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# SOME EXTENSIONS OF AGI-RINGS

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*This paper is dedicated to my Professor Najib Mahdou*

Abstract. S. Sharmila and S. Hegde defined in [9] a ring to be AGI-ring if each of its elements can be written as a sum of a finite number of idempotents. This paper aims at the study of the notion of AGI-ring in various contexts of commutative rings such as direct product, homomorphic images, trivial ring extensions, amalgamations, bi-amalgamations and pullbacks. Our results generate new original classes of rings satisfying this property.

*Keywords.* AGI-rings, Amalgamated algebras, Boolean rings, Pullbacks, Trivial ring extensions.
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## 1. Introduction and preliminaries

Throughout the whole paper, we assume that all rings are commutative with nonzero unity. For the convenience of the reader, we fix notation for the rest of the paper. For a ring $A$, the ideal of all nilpotent elements of $A$ is denoted by $Nilp(A)$, we denote, also, by $Idem(A)$ the set of all idempotent elements of $A$. Recall that an ideal $I$ of a ring $A$ is nil if $I \subseteq Nilp(A)$ and a ring $R$ is Boolean if every element $x \in R$ is an idempotent (i.e., $x^2 = x$ for all $x \in R$). It is clear that $R$ is Boolean if and only if $Idem(R) = R$.

Hirano and Tominaga, considered in [6], the class of rings in which every element is the sum of two idempotents. In [2], the authors called such rings BB-rings and studied this property in various contexts of extension rings such as homomorphic images, direct product, and amalgamations of algebras. In [6, Example, p. 161 ], it is proved that such a ring need not be Boolean or even commutative. Also, the authors gave in [6] a characterisation of a ring $R$ to be BB-ring: A ring $R$ is BB-ring if and only if for all $r \in R,\ r^3 = r$.

In [10], Vasantha Kandasamy defined n-Boolean rings which are a generalization of Boolean ring and obtained conditions for a ring to be n-Boolean. Recall that a ring $R$ with identity is said to be n-Boolean if for every $x \in R,\ x^{2n} = x$, for some natural number $n$, and when $n = 1$ we trivially get $R$ to be Boolean

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ring. It is clear that every Boolean ring is n-Boolean and n-Boolean ring need not be a Boolean ring, one can see [10, Proposition 3]. It is shown also, that every n-Boolean ring has characteristic two see [10, Theorem 9].

S. Sharmila and S. Hegde defined in [9] a ring to be an AGI-ring if each of its elements can be written as a sum of a finite number of idempotents. The following implications always hold but cannot be reversed in general.

$$\textbf{Boolean ring} \Longrightarrow \textbf{BB-ring} \Longrightarrow \textbf{AGI-ring}$$

Let $A$ be a ring and $M$ be an $A$-module. The trivial ring extension of $A$ by $M$ (also called the idealization of $M$ over $A$) is the ring $R = A \ltimes M$ whose underlying group is $A \times M$ with multiplication given by $(a, m)(b, n) = (ab, an + bm)$ for each $a, b \in A$ and $m, n \in M$. For more details about idealization one can see [1, 8].

Let $A$ be a commutative ring with nonzero unity and $I$ be an ideal of $A$, D'Anna and Fontana introduced in [5] the amalgamated duplication of a ring $A$ along an ideal $I$ as a subring of $A \times A$ defined by $A \bowtie I := \{(a, a + i) \mid a \in A,\ i \in I\}$. This construction is a special case of the following amalgamated algebras.

Let $(A, B)$ be a pair of rings, $J$ be an ideal of $B$ and $f : A \to B$ be a ring homomorphism. D'Anna, Finocchiaro and Fontana introduced in [4], a subring of $A \times B$ denoted: $A \bowtie^f J := \big\{(a, f(a) + j) \mid a \in A,\ j \in J\big\}$, called the amalgamation of $A$ with $B$ along $J$ with respect to $f$.

Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let $J$ (resp., $J'$) be an ideal of $B$ (resp., $C$) such that $f^{-1}(J) = g^{-1}(J')$. *Kabbaj*, *Louartiti* and *Tamekkante* defined and studied in [7] the following subring: $A \bowtie^{f,g} (J, J') := \big\{(f(a)+j, g(a)+j') \mid a \in A,\ j \in J,\ j' \in J'\big\}$ of $B \times C$, called the bi-amalgamation of $A$ with $(B, C)$ along $(J, J')$ with respect to $(f, g)$. This ring construction is a natural generalization of the amalgamated algebras along an ideal introduced and studied by M. D'Anna, C. A. Finocchiaro and M. Fontana in [4].

