpha^{-1}(s_1)\big)R\big(\nu_\alpha^{-1}(s_2)\big)S\big(\nu_\alpha^{-1}(t_1)\big)S\big(\nu_\alpha^{-1}(t_2)\big)\\ &= \big[R\big(\nu_\alpha^{-1}(s_1)\big)S\big(\nu_\alpha^{-1}(t_1)\big)\big]\big[R\big(\nu_\alpha^{-1}(s_2)\big)S\big(\nu_\alpha^{-1}(t_2)\big)\big]\\ &= T\big(\nu_\alpha^{-1}(s_1),\, \nu_\alpha^{-1}(t_1)\big)\, T\big(\nu_\alpha^{-1}(s_2),\, \nu_\alpha^{-1}(t_2)\big),\end{aligned}$$

which proves the double $\mathcal{K}_\phi^\alpha$-semigroup property.

Example 3.9. Let A, B be two bounded linear operators. Consider

$$\nu_\alpha(t) = \alpha sinh(t) \quad \text{and} \quad T(s,t) = e^{\alpha\left(Asinh(s)+Bsinh(t)\right)},$$

It follows that $\nu_\alpha^{-1}(t) = argsh\left(\frac{t}{\alpha}\right)$. Thus

$$\begin{aligned}&T\left(argsh\left(\frac{s_1+s_2}{\alpha}\right), argsh\left(\frac{t_1+t_2}{\alpha}\right)\right) = e^{A(s_1+s_2)+B(t_1+t_2)}\\ &= e^{As_1+Bt_1}e^{As_2+Bt_2}\\ &= T\left(argsh\left(\frac{s_1}{\alpha}\right), argsh\left(\frac{t_1}{\alpha}\right)\right) T\left(argsh\left(\frac{s_2}{\alpha}\right), argsh\left(\frac{t_2}{\alpha}\right)\right).\end{aligned}$$

Definition 3.10. Consider a Banach space X and a family $\{T(s,t)\}_{s,t\geq 0}$forming a double $\mathcal{K}_\phi^\alpha$-semigroup. Then:

(1) The family $\{T(s,t)\}_{s,t\geq 0}$ is said to be uniformly continuous if

$$\lim_{(s,t)\to(0^+,0^+)} \|T(s,t) - I\| = 0.$$

(2) The family $\{T(s,t)\}_{s,t\geq 0}$ is said to be a double C_0-$\mathcal{K}_\phi^\alpha$-semigroup if, for every $x \in X$,

$$\lim_{(s,t)\to(0^+,0^+)} \|T(s,t)x - x\| = 0.$$

Lemma 3.11. *Let $\{T(s,t)\}_{s,t\geq 0}$ be a double $\mathcal{K}_\phi^\alpha$-semigroup. Assume that $\nu_\alpha^{-1}(0)$ exists and $\nu_\alpha^{-1}(0) = 0$. Then $\{T(s,t)\}_{s,t\geq 0}$ is a C_0-$\mathcal{K}_\phi^\alpha$-semigroup if and only if both $\{T(s,0)\}_{s\geq 0}$ and $\{T(0,t)\}_{t\geq 0}$ are C_0-$\mathcal{K}_\phi^\alpha$-semigroups.*

Proof. Assume that $(T(s,t))_{s,t\geq 0}$ is a double C_0-$\mathcal{K}_\phi^\alpha$-semigroup. Then, in particular, by taking $s = 0$ or $t = 0$, we obtain that $\{T(s,0)\}_{s\geq 0}$ and $\{T(0,t)\}_{t\geq 0}$ are C_0-$\mathcal{K}_\phi^\alpha$-semigroups.

Conversely, for any $s,t \geq 0$ we have

$$\begin{aligned}T(s,t) &= T\big(\nu_\alpha^{-1}(\nu_\alpha(s)+0),\, \nu_\alpha^{-1}(0+\nu_\alpha(t))\big)\\ &= T\big(s, \nu_\alpha^{-1}(0)\big)\, T\big(\nu_\alpha^{-1}(0), t\big) = T\big(\nu_\alpha^{-1}(0), t\big)T\big(s, \nu_\alpha^{-1}(0)\big)\\ &= T(s,0)\, T(0,t) = T(0,t)\, T(s,0).\end{aligned}$$

Suppose now that $\{T(s,0)\}_{s\geq 0}$ and $\{T(0,t)\}_{t\geq 0}$ are C_0-$\mathcal{K}_\phi^\alpha$-semigroups. Then, for any $s,t\geq 0$ and $x\in X$, we have

$$\begin{aligned}\|T(s,t)x-x\| &= \|T(s,0)T(0,t)x-T(s,0)x+T(s,0)x-x\|\\ &= \|T(s,0)(T(0,t)x-x)+(T(s,0)x-x)\|\\ &\leq \|T(s,0)\|\,\|T(0,t)x-x\|+\|T(s,0)x-x\|.\end{aligned}$$

By the uniform boundedness principle, there exist constants $a>0$ and $M>0$ such that

$$\|T(s,0)\|\leq M \quad \text{for all } s\in(0,a).$$

Then, for any $t\geq 0$, $s\in(0,a)$, and $x\in X$,

$$\|T(s,t)x-x\|\leq M\,\|T(0,t)x-x\|+\|T(s,0)x-x\|,$$

which approaches zero as $(s,t)\to(0^+,0^+)$. This shows that $\{T(s,t)\}_{s,t\geq 0}$ is indeed a double C_0-$\mathcal{K}_\phi^\alpha$-semigroup. □

