

ON THE QUASIMINIMIZING CONSTANT FOR THE MINIMUM OF TWO QUASISUPERMINIMIZERS ON $\mathbf{R}^n$ EQUIPPED WITH A p -ADMISSIBLE WEIGHT

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ABSTRACT. By adding a p -admissible radial weight on the n -dimensional real line, we extend the higher dimensional examples given in Björn–Björn–Mirumbe [9] for integral values of p to all values of p to demonstrate a phenomenon that was shown in Björn–Björn–Korte (“*Minima of quasisuperminimizers*”, Nonlinear Anal. **155** (2017), 264–284) stating that $u := \min\{u_1, u_2\}$ is a $\bar{Q}$ -quasisuperminimizer if u_1 and u_2 are Q -quasisuperminimizers and the optimal quasiminimizer constant is $\bar{Q} = 2Q^2/(Q + 1)$, as an extension of the one-dimensional examples therein showing that $\bar{Q}$ is close to optimal.

1. INTRODUCTION AND PRELIMINARIES

Let $\Omega \subset \mathbf{R}^n$ be a nonempty open set and $1 < p < \infty$. A function $u \in W_{\text{loc}}^{1,p}(\Omega)$ is a Q -quasi(super)minimizer in Ω , with $Q \geq 1$, if

$$\int_{\varphi \neq 0} |\nabla u|^p dx \leq Q \int_{\varphi \neq 0} |\nabla(u + \varphi)|^p dx \quad (1.1)$$

for all (nonnegative) $\varphi \in W_0^{1,p}(\Omega)$. A function $u \in W_{\text{loc}}^{1,p}(\Omega)$ is a Q -quasiminimizer in Ω if and only if it is both a Q -quasisubminimizer in Ω and a Q -quasisuperminimizer in Ω . Quasiminimizers were introduced by Giaquinta–Giusti [12] as a tool for a unified treatment of variational integrals, elliptic equations and systems, and quasiregular mappings on $\mathbf{R}^n$. For degenerate partial differential equations that is those with various types of singularities in the coefficients, it is natural to search for solutions in weighted Sobolev spaces, for details see [1] and the references therein. As noted in [9] Quasi(super)minimizers have an interesting theory already in the one-dimensional case, see e.g. [12] and Martio–Sbordone [16]. Kinnunen–Martio [15] showed that one can build a rich potential theory based on quasiminimizers. Given that $\mathbf{B} = B(0, 1)$ is the unit ball in $\mathbf{R}^n$. Björn–Björn [3] studied radial quasiminimizers of the power type $u(x) = |x|^\alpha$ on the unit ball $\mathbf{B}$ in $\mathbf{R}^n$ for $1 < p \neq n < \infty$ and also considered the conformal case where $p = n$. We note here that quasisuperharmonic functions are related to quasisuperminimizers

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in a similar way as superharmonic functions are related to supersolutions. See Björn–Björn–Korte [8] for further references.

Kinnunen–Martio [15, Lemmas 3.6 and 3.7] showed that if u_j is a Q_j -quasisuperminimizer in $\Omega \subset \mathbf{R}^n$ (or in a metric space), $j = 1, 2$, then $u := \min\{u_1, u_2\}$ is a $\min\{Q_1 + Q_2, Q_1 Q_2\}$ -quasisuperminimizer. Björn–Björn–Korte [8] improved upon this result in the following way.

Theorem 1.1. (Theorem 1.2 in [8]) *Let u_j be a Q_j -quasisuperminimizer in $\Omega \subset \mathbf{R}^n$ (or in a metric space), $j = 1, 2$. Then $u := \min\{u_1, u_2\}$ is a $\bar{Q}$ -quasisuperminimizer in Ω , where*

$$\bar{Q} = \begin{cases} 1, & \text{if } Q_1 = Q_2 = 1, \\ (Q_1 + Q_2 - 2) \frac{Q_1 Q_2}{Q_1 Q_2 - 1}, & \text{otherwise.} \end{cases} \quad (1.2)$$

In particular, if $Q_1 = Q_2$, then $\bar{Q} = 2Q_1^2/(Q_1 + 1)$.

The first examples showing that u is (in general) *not* a $\max\{Q_1, Q_2\}$ -quasisuperminimizer were given on $\mathbf{R}$ in [8]. More precisely, if $1 < Q_1 \leq Q_2$ then there are Q_j -quasisuperminimizers u_j on $(0, 1) \subset \mathbf{R}$ such that $u := \min\{u_1, u_2\}$ is not a Q_2 -quasisuperminimizer. Estimates and concrete examples, showing that the constant $\bar{Q}$ above is not too far from being optimal, were given in [8]. All examples therein were on $(0, 1) \subset \mathbf{R}$. The examples in [8] (giving the best lower bounds) were based on power functions $x \mapsto x^\alpha$ and reflections of such functions. The first examples in higher dimensions, i.e. for subsets of $\mathbf{R}^n$, $n \geq 2$ demonstrating this phenomenon were given in [9]. For the functions, in one dimension and also in the higher-dimensional case on $\mathbf{R}^n$, $n \geq 2$, optimal quasi(super)minimizing constants $Q(\alpha, p, n)$ were obtained in Björn–Björn [3], and some of these formulas for $Q(\alpha, p, n)$ (with $n = 1$) were used in the calculations in [8] on $\mathbf{R}$.