Recall that pullback can be defined as follows: Let $T$ be a ring, $M$ is a nonzero ideal of $T$, $\pi$ is the natural surjection $\pi : T \to T/M$ and $D$ is a subring of $T/M$. Then $R := \pi^{-1}(D)$ is a subring of $T$ and $M$ is a common ideal of $R$ and $T$ and $D = R/M$. $R$ is called a pullback ring associated to the following pullback diagram:

$$\begin{array}{ccc} R = \pi^{-1}(D) & \xrightarrow{\pi/R} & D = R/M \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ T & \xrightarrow{\pi} & T/M \end{array}$$

where $i$ and $j$ are the natural injections.

## 2. About AGI-rings

We start with the main definition introduced and studied by *S. Sharmila* and *S. Hegde* in [9]:

**Definition 2.1.** Let $R$ be a ring and let $I(R)$ denote the set of all elements of $R$ which can be written as a sum of a finite number of idempotents of $R$. If $I(R) = R$ then we say that $R$ is a ring additively generated by its idempotents (for short AGI- ring).

That is, for all $x \in A$, there exists some positive integer $n \geq 1$ such that, $x = e_1 + e_2 + \cdots + e_n$, where $e_i^2 = e_i \in Idem(A)$ for $i = 1, 2, \cdots, n$.

**Examples 2.2.**

(1) $\mathbb{Z}/n\mathbb{Z}$, the ring of integers modulo $n$, is an AGI-ring but not Boolean ring.
(2) Every Boolean ring is an AGI-ring but the converse is not true.
(3) Every BB-ring is an AGI-ring.

The first result studies the homomorphic image and direct product of AGI-rings.

**Proposition 2.3.** *Let $A$ be a commutative ring and $I$ an ideal of $A$.*

(1) *Any homomorphic image of AGI-ring is, again, an AGI-ring.*
(2) *If $A$ is an AGI-ring then so is $A/I$. The converse holds when $I$ is a nil ideal. In particular, $A$ is an AGI-ring if and only if $A/Nilp(A)$ so is.*
(3) $A = \prod_{k=1}^{k=n} A_k$ *is an AGI-ring if and only if so is $A_k$, for all $k = 1, \cdots, n$.*

*Proof.*

(1) this is due to the fact that every homomorphic image of an idempotent element is, again, an idempotent.
(2) It is clear that every homomorphic image of idempotent element is, again, idempotent. Then every homomorphic image of AGI-ring is, again, AGI-ring. For the converse, it is well known that idempotents lift modulo every nil ideal. Thus, when $I$ is a nil ideal of a ring $A$ and $A/I$ is an AGI-ring then so is $A$. The "in particular" is easy to see since $Nilp(A)$ is a nil ideal of $A$.
(3) If $A = \prod_{k=1}^{k=n} A_k$ is an AGI-ring, then each $A_k$ so is, being a homomorphic image of $A$ by the canonical projections. Conversely, assume that each $A_k$ is an AGI-ring. Without loss of generality, we can suppose that $A = A_1 \times A_2$. Let $x = (x_1, x_2) \in A$, then $x_1 = e_1 + e_2 + \cdots + e_m$ and $x_2 = f_1 + f_2 + \cdots + f_n$ with the $e_i$ and $f_j$ are all idempotents for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$, we can assume that $m < n$. Then $x = (x_1, x_2) = (e_1, f_1) + \cdots + (e_m, f_m) + (0, f_{m+1}) + \cdots + (0, f_n)$ is a finite sum of idempotents, which completes the proof.

□

**Remark 2.4.** Let $A$ be a commutative ring. Then, the polynomial ring $A[X]$ and the ring of formal power series $A[[X]]$ are never AGI-rings.