Example 3.12. Let A and B be two bounded linear operators. Let us consider

$$T(s,t)=e^{Asinh(\alpha s)+Bsinh(\alpha s)}$$

From Example 2.9 we have $T(s,0)=e^{Asinh(\alpha s)}$ and $T(0,t)=e^{Bsinh(\alpha t)}$ are $\mathcal{K}_\phi^\alpha$-semigroups, and we have $\lim\limits_{s\to 0^+}T(s,0)x=x$ and $\lim\limits_{t\to 0^+}T(0,t)x=x$.
Then $\{T(s,0)\}_{s\geq 0}$ and $\{T(0,t)\}_{t\geq 0}$ are C_0-$\mathcal{K}_\phi^\alpha$-semigroups. Thus $\{T(s,t)\}_{s,t\geq 0}$ is a double C_0-$\mathcal{K}_\phi^\alpha$-semigroup.

Proposition 3.13. *1. Let $\{T(s,t)\}_{s,t\geq 0}$ be a double C_0- $\mathcal{K}_\phi^\alpha$-semigroup. Then the family $\{S(s,t)\}_{s,t\geq 0}$ defined as follows*

$$S(s,t)=T\left(\nu_\alpha^{-1}(s),\nu_\alpha^{-1}(t)\right),\textit{for all}\quad s,t\geq 0$$

is a double C_0-semigroup.
2. Let $\{S(s,t)\}_{s,t\geq 0}$ be a double C_0-semigroup. Then the family $\{T(s,t)\}_{s,t\geq 0}$ defined as follows

$$T(s,t)=S\left(\nu_\alpha(s),\nu_\alpha(t)\right),\textit{for all}\quad s,t\geq 0$$

is a double C_0-$\mathcal{K}_\phi^\alpha$-semigroup.

Proof. 1. Assume that $\{T(s,t)\}_{s,t\geq 0}$ is a double C_0- $\mathcal{K}_\phi^\alpha$-semigroup.
It follows that

$$S(0,0)=T\left(\nu_\alpha^{-1}(0),\nu_\alpha^{-1}(0)\right)=T(0,0)=I.$$

For all $t_1,t_2,s_1,s_2\geq 0$,

$$\begin{aligned}S(s_1+s_2,t_1+t_2) &= T\left(\nu_\alpha^{-1}(s_1+s_2),\nu_\alpha^{-1}(t_1+t_2)\right)\\ &= T\left(\nu_\alpha^{-1}(s_1,t_1)\right)T\left(\nu_\alpha^{-1}(s_2,t_2)\right)\\ &= S(s_1,t_1)S(s_2,t_2).\end{aligned}$$

Then $\{S(s,t)\}_{t\geq 0}$ is double-$\mathcal{K}_\phi^\alpha$-semigroup.
Assume that $(\nu_\alpha^{-1}(s),\nu_\alpha^{-1}(t))\to(0,0)$ as $(s,t)\to(0,0)$. For some fixed $a>0$, let $\alpha\in(0,a]$, and define

$$(h,k)=(\nu_\alpha^{-1}(s),\nu_\alpha^{-1}(t))\quad\text{with } s,t>0.$$

Then $(h,k)\to(0^+,0^+)$ as $(s,t)\to(0^+,0^+)$.

For any $x\in X$, we have

$$\begin{aligned}\lim_{(s,t)\to(0^+,0^+)} S(s,t)x &= \lim_{(s,t)\to(0^+,0^+)} T(\nu_\alpha^{-1}(s),\nu_\alpha^{-1}(t))x\\ &= \lim_{(h,k)\to(0^+,0^+)} T(h,k)x\\ &= x.\end{aligned}$$

2. Similar to 1 □

Proposition 3.14. *Consider a Banach space X and a family $\{T(s,t)\}_{s,t\geq 0}$ forming a C_0-$\mathcal{K}_\phi^\alpha$-semigroup. Then there exist constants $\omega\geq 0$ and $M\geq 1$ such that*

$$\|T(s,t)\|\leq M\, e^{\omega(\nu_\alpha(s)+\nu_\alpha(t))} \quad \textit{for all } s,t\geq 0.$$

Proof. Let $\{T(s,t)\}_{s,t\geq 0}$ be a double C_0-$\mathcal{K}_\phi^\alpha$-semigroup on a Banach space X. By Lemma 3.11, we have $\{T(s,0)\}_{s\geq 0}$ and $\{T(0,t)\}_{t\geq 0}$ are C_0-$\mathcal{K}_\phi^\alpha$-semigroups. Hence, there exist constants $\omega_1,\omega_2\geq 0$ and $M_1,M_2\geq 1$ such that

$$\|T(s,0)\|\leq M_1 e^{\omega_1\nu_\alpha(s)} \quad\text{and}\quad \|T(0,t)\|\leq M_2 e^{\omega_2\nu_\alpha(t)}.$$

For all $s,t\geq 0$,

$$\|T(s,t)\|=\|T(s,0)T(0,t)\|\leq\|T(s,0)\|\,\|T(0,t)\|\leq M_1M_2e^{\omega_1\nu_\alpha(s)+\omega_2\nu_\alpha(t)}.$$