However, it was noted in [9] that since power-type functions $x \mapsto |x|^\alpha$ only have point singularities then it is difficult for one to use them for higher-dimensional analogues of the examples in [8]. Instead the examples therein were based on log-power functions $\log^\alpha |x|$ and $(-\log |x|)^\alpha$. Since $\log |ex| = 1 - (-\log |x|)$, then the authors were able to scale and translate them and create higher-dimensional examples on annuli, in the spirit of [8] given the fact that quasiminimizers are invariant under scaling (Lemma 2.1 in [8]). In particular they obtained the following result.

Theorem 1.2. (Theorem 1.2 in [9]) *Let $p = n \geq 2$ and $1 < Q_1 \leq Q_2$ be given. Then there are functions u_1 and u_2 on $A := \{x \in \mathbf{R}^n : 1/e < |x| < 1\}$ such that u_j is a Q_j -quasisuperminimizer in A , $j = 1, 2$, but $\min\{u_1, u_2\}$ is not a Q_2 -quasiminimizer in A .*

We note here that Theorem 1.2 as stated above shows that p takes on only integral values greater or equal to 2. However, it is noted by the authors at the end of section 3 in [9] that this Theorem is still valid for all values of p and for this case a value of $p = 1.2$ was included in Table 3 in [9]. In this paper we extend the results on power-type quasiminimizers on $\mathbf{R}^n$ for the conformal dimension $p = n$ given in Björn–Björn [3] to the weighted $\mathbf{R}^n$ with a p -admissible weight

$w(x) = |x|^\beta$ (this is done so as to have p be written in terms of n and β). We use the obtained quasiminimizing constants to extend Theorem 1.2 to cover all values of $p > 1$ thereby proving that;

Theorem 1.3. *Let $p = n + \beta \geq 1$ and $1 < Q_1 \leq Q_2$ be given. Then there are functions u_1 and u_2 on $A := \{x \in \mathbf{R}^n : 1/e < |x| < 1\}$ such that u_j is a Q_j -quasisuperminimizer with respect to $w(x)$ in A , $j = 1, 2$, but $\min\{u_1, u_2\}$ is not a Q_2 -quasiminimizer with respect to $w(x)$ in A .*

Along the way letting $A_{r_1, r_2} = \{x \in \mathbf{R}^n : 0 \leq r_1 < r_2 \leq \infty\}$ we prove the following as one of our results

Theorem 1.4. *Let $p = n + \beta > 1$, $1 < \gamma \leq \infty$, $\alpha > 1 - \frac{1}{n+\beta}$ and $u(x) = (-\log x)^\alpha$. Then u is a quasiminimizer with respect to the weight $w(x)$ in $A_{1, \gamma}$ with*

$$Q_{\alpha, n, \beta} = \frac{\alpha^{n+\beta}}{(n + \beta)(\alpha - 1) + 1} \quad (1.3)$$

being the best quasiminimizing constant.

Just as in [8] and [9] we also give lower bounds for the increase in the quasisuperminimizing constant and show that these lower bounds are the same as in the 1-dimensional case considered in [8] and those of the higher dimensional case with p a positive integer greater than 2 [9], see Section 3.

2. p - ADMISSIBLE WEIGHTS AND QUASI(SUB/SUPER)MINIMIZERS

Suppose w is a locally integrable, nonnegative function in $\mathbf{R}^n$ that satisfies $w(x) > 0$. Then a Radon measure μ is canonically associated with the weight w on a measurable set $E \subset \mathbf{R}^n$ as

$$\mu(E) = \int_E w(x) dx.$$

Thus $d\mu(x) = w(x)dx$ for $w(x)$ a p -admissible weight, where dx is the n -dimensional Lebesgue measure. The following definition is an extract from section 1.1 of [11].

Definition 2.1. The weight $w(x)$ or μ is p -admissible if the following four conditions are satisfied.