*Proof.* Suppose that $A[X]$ is AGI-ring. Thus $X = e_1 + e_2 + \cdots + e_n$ for some natural number $n$ and $e_i \in Idem(A[X])$ for $i = 1, 2, \cdots, n$, but $Idem(A[X]) = Idem(A)$. Thus, we get $X \in A$ which is impossible.
The same argument to show that $A[[X]]$ is never AGI-ring. □

The next example shows that the condition "$I$ is a nil ideal" is essential for the converse of the item (2) in Proposition 2.3.

**Example 2.5.** Let $A$ be an AGI-ring (for instance, $A = \mathbb{Z}/n\mathbb{Z}$ for some positive integer $n$). Consider the surjection $s: \ A[X] \to A[X]/(X) \simeq A$. By Remark 2.4, $A[X]$ is never AGI-ring however, the image $A[X]/(X) \simeq A$ is an AGI-ring, one can see that $I = (X)$ is not a nil ideal.

Now, in the rings $R[X]/(X^n)$ and $R[[X]]/(X^n)$ we establish the following result.

**Theorem 2.6.** *For a commutative ring $A$ and a positive integer $n$, the following statements are equivalent.*

(1) *$A$ is an AGI-ring.*
(2) *$A[X]/(X^n)$ is an AGI-ring.*
(3) *$A[[X]]/(X^n)$ is an AGI-ring.*

*Proof.*
(1) $\Leftrightarrow$ (2): Consider the canonical surjection $f: R = A[X]/(X^n) \twoheadrightarrow A$, such that $f(a_0 + a_1X + \cdots + a_{n-1}X^{n-1} + (X^n)) = a_0$. Clearly, $Ker(f) = I_X$ where $I_X$ is the ideal generated by $X + (X^n)$ and $Ker(f)$ is the kernel of $f$. Since $f$ is a surjective then $R/I_X \simeq A$. But $I_X$ is nil ideal. Hence, the result follows by Proposition 2.3.
(1) $\Leftrightarrow$ (3): The proof is similar to (1) $\Leftrightarrow$ (2). □

The following result examines the localization of AGI-rings.

**Proposition 2.7.** *Let $A$ be a ring. Then:*

(1) *$Idem(A)$ is closed under multiplication.*
(2) *If $S \subseteq Idem(A)$ is a multiplicatively closed set and $A$ is an AGI-ring, then $S^{-1}A$ is an AGI-ring.*

*Proof.*

(1) Let $e, f \in Idem(A)$. Then $(ef)^2 = e^2f^2 = ef$.
(2) Let $\dfrac{x}{s} \in S^{-1}A$ for some $x \in A$ and $s \in S \subseteq Idem(A)$. Since $A$ is an AGI-ring, we can write $x = \sum_{i=1}^{i=n} e_i$ for some $n \in \mathbb{N}$ and $e_i \in Idem(A)$. We obtain that $\dfrac{x}{s} = \sum_{i=1}^{i=n} \frac{e_i}{s}$ and note that every $\dfrac{e_i}{s}$ is idempotent. Hence, $S^{-1}A$ is an AGI-ring.

□

Let $A$ be a commutative ring and let $T_n(A)$ denotes the upper triangular matrices $n \times n$ over a ring $A$. Then the following hold.

**Theorem 2.8.** *Let $A$ be a commutative ring and $n$ be a positive integer. Then, $A$ is an AGI-ring if and only if the upper triangular ring $T_n(A)$ is an AGI-ring.*

*Proof.* If $I$ is the ideal of $T_n(A)$ consisting of all matrices with zeros along the main diagonal, then clearly $I$ is nil. Moreover, the quotient $T_n(A)/I$ is isomorphic to the direct product of $n$ copies of $A$. But a finite direct product of $A$ is an AGI-ring if and only if $A$ so is and the proof follows by Proposition 2.3. □

## 3. AGI-rings and amalgamated algebras

The next result investigates the AGI-rings in amalgamated algebras along an ideal.