Let $\omega=\max(\omega_1,\omega_2)$ and $M=M_1M_2$. Then,

$$\|T(s,t)\|=\|T(s,0)T(0,t)\|\leq Me^{\omega(\nu_\alpha(s)+\nu_\alpha(t))}.$$

□

Definition 3.15. Consider a Banach space X and a family $\{T(s,t)\}_{s,t\geq 0}$ forming a double $\mathcal{K}_\phi^\alpha$-semigroup. The operator $\{T(s,t)\}_{s,t\geq 0}$ is said to be strongly continuous if, for all $x\in X$ and all $s_0,t_0\geq 0$,

$$\lim_{(s,t)\to(s_0,t_0)}\|T(s,t)x-T(s_0,t_0)x\|=0,$$

where the limit is taken as $(s,t)\to(0^+,0^+)$ when $(s_0,t_0)\to(0,0)$.

Corollary 3.16. *Consider a Banach space X and a family $\{T(s,t)\}_{s,t\geq 0}$ forming a double $\mathcal{K}_\phi^\alpha$-semigroup. Then $\{T(s,t)\}_{s,t\geq 0}$ is strongly continuous if and only if it is a C_0-$\mathcal{K}_\phi^\alpha$-semigroup.*

Proof. Assume that $\{T(s,t)\}_{s,t\geq 0}$ is strongly continuous, then it is a C_0-$\mathcal{K}_\phi^\alpha$-semigroup.
Conversely, let $s_0,t_0\geq 0$.

(1) For $(s,t)\in[0,s_0)\times[0,t_0)$ and $x\in X$.

$$\begin{aligned}\|T(s,t)x-T(s_0,t_0)x\| &\leq \|T(s,t)\|\,\|x-T(s_0-s,t_0-t)x\|\\ &\leq Me^{\omega\left(\nu_\alpha(s)+\nu_\alpha(t)\right)}\,\|x-T(s_0-s,t_0-t)x\|.\end{aligned}$$

Since

$$\lim_{(s,t)\to(s_0,t_0)} e^{\omega\left(\nu_\alpha(s)+\nu_\alpha(t)\right)}\,\|x-T(s_0-s,t_0-t)x\|=0,$$

it follows that

$$\lim_{(s,t)\to(s_0,t_0)} \|T(s,t)x - T(s_0,t_0)x\| = 0.$$

(2) For $(s,t) \in [0, s_0) \times (t_0, t_0+1]$ and $x \in X$.

$$\begin{aligned}\|T(s,t)x - T(s_0,t_0)x\| &= \|T(s,t_0)\ (T(0,t-t_0)x - T(s_0-s,0)x)\ \| \\ &\leq \|T(s,t_0)\|\,\|T(0,t-t_0)x - T(s_0-s,0)x\| \\ &\leq Me^{\omega\left(\nu_\alpha(s)+\nu_\alpha(t_0)\right)}\ \|T(0,t-t_0)x - T(s_0-s,0)x\|.\end{aligned}$$

Since

$$\lim_{(s,t)\to(s_0,t_0)} \|T(0,t-t_0)x - T(s_0-s,0)x\| = 0,$$

it follows that

$$\lim_{(s,t)\to(s_0,t_0)} \|T(s,t)x - T(s_0,t_0)x\| = 0.$$

(3) For $(s,t) \in (s_0, s_0+1] \times [0, t_0)$ and $x \in X$.

$$\begin{aligned}\|T(s,t)x - T(s_0,t_0)x\| &= \|T(s_0,t)\ (T(s-s_0,0)x - T(0,t_0-t)x)\ \| \\ &\leq \|T(s_0,t)\|\,\|T(s-s_0,0)x - T(0,t_0-t)x\| \\ &\leq Me^{\omega\left(\nu_\alpha(s_0)+\nu_\alpha(t)\right)}\ \|T(s-s_0,0)x - T(0,t_0-t)x\|.\end{aligned}$$

From

$$\lim_{(s,t)\to(s_0,t_0)} \|T(s-s_0,0)x - T(0,t_0-t)x\| = 0,$$

we conclude that

$$\lim_{(s,t)\to(s_0,t_0)} \|T(s,t)x - T(s_0,t_0)x\| = 0.$$

(4) For $(s,t) \in (s_0, s_0+1] \times (t_0, t_0+1]$ and $x \in X$.

$$\begin{aligned}\|T(s,t)x - T(s_0,t_0)x\| &= \|T(s_0,t_0)\,(T(s-s_0,t-t_0)x - x)\| \\ &\leq \|T(s_0,t_0)\|\,\|T(s-s_0,t-t_0)x - x\| \\ &\leq Me^{\omega\left(\nu_\alpha(s_0)+\nu_\alpha(t_0)\right)}\ \|T(s-s_0,t-t_0)x - x\|.\end{aligned}$$