- (a) $0 < w(x) < \infty$ almost everywhere in $\mathbf{R}^n$ and the measure μ is doubling that is $\exists$ a constant $C > 0$ such that $\mu(2B) \leq C\mu(B)$ whenever B is a ball in $\mathbf{R}^n$ where $2B$ is a ball whose radius is twice that of B .
- (b) If D is an open set and $\varphi_i \in C^\infty(D)$ is a sequence of functions such that $\int_D |\varphi_i|^p d\mu \rightarrow 0$ and $\int_D |\nabla \varphi_i - v|^p d\mu \rightarrow 0$ as $i \rightarrow \infty$ where v is a vector-valued measurable function in $L^p(D, \mu : \mathbf{R}^n)$ then $v = 0$.
- (c) There are constants $\gamma > 1$ and $C_1 > 0$ such that

$$\left(\frac{1}{\mu(B)} \int_B |\varphi|^{\gamma p} d\mu \right)^{\frac{1}{\gamma p}} \leq C_1 r \left(\frac{1}{\mu(B)} \int_B |\varphi|^p d\mu \right)^{\frac{1}{p}}.$$

whenever $B = B(x_0, r)$ is a ball in $\mathbf{R}^n$ and $\varphi \in C_0^\infty(B)$,

(d) There is a constant $C_2 > 0$ such that

$$\int_B |\varphi - \varphi_B|^p d\mu \leq C_2 r^p \int_B |\nabla \varphi|^p d\mu$$

whenever $B = B(x_0, r)$ is a ball in $\mathbf{R}^n$ and $\varphi \in C^\infty(B)$ is bounded and

$$\varphi_B = \frac{1}{\mu(B)} \int_B \varphi d\mu$$

In this definition, the meaning of the measure μ being doubling is adopted from Definition 2.6 in [7]. This is one of the many different equivalent definitions of the p -admissible weights in literature; see, e.g. Hajlasz and Koskela [10] in which it is remarked that the four conditions can be reduced to only two: ω is p -admissible $1 < p < \infty$ if and only if w is doubling and there exists constants $C > 0$ and $\lambda \geq 1$ such that

$$\frac{1}{\mu(B)} \int_B |\varphi - \varphi_B| d\mu \leq CR \left(\frac{1}{\mu(\lambda B)} \int_{\lambda B} |\varphi|^p d\mu \right)^{\frac{1}{p}},$$

(This is similar to definition 2 in Björn–Björn [6]) and Proposition A.17 in Björn–Björn [5].

Definition 2.2. (Definition 2.1 in [3]) A function $u : \Omega \rightarrow (-\infty, \infty]$ is Q -quasisuperharmonic in Ω if u is not identically ∞ in any component of Ω , u is lower semicontinuously regularized, and $\min\{u, k\}$ is a Q -quasisuperminimizer in Ω for every $k \in \mathbf{R}$. A function $u : \Omega \rightarrow (-\infty, \infty]$ is Q -quasisubharmonic in Ω if $-u$ is Q -quasisuperharmonic in Ω .

We note here that the definition of quasiminimizers (and quasisub- and quasisuperminimizers) above is one of several equivalent possibilities, see Proposition 3.2 in A. Björn [2]. In particular, we will use that it is enough to require (1.1) to hold for all (nonnegative) $\varphi \in \text{Lip}_c(\Omega)$, where $\text{Lip}_c(\Omega)$ denotes the space of all Lipschitz functions with compact support in Ω .

When $Q = 1$ we usually drop “quasi” and say (sub/super)minimizer. Being a (sub/super)minimizer is the same as being a (weak) (sub/super)solution of the p -Laplace equation

$$-\text{div}(|\nabla u|^{p-2} \nabla u) = 0,$$

see Chapter 5 in Heinonen–Kilpeläinen–Martio [11]. The function u is a supersolution of this equation if the left-hand side is nonnegative in a weak sense.

By Giaquinta–Giusti [13, Theorem 4.2], a Q -quasiminimizer can be modified on a set of measure zero so that it becomes locally Hölder continuous in Ω . This continuous Q -quasiminimizer is called a Q -quasiharmonic function, and a p -harmonic function is a continuous minimizer.

If u is a quasisuperminimizer, then $au + b$ is also a quasisuperminimizer whenever $a \geq 0$ and $b \in \mathbf{R}$. Also, u is a Q -quasiminimizer if and only if it is both a Q -quasisubminimizer and a Q -quasisuperminimizer. Quasisuperminimizers are invariant under scaling see Lemma 2.1 in [9] reproduced here for completeness,

Lemma 2.3. Let $\Omega \subset \mathbf{R}^n$ be open and $\tau > 0$. If $u : \Omega \rightarrow \mathbf{R}^n$ is a Q -quasisuperminimizer in Ω , then $v(x) := u(\tau x)$ is a Q -quasisuperminimizer in

$$\Omega_\tau := \{x : \tau x \in \Omega\}.$$

Proof. See Lemma 2.1 in [9]. □

The following definition will play a role in Section 3. The finiteness requirement is important for this to be useful, and is always fulfilled when $\overline{G}$ is a compact subset of Ω , by the definition of quasiminimizers.

Definition 2.4. If u is a Q -quasiminimizer in $\Omega \subset \mathbf{R}^n$ we say that u has the *maximal p -energy allowed by Q* on the open set $G \subset \Omega$ with respect to a p -admissible weight $w(x)$ if

$$\int_G |\nabla u|^p d\mu = Q \int_G |\nabla v|^p d\mu < \infty,$$

where $d\mu = |x|^\beta dx$ and v is the minimizer in G with boundary values $v = u$ on ∂G .