**Theorem 3.1.** *Let $(A, B)$ be a pair of commutative rings, $f : A \to B$ be a ring homomorphism and $J$ be an ideal of $B$. Then.*

1. *If $A \bowtie^f J$ is an AGI-ring, then so are $A$ and $f(A) + J$.*
2. *If $J$ is nil ideal of $B$. Then $A \bowtie^f J$ is an AGI-ring if and only if so is $A$.*
3. *If $J \subseteq Idem(B)$. Then $A \bowtie^f J$ is an AGI-ring if and only if so is $A$.*

*Proof.*

1. By [4, Proposition 5.1 (3)], $A$ and $f(A) + J$ are homomorphic images of $A \bowtie^f J$. Then in view of Proposition 2.3 the result follows.
2. In view of [4, Proposition 5.1 (3)], $\dfrac{A \bowtie^f J}{0 \times J} \simeq A$. Since $J$ is a nil ideal of $B$, then $0 \times J$ is a nil ideal of $A \bowtie^f J$ and the Proposition 2.3(2) completes the proof.
3. If $A \bowtie^f J$ is an AGI-ring then so is $A$ by the first item. Conversely, assume that $A$ is an AGI-ring, $J \subseteq Idem(B)$ and let $(a, j) \in A \times J$. Then there exists a positive integer $n$ such that $a = e_1 + e_2 + \cdots + e_n$ with $e_i \in Idem(A)$ for $i \in \{1, \cdots, n\}$. Thus $(a, f(a) + j) = (e_1, f(e_1)) + (e_2, f(e_2)) + \cdots + (e_n, f(e_n)) + (0, j)$, it is clear that each $(e_i, f(e_i))$ is an idempotent of the ring $A \bowtie^f J$. Furthermore, $(0, j)$ is an idempotent of $A \bowtie^f J$ since $J \subseteq Idem(B)$. Therefore, $A \bowtie^f J$ is an AGI-ring.

□

**Remark 3.2.** in Theorem 3.1 (3), the assumption "$J \subseteq Idem(B)$" can be replaced by the statement "$J$ is an AGI-ring", as a ring without a unity, and hence each element in $J$ can be written as a sum of a finite number of idempotents.

The following example shows the failure of Theorem 3.1 in the case the assumptions ("$J \subseteq Idem(B)$" or "$J$ is a nil ideal") is not satisfied.

**Example 3.3.** Let $A$ be any AGI-ring (for instance, take $A := \mathbb{Z}/n\mathbb{Z}$ for some positive integer $n$), $B := A[X]$ be the polynomial ring with coefficient in $A$, $J := XA[X]$ be an ideal of $B$ and $f : A \hookrightarrow B$ be the natural injection. Then:

1. $J \not\subseteq Idem(B)$.
2. $J$ is not a nil ideal.
3. $A$ is an AGI-ring.
4. $A \bowtie^f J$ is not AGI-ring.

*Proof.*
(1), (2) and (3) Straightforward.
(4) We claim that $A \bowtie^f J$ is not AGI-ring. Indeed, $f(A)+J = A+XA[X] = A[X]$ is not AGI-ring by Remark 2.4. Then, in virtue of Theorem 3.1, $A \bowtie^f J$ is not AGI-ring. □

**Example 3.4.** Let $T$ be be a ring and $J$ an ideal of $T$ and let $D$ be a subring of $T$ such that $J \cap D = (0)$. If $J \subset Nilp(T)$ or $J \subset Idem(T)$. Then $D + J$ is an AGI-ring if and only if $D$ so is.

*Proof.* In view of [4, Example 2.6], $D+J$ is isomorphic to $D \bowtie^i J$ where $i : D \hookrightarrow T$ is the natural embedding. Thus, by Theorem 3.1, $D + J$ is an AGI-ring if and only if $D$ so is. □

As an immediate conclusion of the Theorem 3.1 we obtain the following.

**Corollary 3.5.** *Let $(A, B)$ be a pair of commutative rings, $f : A \to B$ be a ring homomorphism and $J$ be an ideal of $B$. If $B$ is Boolean ring, then $A \bowtie^f J$ is an AGI-ring if and only if so is $A$.*

*Proof.* When $B$ is Boolean, we get $J \subseteq Idem(B) = B$ and by Theorem 3.1(3) the result follows. □

In amalgamated duplication we get the following result.

**Corollary 3.6.** *Let $A$ be a commutative ring, $I$ be an ideal of $A$ such that .*
*(1) If $I \subset Idem(A)$, then $A \bowtie I$ is an AGI-ring if and only if so is $A$.*
*(2) If $I$ is nil ideal, then $A \bowtie I$ is an AGI-ring if and only if so is $A$.*

*Proof.* In this case we have $f = id_A$ and $f(A) + I = A$. The result follows by Theorem 3.1. □

The next result shows that the characterization for $A \bowtie^f J$ to be an AGI-ring can be returned to the case where A is a reduced ring and $J \cap Nilp(B) = (0)$.