Since

$$\lim_{(s,t)\to(s_0,t_0)} \|T(s-s_0,t-t_0)x - x\| = 0,$$

we conclude that

$$\lim_{(s,t)\to(s_0,t_0)} \|T(s,t)x - T(s_0,t_0)x\| = 0.$$

Hence, $\{T(s,t)\}_{s,t\geq 0}$ is strongly continuous. □

Definition 3.17. The conformable $\mathcal{K}_\phi^\alpha$-derivative of $T(s,t)$ at $(t,s) = (0,0)$ is called the $\mathcal{K}_\phi^\alpha$-infinitesimal generator of the double $\mathcal{K}_\phi^\alpha$-semigroup $\{T(s,t)\}_{s,t\geq 0}$, with domain equals

$$D(A) = \{x \in X : T(\cdot,\cdot)x \text{ is } \mathcal{K}_\phi^\alpha\text{-differentiable at } (0,0)\},$$

and defined, for all $x \in D(A)$, by

$$Ax = \mathcal{K}_{\phi}^{\alpha}\big(T(0,0)x\big).$$

Lemma 3.18. *Consider a Banach space X and a family $\{T(s,t)\}_{s,t\geq 0}$ forming a double C_0-$\mathcal{K}_{\phi}^{\alpha}$-semigroup. Then the $\mathcal{K}_{\phi}^{\alpha}$-infinitesimal generator A satisfies*

$$Ax = \left(\lim_{s\to 0^+} \mathcal{K}_{\phi,s}^{\alpha} T(s,0)x, \lim_{t\to 0^+} \mathcal{K}_{\phi,t}^{\alpha} T(0,t)x\right).$$

Proof. Let $x \in D(A)$. Then $T(\cdot,\cdot)x$ is $\mathcal{K}_{\phi}^{\alpha}$-differentiable at $(0,0)$.
Thus $\mathcal{K}_{\phi}^{\alpha}(T(s,t)x)$ exists for $(s,t) \in]0,a[\times]0,b[$, $a,b>0$, and

$$\mathcal{K}_{\phi}^{\alpha}(T(0,0)x) = \lim_{(s,t)\to(0^+,0^+)} \mathcal{K}_{\phi}^{\alpha}(T(s,t)x) \text{ exists.}$$

Let $(s,t) \in]0,a[\times]0,b[$. Then $\mathcal{K}_{\phi,s}^{\alpha}(T(s,t)x)$ and $\mathcal{K}_{\phi,t}^{\alpha}(T(s,t)x)$ exists, and we have

$$\begin{aligned}
\mathcal{K}_{\phi}^{\alpha}(T(0,0)x) &= \lim_{(s,t)\to(0^+,0^+)} \mathcal{K}_{\phi}^{\alpha}(T(s,t)x),\\
&= \lim_{(s,t)\to(0^+,0^+)} \Big(\mathcal{K}_{\phi,s}^{\alpha}(T(s,t)x), \mathcal{K}_{\phi,t}^{\alpha}(T(s,t)x)\Big),\\
&= (l_1,l_2) \in X\times X,
\end{aligned}$$

with

$$\lim_{(s,t)\to(0^+,0^+)} \mathcal{K}_{\phi,s}^{\alpha}(T(s,t)x) = l_1, \qquad \lim_{(s,t)\to(0^+,0^+)} \mathcal{K}_{\phi,t}^{\alpha}(T(s,t)x) = l_2.$$

Let us show that

$$\lim_{s\to 0^+} \mathcal{K}_{\phi,s}^{\alpha}(T(s,0)x) = l_1, \quad \lim_{t\to 0^+} \mathcal{K}_{\phi,t}^{\alpha}(T(0,t)x) = l_2.$$

For any $(s,t) \in]0,a[\times]0,b[$, we have

$$\begin{aligned}
\mathcal{K}_{\phi,s}^{\alpha}(T(s,t)x) &= \lim_{\varepsilon\to 0} \frac{T\left(s+\frac{\varepsilon}{\phi(s,\alpha)},t\right)x - T(s,t)x}{\varepsilon}\\
&= \lim_{\varepsilon\to 0} \frac{T(0,t)\left(T(s+\frac{\varepsilon}{\phi(s,\alpha)},0)x - T(s,0)x\right)}{\varepsilon}\\
&= T(0,t)\left[\lim_{\varepsilon\to 0} \frac{T(s+\frac{\varepsilon}{\phi(s,\alpha)},0)x - T(s,0)x}{\varepsilon}\right]\\
&= T(0,t)\Big[\mathcal{K}_{\phi,s}^{\alpha}(T(s,0)x)\Big],
\end{aligned}$$

and similarly,

$$\mathcal{K}_{\phi,t}^{\alpha}(T(s,t)x) = T(s,0)\Big[\mathcal{K}_{\phi,t}^{\alpha}(T(0,t)x)\Big].$$

Since $\{T(s,t)\}_{s,t\geq 0}$ is double C_0-$\mathcal{K}_{\phi}^{\alpha}$-semigroup, by Proposition 3.14 we have that $\{T(0,t))\}_{t\geq 0}$ is a C_0-$\mathcal{K}_{\phi}^{\alpha}$-semigroup. Hence, for any $s \in]0,a[$,

$$\begin{aligned}
\lim_{t\to 0^+} \mathcal{K}_{\phi,s}^{\alpha}(T(s,t)x) &= \lim_{t\to 0^+} T(0,t)\Big[\mathcal{K}_{\phi,s}^{\alpha}(T(s,0)x)\Big]\\
&= \mathcal{K}_{\phi,s}^{\alpha}(T(s,0)x),
\end{aligned}$$

and similarly, for all $t \in]0, b[$,

$$\lim_{s \to 0^+} \mathcal{K}^{\alpha}_{\phi,t}(T(s,t)x) = \mathcal{K}^{\alpha}_{\phi,t}(T(0,t)x).$$