For further discussion on quasi(super)minimizers, as well as references to the literature, we refer to Björn–Björn [3],[6], and Björn–Björn–Korte [8]. We will mainly be interested in radially symmetric functions on weighted $\mathbf{R}^n$, $n \geq 2$ with the radial weight $w(x) = |x|^\beta$. We shall consider only the conformal case $p = n + \beta$ where the logarithm $\log |x|$ is the p -harmonic function and the log-powers are the quasiminimizers. The quasiminimizing constants obtained in this section will be useful in section 3 to prove Theorem 1.3. Using some of the notations developed in [9], we define the annulus

$$A_{r_1, r_2} = \{x \in \mathbf{R}^n : r_1 < |x| < r_2\}, \quad \text{where } 0 \leq r_1 < r_2 \leq \infty.$$

Lemma 2.5. Let $u(x) = (\log |x|)^\alpha$. Then $u(x)$ is a superminimizer with respect to the weight $w(x) = |x|^\beta$ in $A_{1, \infty}$ if and only if $0 \leq \alpha \leq 1$ and u is a subminimizer in $A_{1, \infty}$ if and only if $\alpha \leq 0$ or $\alpha \geq 1$.

Proof. Since the weight $w(x) = |x|^\beta$ and $u(x) = (\log |x|)^\alpha$, then we have $-\operatorname{div}(|x|^\beta |\nabla u|^{n+\beta-2} \nabla u)$ given by

$$-\operatorname{div}(|x|^\beta |\nabla u(x)|^{n+\beta-2} \nabla u(x)) = \alpha(1 - \alpha)(n + \beta - 1)|\alpha|^{n+\beta-2}|x|^{-n} (\log |x|)^{(\alpha-1)(n+\beta-1)-1} \quad (2.1)$$

for $x \in A_{1, \infty}$. The sign of (2.1) is determined by the function $\alpha(1 - \alpha)$ since $p = n + \beta > 1$. (2.1) is nonpositive if and only if $0 \leq \alpha \leq 1$ that is u is a superminimizer in this range and nonnegative throughout $A_{1, \infty}$ if and only if $\alpha \leq 0$ or $\alpha \geq 1$ thus a subminimizer there. □

Lemma 2.6. Let $p = n + \beta > 1$, $0 \leq \gamma < 1$, and $u(x) = (-\log |x|)^\alpha$. Then $u(x)$ is a superminimizer in $A_{\gamma, 1}$ with respect to the weight $w(x) = |x|^\beta$ if and only if $0 \leq \alpha \leq 1$ and u is a subminimizer in $A_{\gamma, 1}$ if and only if $\alpha \leq 0$ or $\alpha \geq 1$.

Proof. Given the weight $w(x) = |x|^\beta$ and $u(x) = (-\log |x|)^\alpha$, then we have $-\operatorname{div}(|x|^\beta |\nabla u|^{n+\beta-2} \nabla u)$ given by

$$-\operatorname{div}(|x|^\beta |\nabla u(x)|^{n+\beta-2} \nabla u(x)) = \alpha(1-\alpha)(n+\beta-1)|\alpha|^{n+\beta-2}|x|^{-n}(-\log |x|)^{(\alpha-1)(n+\beta-1)-1} \quad (2.2)$$

for $x \in A_{\gamma,1}$. The sign of (2.2) is determined by the function $\alpha(1-\alpha)$ since $p = n + \beta > 1$. (2.2) is nonpositive if and only if $0 \leq \alpha \leq 1$ that is u is a superminimizer in this range and nonnegative throughout $A_{\gamma,1}$ if and only if $\alpha \leq 0$ or $\alpha \geq 1$ thus a subminimizer there. $\square$

The optimal quasiminimizing and quasisub/superminimizing constant for the function $u = (\log |x|)^\alpha$ on the annuli $A_{1,\gamma}$, $\gamma > 1$ with respect to the weight $w(x) = |x|^\beta$ is given by the following result which includes Theorem 1.4 whose proof relies on the methods developed in [9].

Theorem 2.7. *Let $p = n + \beta > 1$, $1 < \gamma \leq \infty$, $\alpha > 1 - \frac{1}{n+\beta}$ and $u(x) = (\log x)^\alpha$. Then u is a quasiminimizer with respect to the weight $w(x) = |x|^\beta$ in $A_{1,\gamma}$ with*

$$Q_{\alpha,n,\beta} = \frac{\alpha^{n+\beta}}{(n+\beta)(\alpha-1)+1} \quad (2.3)$$

being the best quasiminimizer constant. Moreover, u is a Q -quasi(sub/super)minimizer in $A_{1,\gamma}$ if $\gamma < \infty$ as given in Table 1, where Q in the table is the best quasi(sub/super)minimizing constant.