**Theorem 3.7.** *Let $f : A \longrightarrow B$ be a ring homomorphism and let $J$ be an ideal of $B$. Set $\overline{A} = A/Nilp(A)$, $\overline{B} = B/Nilp(B)$, $\pi : B \to \overline{B}$ the canonical surjection and $\overline{J} = \pi(J)$, consider the ring homomorphism $\overline{f} : \overline{A} \to \overline{B}$ such that $\overline{x} \mapsto \overline{f}(\overline{x}) = \overline{f(x)}$, then : $A \bowtie^f J$ is an AGI-ring if and only if $\overline{A} \bowtie^{\overline{f}} \overline{J}$ so is.*

*Proof.* It is easy to see that $\overline{f}$ is well defined and it is a ring homomorphism. Consider the map :

$$\begin{array}{rcl} \phi : \; A \bowtie^f J/Nilp(A \bowtie^f J) & \longrightarrow & \overline{A} \bowtie^{\overline{f}} \overline{J} \\ \overline{(a, f(a) + j)} & \longmapsto & (\overline{a}, \overline{f}(\overline{a}) + \overline{j}) \end{array}$$

It is an isomorphism, one can see the proof of Theorem 2.9 in [3].
⇒) if $A \bowtie^f J$ is an AGI-ring then by Proposition 2.3, $A \bowtie^f J/Nilp(A \bowtie^f J)$ is an AGI-ring, then $\overline{A} \bowtie^{\overline{f}} \overline{J}$ so is.

$\Leftarrow$) If $\overline{A} \bowtie^{\overline{f}} \overline{J}$ is an AGI-ring, then $A \bowtie^f J/Nilp(A \bowtie^f J)$ so is. Hence, by Proposition 2.3, $A \bowtie^f J$ is an AGI-ring. □

Let $A$ be a commutative ring and $E$ be an $A$-module. Set $R = A \ltimes E$ be the trivial ring extension of $A$-module $E$. Let $f : A \hookrightarrow R$ be the canonical embedding. After identifying $E$ with $\{0\} \ltimes E$, $E$ becomes an ideal of $R$ and $(\{0\} \ltimes E)^2 = 0$, it is known that $A \ltimes E$ coincides with $A \bowtie^f E$, for more details one can see [4, Remark 2.8]. The following corollary examine the AGI-ring in trivial ring extension.

**Corollary 3.8.** *$A \ltimes E$ is an AGI-ring if and only if so is $A$.*

*Proof.* Let $J := \{0\} \ltimes E$. Since $J^2 = 0$, then $J$ is nil ideal and the result follows by Theorem 3.1(2). □

## 4. AGI-rings and bi-amalgamated algebras

Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let $J$ (resp., $J'$) be an ideal of $B$ (resp., $C$) such that $f^{-1}(J) = g^{-1}(J') = I$. Recall that the bi-amalgamation [7] of $A$ with $(B, C)$ along $(J, J')$ with respect to $(f, g)$ is the following subring of $B \times C$:

$$A \bowtie^{f,g} (J, J') = \{(f(a) + j, g(a) + j') \mid a \in A,\ j \in J,\ j' \in J'\}.$$

The next result investigate the AGI-rings in bi-amalgamated algebras.

**Theorem 4.1.** *Let $A \bowtie^{f,g} (J, J')$ be the bi-amalgamation of $A$.*
(1) *If $A \bowtie^{f,g} (J, J')$ is an AGI-ring, then so are $f(A) + J$ and $g(A) + J'$.*
(2) *Suppose that $J$ is a nil ideal of $B$. Then $A \bowtie^{f,g} (J, J')$ is an AGI-ring if and only if so is $g(A) + J'$*
(3) *Suppose that $J'$ is a nil ideal of $C$. Then $A \bowtie^{f,g} (J, J')$ is an AGI-ring if and only if so is $f(A) + J$.*

*Proof.* (1) By [7, Proposition 4.1 (2)] $f(A) + J$ and $g(A) + J'$ are homomorphic images of $A \bowtie^{f,g} (J, J')$. Then by Proposition 2.3(1) the result follows.