Let $\varepsilon > 0$. There exist $0 < \delta_1 \leq a$ and $0 < \delta_2 \leq b$ such that for $(s,t) \in]0, \delta_1[\times]0, \delta_2[$,

$$\left\| \mathcal{K}^{\alpha}_{\phi,s}(T(s,t)x) - l_1 \right\| \leq \frac{\varepsilon}{2},$$

and there exist $0 < \eta_1 \leq b$ such that for $t \in]0, \eta_1[$ and all $s \in]0, a[$,

$$\left\| \mathcal{K}^{\alpha}_{\phi,s}(T(s,t)x) - \mathcal{K}^{\alpha}_{\phi,s}(T(s,0)x) \right\| \leq \frac{\varepsilon}{2}.$$

Let $\gamma_1 = \min\{\delta_1, \eta_1\}$ and $t \in]0, \gamma_1[$, then for all $s \in]0, \delta_1[$,

$$\begin{aligned} \left\| \mathcal{K}^{\alpha}_{\phi,s}(T(s,0)x) - l_1 \right\| &\leq \left\| \mathcal{K}^{\alpha}_{\phi,s}(T(s,0)x) - \mathcal{K}^{\alpha}_{\phi,s}(T(s,t)x) \right\| + \left\| \mathcal{K}^{\alpha}_{\phi,s}(T(s,t)x) - l_1 \right\| \\ &\leq \varepsilon. \end{aligned}$$

It follows that

$$\lim_{s \to 0^+} \mathcal{K}^{\alpha}_{\phi,s}(T(s,0)x) = l_1.$$

Thus

$$\begin{aligned} \lim_{(s,t) \to (0^+,0^+)} \mathcal{K}^{\alpha}_{\phi,s}(T(s,t)x) &= \lim_{s \to 0^+} \lim_{t \to 0^+} \mathcal{K}^{\alpha}_{\phi,s}(T(s,t)x) \\ &= \lim_{s \to 0^+} \mathcal{K}^{\alpha}_{\phi,s}(T(s,0)x). \end{aligned}$$

Similarly,

$$\lim_{t \to 0^+} \mathcal{K}^{\alpha}_{\phi,t}(T(0,t)x) = l_2.$$

Which implies that

$$\begin{aligned} \lim_{(s,t) \to (0^+,0^+)} \mathcal{K}^{\alpha}_{\phi,t}(T(s,t)x) &= \lim_{t \to 0^+} \lim_{s \to 0^+} \mathcal{K}^{\alpha}_{\phi,t}(T(s,t)x) \\ &= \lim_{t \to 0^+} \mathcal{K}^{\alpha}_{\phi,t}(T(0,t)x). \end{aligned}$$

□

Let $\{T(s,t)\}_{s,t \geq 0}$ be a double C_0-$\mathcal{K}^{\alpha}_{\phi}$-semigroup. Then $\{T(s,0)\}_{s \geq 0}$ and $\{T(0,t)\}_{t \geq 0}$ are C_0-$\mathcal{K}^{\alpha}_{\phi}$-semigroups. Let A_1 and A_2 be two linear operators defined by

$$D(A_1) = \left\{ x \in X : \lim_{s \to 0^+} \mathcal{K}^{\alpha}_{\phi,s}(T(s,0)x) \text{ exists} \right\},$$

$$D(A_2) = \left\{ x \in X : \lim_{t \to 0^+} \mathcal{K}^{\alpha}_{\phi,t}(T(0,t)x) \text{ exists} \right\},$$

and

$$A_1 x = \lim_{s \to 0^+} \mathcal{K}^{\alpha}_{\phi,s}(T(s,0)x), \quad \text{for all } x \in D(A_1),$$

$$A_2 x = \lim_{t \to 0^+} \mathcal{K}^{\alpha}_{\phi,t}(T(0,t)x), \quad \text{for all } x \in D(A_2).$$

Clearly, A_1 and A_2 are the $\mathcal{K}^{\alpha}_{\phi}$-infinitesimal generators of the C_0-$\mathcal{K}^{\alpha}_{\phi}$-semigroups $\{T(s,0)\}_{s \geq 0}$ and $\{T(0,t)\}_{t \geq 0}$, respectively.

Theorem 3.19. *Let $\{T(s,t)\}_{s,t\geq 0}$ be a double C_0-$\mathcal{K}_\phi^\alpha$-semigroup. We define the $\mathcal{K}_\phi^\alpha$-infinitesimal generator A as a linear transformation*

$$A : \mathbb{R}_+^2 \longrightarrow \mathcal{B}(D(A_1) \cap D(A_2), X)$$

given by

$$A(h,k) = hA_1 + kA_2,$$

where A_1 and A_2 denote the $\mathcal{K}_\phi^\alpha$-infinitesimal generators of the C_0-$\mathcal{K}_\phi^\alpha$-semigroups $(T(s,0))_{s\geq 0}$ and $(T(0,t))_{t\geq 0}$, respectively.