Proof. Let $\varphi(r) = (\log r)^\alpha$, $1 < r_1 < r_2 < \gamma$, $G = \{r : 1 < r_1 < r < r_2 \leq \infty\}$, $s_1 = \log r_1$, $s_2 = \log r_2$ and $S = \frac{s_2}{s_1}$. The p -energy of u with respect to the weight $w(x) = |x|^\beta$ in A_{r_1,r_2} is given by

$$I_u(A_{r_1,r_2}) := \int_{\Omega} |\nabla u|^{n+\beta} d\mu = c_{n-1} \int_{r_1}^{r_2} |\varphi'(r)|^{n+\beta} r^{n+\beta-1} dr =: c_{n-1} \hat{I}_\varphi(G),$$

where c_{n-1} is the $(n-1)$ -dimensional surface area on the sphere $\mathbf{S}^{n-1}$. Now,

$$\begin{aligned} \hat{I}_\varphi(G) &= \int_{r_1}^{r_2} \alpha^{n+\beta} (\log r)^{(n+\beta)(\alpha-1)} \frac{dr}{r} = \int_{s_1}^{s_2} \alpha^{n+\beta} t^{(n+\beta)(\alpha-1)} dt \\ &= Q_{\alpha,n,\beta} (s_2^{(\alpha-1)(n+\beta)+1} - s_1^{(\alpha-1)(n+\beta)+1}) = Q_{\alpha,n,\beta} s_1^{(\alpha-1)(n+\beta)+1} (S^{(\alpha-1)(n+\beta)+1} - 1). \end{aligned} \quad (2.4)$$

The minimizer is given by $\psi(r) = \log r$ and we note that the p -energy integral with respect to the weight $w(x) = |x|^\beta$ is equivalent to the one above ($\hat{I}_\varphi(G)$) with $\alpha = 1$ thus,

$$\hat{I}_\psi(G) = s_1(S-1).$$

We want to compare the energy $\hat{I}_\psi(G)$ with the the energy $\hat{I}_\eta(G)$ of the minimizer $\eta = a\psi + b$ where we choose a and b so that $\eta = \varphi$ on ∂G . As $s_1^\alpha = as_1 + b$ and $s_2^\alpha = as_2 + b$ from where,

$$a = \frac{s_2^\alpha - s_1^\alpha}{s_2 - s_1} = s_1^{\alpha-1} \left(\frac{S^\alpha - 1}{S - 1} \right)$$

thus $\hat{I}_\eta(G) = |a|^{n+\beta} \hat{I}_\psi(G)$. It is known that η minimises the $\hat{I}$ energy given by the boundary values of φ on ∂G . Now we calculate the quotient

$$k(S) := \frac{\hat{I}_\varphi(G)}{\hat{I}_\eta(G)} = \frac{\hat{I}_\varphi(G)}{|a|^{n+\beta} \hat{I}_\psi(G)} = Q_{\alpha,n,\beta} \frac{S^{(n+\beta)(\alpha-1)+1} - 1}{S - 1} \left(\frac{S - 1}{S^\alpha - 1} \right)^{n+\beta}, \quad (2.5)$$

which only depends on S . Let $k(S)$ be re-written in the form

$$k(S) = Q_{\alpha,n,\beta} \frac{(S^{(n+\beta)(\alpha-1)+1} - 1)(S - 1)^{n+\beta-1}}{(S^\alpha - 1)^{n+\beta}}. \quad (2.6)$$

Let $\gamma := \frac{(S^{(n+\beta)(\alpha-1)+1} - 1)(S - 1)^{n+\beta-1}}{(S^\alpha - 1)^{n+\beta}}$. The derivative $k'(S)$ of $k(S)$ is given by

$$k'(S) = Q_{\alpha,n,\beta} \cdot \gamma \cdot \left(\frac{n + \beta - 1}{S - 1} + \frac{((n + \beta)(\alpha - 1) + 1)S^{(n+\beta)(\alpha-1)}}{S^{(n+\beta)(\alpha-1)+1} - 1} - \frac{\alpha(n + \beta)S^{\alpha-1}}{S^\alpha - 1} \right), \quad (2.7)$$

Since by hypothesis $n + \beta > 1$ and $(n + \beta)(\alpha - 1) + 1 > 0$ then the derivative is positive for all values of $S > 1$. Thus

$$\sup_{S>1} k(S) = \lim_{S \rightarrow \infty} k(S) = Q_{\alpha,n,\beta}. \quad (2.8)$$

Comparing u with $x \mapsto \eta(|x|)$ shows that the quasiminimizing constant for u with respect to the weight $w(x) = |x|^\beta$ cannot be less than $Q_{\alpha,n,\beta}$.