(2) In virtue of [7, Proposition 4.1 (2)], $\dfrac{A \bowtie^{f,g} (J, J')}{J \times 0} \simeq g(A) + J'$. Since $J$ is nil, then so is $J \times 0$, and the proof follows by Proposition 2.3(2).

(3) In virtue of [7, Proposition 4.1 (2)], $\dfrac{A \bowtie^{f,g} (J, J')}{0 \times J'} \simeq f(A) + J$. Since $J'$ is nil, then so is $0 \times J'$, and the proof follows by Proposition 2.3(2). □

In the next result we study the transfer of AGI-ring between a ring $A$ and its bi-amalgamation $A \bowtie^{f,g} (J, J')$.

**Theorem 4.2.** *Let $A \bowtie^{f,g} (J, J')$ be the bi-amalgamation of $A$, let $I = f^{-1}(J) = g^{-1}(J')$.*
(1) *If $J$ and $J'$ are nil ideals and $A$ is an AGI-ring, then $A \bowtie^{f,g} (J, J')$ so is.*
(2) *If $I$ is nil ideal of $A$ and $A \bowtie^{f,g} (J, J')$ is an AGI-ring, then so is $A$.*
(3) *Suppose that $I, J$ and $J'$ are all nil ideals. Then $A \bowtie^{f,g} (J, J')$ is an AGI-ring*

*if and only if so is A.*

*Proof.* (1) In virtue of [7, Proposition 4.1 (3)], $\dfrac{A}{I} \simeq \dfrac{A \bowtie^{f,g} (J, J')}{J \times J'}$. Then, $A$ is an AGI-ring implies that $\dfrac{A}{I}$ so is, which implies that $\dfrac{A \bowtie^{f,g} (J, J')}{J \times J'}$ is an AGI-ring. But $J \times J'$ is nil since $J$ and $J'$ are. Hence, the Proposition 2.3 completes the proof.
(2) In virtue of [7, Proposition 4.1 (3)], $A \bowtie^{f,g} (J, J')$ is an AGI-ring implies that $\dfrac{A \bowtie^{f,g} (J, J')}{J \times J'}$ is an AGI-ring which implies that $\dfrac{A}{I}$ is an AGI-ring. But $I$ is nil, then the Proposition 2.3 completes the proof.
(3) In virtue of [7, Proposition 4.1 (3)], and since $I, J$ and $J'$ are nil ideals, then the statements (1) and (2) complete the proof. □

## 5. AGI-rings and pullback

Recall that pullback can be defined as follows: Let $T$ be a ring, $M$ is a nonzero ideal of $T$, $\pi$ is the natural surjection $\pi : T \to T/M$ and $D$ is a subring of $T/M$. Then $R := \pi^{-1}(D)$ is a subring of $T$ and $M$ is a common ideal of $R$ and $T$ and $D = R/M$. $R$ is called a pullback ring associated to the following pullback diagram:

$$\begin{array}{ccc} R = \pi^{-1}(D) & \xrightarrow{\pi/R} & D = R/M \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ T & \xrightarrow{\pi} & T/M \end{array}$$

where $i$ and $j$ are the natural injections.

We assume that $R \subset T$ and we refer to this as a diagram of type $\triangle$. Our next result investigates the AGI-rings in pullback of type $\triangle$.

**Theorem 5.1.** *For a diagram of type $\triangle$:*

(1) *If $R$ is an AGI-ring, then so is $D$.*
(2) *Assume that $Idem(R) = Idem(T)$. If $T$ is an AGI-ring, then so is $R$.*

*Proof.* (1) This follows from Proposition 2.3, as $D$ is a homomorphic image of $R$.
(2) Suppose that $Idem(R) = Idem(T)$ and $T$ is an AGI-ring. We claim that $R$ is an AGI-ring. Indeed, consider an element $r \in R$. Clearly $r \in T$ which is an AGI-ring. So, $r = e_1 + e_2 + \cdots + e_n$ with $e_i \in Idem(T)$ for $i = 1, 2, \cdots, n$. Since $Idem(R) = Idem(T)$, then $e_i \in Idem(R)$ for $i = 1, 2, \cdots, n$. Which completes the proof. □

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