Proof. Let $x \in D(A) = D(A_1) \cap D(A_2)$. Then the derivative $\mathcal{K}_\phi^\alpha(T(0,0)x)$ exists and defines a linear transformation

$$\psi(\cdot,\cdot) : \mathbb{R}_+^2 \longrightarrow X$$

such that

$$\psi(h,k) = \mathcal{K}_\phi^\alpha(T(0,0)x)(h,k)^\mathsf{T} = hA_1x + kA_2x.$$

Now, define

$$\widehat{\psi}(\cdot,\cdot) : \mathbb{R}_+^2 \longrightarrow \mathcal{B}\big(D(A_1) \cap D(A_2), X\big)$$

by

$$\widehat{\psi}(h,k) = hA_1 + kA_2.$$

Then $\widehat{\psi}(\cdot,\cdot)$ is a linear transformation and, for all $(h,k) \in \mathbb{R}_+^2$ and $x \in D(A_1) \cap D(A_2)$, we have

$$\psi(h,k) = \widehat{\psi}(h,k)x.$$

Hence, for every $x \in D(A_1) \cap D(A_2)$,

$$\mathcal{K}_\phi^\alpha(T(0,0)x) = \psi(\cdot,\cdot) = \widehat{\psi}(\cdot,\cdot)x.$$

Thus,

$$A = \widehat{\psi}(\cdot,\cdot),$$

and consequently, the $\mathcal{K}_\phi^\alpha$-infinitesimal generator A can be viewed as the linear transformation

$$A : \mathbb{R}_+^2 \longrightarrow \mathcal{B}\big(D(A_1) \cap D(A_2), X\big), \quad A(h,k) = hA_1 + kA_2.$$

□

Theorem 3.20. *Let $\{T(s,t)\}_{s,t\geq 0}$ be a double C_0-$\mathcal{K}_\phi^\alpha$-semigroup, and let A be its $\mathcal{K}_\phi^\alpha$-infinitesimal generator. Then, for all $x \in D(A)$, we have:*

(1) *For any $t \geq 0$, we have*

$$T(0,t)x \in D(A_1) \quad \text{and} \quad A_1T(0,t)x = T(0,t)A_1x.$$

(2) *For any $s \geq 0$, we have*

$$T(s,0)x \in D(A_2) \quad \text{and} \quad A_2T(s,0)x = T(s,0)A_2x.$$

(3) *For all $(s,t) \in \mathbb{R}_+^2$, we have $T(s,t)x \in D(A)$ and*

$$\mathcal{K}_{\phi,s}^\alpha\big(T(s,t)x\big) = A_1T(s,t)x = T(s,t)A_1x,$$

and

$$\mathcal{K}_{\phi,t}^\alpha\big(T(s,t)x\big) = A_2T(s,t)x = T(s,t)A_2x.$$

(4) *For all* $(s,t) \in \mathbb{R}_+^2$, *we have* $T(s,t)x \in D(A)$ *and, for all* $(h,k) \in \mathbb{R}^2$,

$$\mathcal{K}_\phi^\alpha\big(T(s,t)x\big)(h,k)^\intercal = (A_1, A_2)(h,k)^\intercal T(s,t)x = T(s,t)(A_1,A_2)(h,k)^\intercal x.$$

Proof. (1) Let $x \in D(A) \subset D(A_1)$. Since $\lim\limits_{s\to 0^+} \mathcal{K}_{\phi,s}^\alpha(T(s,0)x)$ exists, the function $\mathcal{K}_\phi^\alpha(T(s,0)x)$ exists in an open interval of the form $]0,a[$, with $a>0$. For $s \in]0,a[$ and $t \geq 0$, we have

$$\mathcal{K}_{\phi,s}^\alpha\big(T(s,0)T(0,t)x\big) = T(0,t)\mathcal{K}_{\phi,s}^\alpha(T(s,0)x).$$

Then,

$$\begin{aligned}\lim_{s\to 0^+} \mathcal{K}_{\phi,s}^\alpha\big(T(s,0)T(0,t)x\big) &= T(0,t)\left(\lim_{s\to 0^+} \mathcal{K}_{\phi,s}^\alpha(T(s,0)x)\right)\\ &= T(0,t)A_1x.\end{aligned}$$

Thus, for any $t \geq 0$, we have $T(0,t)x \in D(A_1)$ and

$$A_1T(0,t)x = T(0,t)A_1x.$$

(2) The same reasoning as in (1) gives, for any $s \geq 0$,

$$T(s,0)x \in D(A_2) \quad \text{and} \quad A_2T(s,0)x = T(s,0)A_2x.$$

(3) Let $(s,t) \in \mathbb{R}_+^2$ and $x \in D(A)$. From (1), we know that $T(0,t)x \in D(A_1)$. Then, by Theorem 2.13, we have

$$\mathcal{K}_{\phi,s}^\alpha\big(T(s,0)T(0,t)x\big) = A_1T(s,0)T(0,t)x = T(s,0)A_1T(0,t)x.$$