To show that $Q_{\alpha,n,\beta}$ will do, let ω be a function such that $\omega - \varphi \in \text{Lip}_c((1, \gamma))$. The open set

$$V = \{x \in (1, \gamma) : \omega(x) \neq \varphi(x)\}$$

can be written as a countable (or finite) union of pairwise disjoint intervals $\{I_j\}_j$. We find from (2.5) and (2.8) that

$$\hat{I}_\varphi(V) = \sum_j \hat{I}_\varphi(I_j) \leq \sum_j Q_{\alpha,n,\beta} \hat{I}_\omega(I_j) = Q_{\alpha,n,\beta} \hat{I}_\omega(V). \quad (2.9)$$

Hence φ is indeed a $Q_{\alpha,n,\beta}$ -quasiminimizer for the energy $\hat{I}$ with respect to the weight $w(x) = |x|^\beta$ on $(1, \gamma)$.

Next, we turn to u . Let v be such that $v - u \in \text{Lip}_c(A_{1,\gamma})$. Also let

$$\Omega = \{x \in A_{1,\gamma} : v(x) \neq u(x)\}.$$

Using polar coordinates $x = (r, \theta)$, where $r > 1$ and $\theta \in \mathbf{S}^{n-1}$, let

$$V_\theta = \{r : (r, \theta) \in \Omega\} \quad \text{and} \quad v_\theta(r) = v(r, \theta).$$

We then find, applying (2.9) to $G = V_\theta$, that

$$\begin{aligned} I_u(\Omega) &= \int_{\mathbf{S}^{n-1}} \hat{I}_\varphi(V_\theta) d\theta \leq \int_{\mathbf{S}^{n-1}} Q_{\alpha,n,\beta} \hat{I}_{v_\theta}(V_\theta) d\theta \\ &= Q_{\alpha,n,\beta} \int_{\Omega} \left| \frac{\partial v}{\partial r} \right|^n d\mu \leq Q_{\alpha,n,\beta} \int_{\Omega} |\nabla v|^n d\mu = Q_{\alpha,n,\beta} I_v(\Omega), \end{aligned}$$

showing that u is indeed a $Q_{\alpha,n,\beta}$ -quasiminimizer in $A_{1,\gamma}$.

It follows directly that the constants in the quasiminimizer column in Table 1 are correct. By Lemma 2.5, u is a subminimizer if $\alpha \geq 1$ and a superminimizer if $1 - 1/(n + \beta) < \alpha \leq 1$. As u is a Q -quasiminimizer if and only if it is both

	quasi-minimizer	quasi-subminimizer	quasi-superminimizer
$1 - \frac{1}{n + \beta} < \alpha < 1$	$Q = Q_{\alpha,n,\beta}$	$Q = Q_{\alpha,n,\beta}$	$Q = 1$
$\alpha = 1$	$Q = 1$	$Q = 1$	$Q = 1$
$\alpha > 1$	$Q = Q_{\alpha,n,\beta}$	$Q = 1$	$Q = Q_{\alpha,n,\beta}$

TABLE 1. Optimal quasiminimizing and quasiminimizing constants with respect to the weight $w(x) = |x|^\beta$ of $\log^\alpha |x|$ in $A_{1,\gamma}$ and $(-\log |x|)^\alpha$ in $A_{\gamma,1}$, as provided by Theorems 2.7 and 2.8.

a Q -quasisubminimizer and a Q -quasisuperminimizer, it follows that $Q_{\alpha,n,\beta}$ is the optimal quasiminimizing constant when $1 - 1/(n + \beta) < \alpha \leq 1$, and the optimal quasiminimizing constant when $\alpha \geq 1$.

Finally, if $\gamma < \infty$, then it follows from (2.10) that

$$\hat{I}_\varphi(A_{1,\gamma}) = Q_{\alpha,n,\beta}(\log \gamma)^{(n+\beta)(\alpha-1)+1}$$

and that the minimizer with the same boundary values has p -energy $(\log \gamma)^{(n+\beta)(\alpha-1)+1}$, i.e. u has the maximal p -energy allowed by $Q_{\alpha,n,\beta}$ on $A_{1,\gamma}$. $\square$

Theorem 2.8. Let $0 \leq \gamma < 1, \alpha > 1 - \frac{1}{n+\beta}$ and $u(x) = (-\log |x|)^\alpha$. Then u is a Q -quasi(sub/super)minimizer with respect to the weight $w(x) = |x|^\beta$ in $A_{\gamma,1}$ with

$$Q_{\alpha,n,\beta} = \frac{\alpha^{n+\beta}}{(n + \beta)(\alpha - 1) + 1}$$

being the best quasiminimizing constant.

Furthermore, u is a Q -quasi(sub/super)minimizer in $A_{1,\gamma}$ as given in Table 1 where the Q in that table is the best quasi(sub/super)minimizing constant. Also, u has the maximal p -energy with respect to weight allowed by $Q_{\alpha,n,\beta}$ on $A_{\gamma,1}$ if $0 < \gamma < 1$.

Proof. Let $\varphi(r) = (-\log r)^\alpha, 0 < r_1 < r_2 < 1, G = \{r : 0 < r_1 < r < r_2 \leq 1\}, s_1 = -\log r_1, s_2 = -\log r_2$ and $S = \frac{s_1}{s_2} > 1$. The p -energy of u with respect to the weight $w(x) = |x|^\beta$ in A_{r_1,r_2} is given by

$$I_u(A_{r_1,r_2}) := \int_\Omega |\nabla u|^{n+\beta} dx = c_{n-1} \int_{r_1}^{r_2} |\varphi'(r)|^{n+\beta} r^{n+\beta-1} dr =: c_{n-1} \hat{I}_\varphi(G),$$

where c_{n-1} is the $(n - 1)$ -dimensional surface area on the sphere $\mathbf{S}^{n-1}$.

$$\begin{aligned} \hat{I}_\varphi(G) &= \int_{r_1}^{r_2} \alpha^{n+\beta} (-\log r)^{(n+\beta)(\alpha-1)} \frac{dr}{r} = - \int_{s_1}^{s_2} \alpha^{n+\beta} t^{(n+\beta)(\alpha-1)} dt \\ &= Q_{\alpha,n,\beta} (s_1^{(\alpha-1)(n+\beta)+1} - s_2^{(\alpha-1)(n+\beta)+1}) = Q_{\alpha,n,\beta} s_2^{(\alpha-1)(n+\beta)+1} (S^{(\alpha-1)(n+\beta)+1} - 1). \end{aligned} \quad (2.10)$$

The minimizer is given by $\psi(r) = -\log r$ and we note that the p -energy integral with respect to the weight $w(x) = |x|^\beta$ is equivalent to the one above with $\alpha = 1$ thus,

$$\hat{I}_\psi(G) = s_2(S - 1).$$

We want to compare the p -energy $\hat{I}_\psi(G)$ with the p -energy $\hat{I}_\eta(G)$ of the minimizer $\eta = a\psi + b$ where we choose a and b so that $\eta = \varphi$ on ∂G . As $s_1^\alpha = as_1 + b$ and $s_2^\alpha = as_2 + b$ from where,

$$a = \frac{s_2^\alpha - s_1^\alpha}{s_2 - s_1} = s_2^{\alpha-1} \left(\frac{S^\alpha - 1}{S - 1} \right)$$

thus $\hat{I}_\eta(G) = |a|^{n+\beta} \hat{I}_\psi(G)$. It is known that η minimises the $\hat{I}$ -energy given by the boundary values of φ on ∂G . Now we calculate the quotient

$$k(S) := \frac{\hat{I}_\varphi(G)}{\hat{I}_\eta(G)} = \frac{\hat{I}_\varphi(G)}{|a|^{n+\beta} \hat{I}_\psi(G)} = Q_{\alpha,n,\beta} \frac{S^{(n+\beta)(\alpha-1)+1} - 1}{S - 1} \left(\frac{S - 1}{S^\alpha - 1} \right)^{n+\beta}, \quad (2.11)$$

which only depends on S . Since equation (2.5) is the same as equation (2.11) then following the same arguments as in the proof of Theorem 2.7 we conclude that $Q = Q_{\alpha,n,\beta}$ is the best quasiminimizing constant.

For the Table 1 constants. By Lemma 2.6, u is a subminimizer if either $\alpha \geq 1$ or $\alpha \leq 0$ and a superminimizer if $1 - 1/(n + \beta) < \alpha \leq 1$. As u is a Q -quasiminimizer if and only if it is both a Q -quasisubminimizer and a Q -quasisuperminimizer, it follows that $Q_{\alpha,n,\beta}$ is the optimal quasisubminimizing constant when $1 - 1/(n + \beta) < \alpha \leq 1$, and the optimal quasisuperminimizing constant when $\alpha \geq 1$.

Finally, if $\gamma < 1$, then it follows from (2.10) that

$$\hat{I}_\varphi(A_{\gamma,1}) = Q_{\alpha,n,\beta} (\log \gamma)^{(n+\beta)(\alpha-1)+1}$$

and that the minimizer with the same boundary values has p -energy $(\log \gamma)^{(n+\beta)(\alpha-1)+1}$, i.e. u has the *maximal p -energy allowed by $Q_{\alpha,n,\beta}$ on $A_{\gamma,1}$* . $\square$

3. THE INCREASE IN THE QUASISUPERMINIMIZING CONSTANT ON WEIGHTED $\mathbf{R}^n$

In this section we are going to make use of the log-power functions considered in section 2 to construct higher dimensional analogues of the examples in Björn–Björn–Korte [8, Section 3] that generalize the higher dimensional analogues in Björn–Björn–Mirumbe [9, Section 3] to all values of p concerning the optimality of (1.2) in Theorem 1.1. For us to achieve this we let $p = n + \beta > 1$ and study quasisuperminimizers with respect to the p -admissible weight $|x|^\beta$ on the annulus

$$A := A_{1/e,1} = \{x \in \mathbf{R}^n : 1/e < |x| < 1\}.$$

Example 3.1. Given $Q > 1$, there are exactly two exponents $1 - 1/(n + \beta) < \alpha' < 1 < \alpha$ such that $Q = Q_{\alpha,n,\beta} = Q_{\alpha',n,\beta}$, where

$$Q_{\alpha,n,\beta} = \frac{\alpha^{n+\beta}}{(n + \beta)(\alpha - 1) + 1}. \quad (3.1)$$