From (1), it follows that

$$T(s,0)A_1T(0,t)x = T(s,0)T(0,t)A_1x.$$

Hence, for all $(s,t) \in \mathbb{R}_+^2$ and $x \in D(A)$,

$$\mathcal{K}_{\phi,s}^\alpha\big(T(s,t)x\big) = A_1T(s,t)x = T(s,t)A_1x.$$

By the same method, we obtain

$$\mathcal{K}_{\phi,t}^\alpha\big(T(s,t)x\big) = A_2T(s,t)x = T(s,t)A_2x.$$

(4) Let $(h,k) \in \mathbb{R}^2$, $(s,t) \in \mathbb{R}_+^2$, and $x \in D(A)$. From (3), we have

$$\begin{aligned}\mathcal{K}_\phi^\alpha(T(s,t)x)(h,k)^\intercal &= \big(\mathcal{K}_{\phi,s}^\alpha(T(s,t)x, \mathcal{K}_{\phi,t}^\alpha(T(s,t)x\big)\,(h,k)^\intercal\\ &= h\mathcal{K}_{\phi,s}^\alpha\,(T(s,t)x) + k\mathcal{K}_{\phi,t}^\alpha(T(s,t)x).\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{K}_\phi^\alpha(T(s,t)x)(h,k)^\intercal &= hA_1T(s,t)x + kA_2T(s,t)x,\\ &= T(s,t)(hA_1+kA_2)x.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{K}_\phi^\alpha(T(s,t)x)(h,k)^\intercal &= (A_1,A_2)(h,k)^\intercal T(s,t)x,\\ &= T(s,t)(A_1,A_2)(h,k)^\intercal x.\end{aligned}$$

□

Theorem 3.21. *Let $\{T(s,t)\}_{s,t\geq 0}$ be a double C_0-$\mathcal{K}_\phi^\alpha$-semigroup, with generator A. Then:*

$$A\left(\int_0^s T(\tau,t)\phi(\tau,\alpha)x\,\mathrm{d}\tau, \int_0^t T(s,\theta)\phi(\theta,\alpha)x\,\mathrm{d}\theta\right) = T(s,t)\left(\phi(s,\alpha),\phi(t,\alpha)\right)x - \left(\phi(0,\alpha),\phi(0,\alpha)\right)x.$$

Proof. We have

$$\begin{aligned}&A\left(\int_0^s T(\tau,t)\phi(\tau,\alpha)x\,\mathrm{d}\tau, \int_0^t T(s,\theta)\phi(\theta,\alpha)x\,\mathrm{d}\theta\right)\\ &=\left(A_1\int_0^s T(\tau,t)\phi(\tau,\alpha)x\,\mathrm{d}\tau, A_2\int_0^t T(s,\theta)\phi(\theta,\alpha)x\,\mathrm{d}\theta\right)\end{aligned}$$

From Theorem 2.15 , we get

$$\begin{aligned}&A\left(\int_0^s T(\tau,t)\phi(\tau,\alpha)x\,\mathrm{d}\tau, \int_0^t T(s,\theta)\phi(\theta,\alpha)x\,\mathrm{d}\theta\right)\\ &=(T(s,t)\phi(s,\alpha)x-\phi(0,\alpha)x, T(s,t)\phi(t,\alpha)x-\phi(0,\alpha)x)\\ &=T(s,t)(\phi(s,\alpha)x,\phi(t,\alpha)x)-(\phi(0,\alpha)x,\phi(0,\alpha)x)\\ &=T(s,t)(\phi(s,\alpha),\phi(t,\alpha))x-(\phi(0,\alpha),\phi(0,\alpha))x.\end{aligned}$$

□

4. Double $\mathcal{K}_\phi^\alpha$-abstract Cauchy Problem

Definition 4.1. Let X be a Banach space, $A : D(A) \subseteq X \to X$ a linear operator and $u_0 \in X$. We consider the following $\mathcal{K}_\phi^\alpha$-Cauchy Problem

$$\begin{cases}\mathcal{K}_\phi^\alpha u(t) = Au(t), & t>0,\\ u(0)=u_0.\end{cases} \tag{4.1}$$

A function $u : [0,\infty) \to X$ is a solution of the $\mathcal{K}_\phi^\alpha$-Cauchy Problem (4.1), if

(1) u is continuous on $[0,\infty)$,

(2) u is continuously $\mathcal{K}_\phi^\alpha$-differentiable on $(0,\infty)$,

(3) $u(t) \in D(A)$ for $t > 0$,

(4) u satisfies (4.1).

Theorem 4.2. *Let X be a Banach space and A the infinitesimal generator of a C_0-$\mathcal{K}_\phi^\alpha$-semigroup $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$. If $u_0 \in D(A)$, then (4.1) has the unique solution $u(t) = T(t)u_0$.*

Proof. It is clear from Theorem 2.13 that $u(t) = T(t)x$ is a solution of (4.1). To prove the uniqueness, let u be a solution of (4.1).

Then

$$\begin{aligned}\mathcal{K}_\phi^\alpha\left[T(t-s)u(s)\right] &= T(t-s)\mathcal{K}_\phi^\alpha u(s) - AT(t-s)u(s)\\ &= T(t-s)\mathcal{K}_\phi^\alpha u(s) - T(t-s)Au(s)\\ &= T(t-s)\left[\mathcal{K}_\phi^\alpha u(s) - Au(s)\right]\\ &= 0.\end{aligned}$$

We know that $\mathcal{I}_\phi^\alpha\left(\mathcal{K}_\phi^\alpha f(t)\right) = f(t) - f(0)$, with $\mathcal{I}_\phi^\alpha$ is the $\mathcal{K}_\phi^\alpha$-integral of f defined by $\mathcal{I}_\phi^\alpha f(t) = \int_0^t \phi(s,\alpha)f(s)ds, t \geq 0$. Then, by applying $\mathcal{I}_\phi^\alpha$ with respect to s, we obtain

$$T(t-t)u(t) - T(t)u(0) = 0.$$

Thus

$$u(t) = T(t)u_0.$$

□

Definition 4.3. Let X be a Banach space, $A_i : D(A_i) \subseteq X \to X, i = 1,2$ a linear operator. A function $u : [0,\infty)\times[0,\infty) \to X$ is a solution of the double $\mathcal{K}_\phi^\alpha$-Cauchy Problem

$$\begin{cases}\mathcal{K}_{\phi,t_i}^\alpha u(t_1,t_2) = A_i u(t_1,t_2), & t_i > 0, i = 1,2\\ u(0,0) = x, & x \in D(A_1)\cap D(A_2),\end{cases} \tag{4.2}$$

if

(1) u is continuous on $[0,\infty)\times[0,\infty)$,

(2) u has continuous $\mathcal{K}_\phi^\alpha$-partial derivatives,

(3) $u(t) \in D(A_i)$,$i = 1,2$ for $s,t > 0$,

(4) u satisfies (4.2).

Theorem 4.4. *Let (A_1, A_2) be the $\mathcal{K}_\phi^\alpha$-infinitesimal generator of a double C_0-$\mathcal{K}_\phi^\alpha$-semigroup $\{T(s,t)\}_{s,t\geq 0}$. Then the Problem (4.2) has the unique solution $u(s,t;x) = T(s,t)x$ for all $x \in D(A_1)\cap D(A_2)$.*

Proof. By Theorem 3.20, we can easily prove that $u(s,t;x) = T(s,t)x$ is a solution of Problem (4.2).
To prove the uniqueness, it suffices to prove that the only solution corresponding to the initial value $x = 0$ is $u(s,t) = 0$.
From Theorem 4.2, we obtain that the systems

$$\begin{cases}\mathcal{K}_{\phi,t}^\alpha f(t) = A_1 f(t), & t > 0,\\ f(0) = 0,\end{cases} \tag{4.3}$$

and

$$\begin{cases}\mathcal{K}_{\phi,t}^\alpha g(t) = A_2 g(t), & t > 0,\\ g(0) = 0,\end{cases} \tag{4.4}$$

have the unique solution $f = 0$and $g = 0$.
Assume that $u(s,t;0)$ is a solution of (4.2) associated zith the initial value $x = 0$.
Let $f_1(s) = T(s,0)u(0,t;0)$, we have

$$\begin{aligned}\mathcal{K}^\alpha_{\phi,s} f_1(s) &= \mathcal{K}^\alpha_{\phi,s} T(s,0)u(0,t;0)\\ &= A_1 T(s,0)u(0,t;0)\\ &= A_1 f_1(s),\end{aligned}$$

and

$$\begin{aligned}f_1(0) &= T(0,0)u(0,t;0)\\ &= u(0,t;0).\end{aligned}$$

Then $f_1(s) = T(s,0)u(0,t;0)$ is a solution of

$$\begin{cases}\mathcal{K}^\alpha_{\phi,s} f(s) = A_1 f(s), & s > 0,\\ f(0) = u(0,t;0),\end{cases}$$

Let $f_2(s) = u(s,t;0)$, we have

$$\begin{aligned}\mathcal{K}^\alpha_{\phi,s} f_2(s) &= \mathcal{K}^\alpha_{\phi,s} u(s,t;0)\\ &= A_1 u(s,t;0)\\ &= A_1 f_2(s),\end{aligned}$$

and $f_2(0) = u(0,t;0)$ Then $f_2(s) = u(s,t;0)$ is a solution of

$$\begin{cases}\mathcal{K}^\alpha_{\phi,s} f(s) = A_1 f(s), & s > 0,\\ f(0) = u(0,t;0),\end{cases}$$

By uniqueness of solution, we get for any $s,t \geq 0, u(s,t;0) = T(s,0)u(0,t;0)$. Using the same method, we show that $g_1(t) = T(0,t)u(s,0;0)$ and $g_2(t) = u(s,t;0)$ are two solutions of

$$\begin{cases}\mathcal{K}^\alpha_{\phi,t} g(t) = A_2 g(t), & t > 0,\\ g(0) = u(s,0;0).\end{cases}$$

By uniqueness of solution, we get for any $s,t \geq 0, u(s,t;0) = T(0,t)u(s,0;0)$.
In conclusion, we obtain

$$\begin{aligned}u(s,t;0) &= T(s,0)u(0,t;0)\\ &= T(s,0)\left(T(0,t)u(0,0;0)\right)\\ &= T(s,0)T(0,t)0 = 0.\end{aligned}$$

□

Example 4.5. Let consider the following double $\mathcal{K}^\alpha_\phi$-Cauchy Problem

$$\begin{cases}\mathcal{K}^\alpha_\phi u(s,t) = Au(s,t), & s,t > 0\\ \mathcal{K}^\alpha_\phi u(s,t) = Bu(s,t), & s,t > 0\\ u(0,0) = x, & x \in D(A) \cap D(B)\end{cases} \tag{4.5}$$

with A and B be two commuting operators and $\phi(t,\alpha) = \alpha \cosh t$. Then for all $s,t \geq 0$ and $x \in D(A) \cap D(B)$, the Problem (4.5) has the unique solution $u(s,t;x) = e^{\alpha(A \sinh s + B \sinh t)}$.

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