This is shown by differentiating (3.1) with respect to α and noting that the derivative is negative for $\alpha < 1$ and positive for $\alpha > 1$, and that $Q_{\alpha,n,\beta} \rightarrow \infty$ as $\alpha \rightarrow 1 - 1/(n + \beta)$ and as $\alpha \rightarrow \infty$.

We let

$$u_Q(x) = \log^\alpha |ex| \quad \text{and} \quad \bar{u}_Q(x) = 1 - (-\log |x|)^{\alpha'}. \quad (3.2)$$

Then $u_Q(y) = \bar{u}_Q(y) = 0$ if $|y| = 1/e$, and $u_Q(y) = \bar{u}_Q(y) = 1$ if $|y| = 1$. By Theorem 2.7 and Lemma 2.3, u_Q is a subminimizer and a Q -quasisuperminimizer with respect to the weight $w(x) = |x|^\beta$ in A . The same is true for $\bar{u}_Q$ by Theorem 2.8.

It follows from Theorems 2.7 and 2.8 that u_Q has the maximal p -energy allowed by Q on each annulus $A_{1/e,\gamma}$, $1/e < \gamma \leq 1$, while $\bar{u}_Q$ has the maximal p -energy allowed by Q on each annulus $A_{\gamma,1}$, $1/e \leq \gamma < 1$. This will be useful when proving our main result Theorem 1.3, which we will do now.

Proof. of Theorem 1.3. Let $1 - \frac{1}{n+\beta} < \alpha_2 < 1 < \alpha_1$ be such that $Q_1 = Q_{\alpha_1,n,\beta}$ and $Q_2 = Q_{\alpha_2,n,\beta}$ as in (2.3) with $u_1 := u_{Q_1}$ and $u_2 := \bar{u}_{Q_2}$ being their corresponding quasiminimizers as given in (3.2). By Theorem 1.1 (Theorem 1.2 in [8]) the function $u := \min\{u_1, u_2\}$ is a $\bar{Q}$ -quasisuperminimizer in A with a $\bar{Q}$ quasisuperminimizing constant given by (1.2). Let $v = \log |ex|$ which is a minimizer on A with the same boundary values as u, u_1 and u_2 . As u_1 and u_2 are subminimizers they are less than v thus $u < v$ in A .

Here below we let $u_j(r) = u_j(x)$ when $r = |x|$ and show that u is not a Q_2 -quasisuperminimizer on A . To do this it suffices to show that the p -energy

$$I_u := \int_A |\nabla u|^p d\mu > Q_2 I_v.$$

where $d\mu = r^\beta dr$ with dr the usual Lebesgue measure on $\mathbf{R}$ and

$$I_v = \int_A |\nabla v|^p d\mu = c_{n-1} \int_{1/e}^1 r^{-(n+\beta)} r^{n+\beta-1} dr = c_{n-1}. \quad (3.3)$$

Let r_0 be the unique number in A such that $u_1(r_0) = u_2(r_0)$ that is the unique solution to

$$1 = (-\log(r_0))^{\alpha_2} + (\log(er_0))^{\alpha_1}. \quad (3.4)$$

(To see that there is a unique solution, we consider $w = u_2 - u_1$ and note that $w(\frac{1}{e}) = w(1) = 0$). Since $w'(\frac{1}{e}) > 0$ and $w'(1) = \infty$, there is at least one $r \in A$ such that $w(r) = 0$. Next, $w'(r) = 0$ if and only if,

$$h(r) := \log(er)(-\log r)^\tau = \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1-\alpha_1}} \quad \text{where } \tau = \frac{1-\alpha_2}{\alpha_1-1} > 0.$$

We note that $h'(r) > 0$ if and only if $0 < r < e^{-\frac{\tau}{\tau+1}}$ and that $h(r)$ attains its maximum at (and only at) $r = e^{-\frac{\tau}{\tau+1}}$. Since $h(\frac{1}{e}) = h(1) = 0$, then we therefore confirm that there are atmost two solutions to $w'(r) = 0$, and thus there can be atmost one solution to (3.4) which must lie in between the two local extrema of w .) Since u_2 is a subminimizer in A we have that;

$$\int_{A_{1/e, r_0}} |\nabla u_1|^p d\mu > \int_{A_{1/e, r_0}} |\nabla u_2|^p d\mu$$

where the strict inequality follows from uniqueness of solutions to obstacle problems (see e.g. Björn–Björn [5, Section 3, Theorem 7.2]) and from the fact that u_1 and u_2 differ on a set of positive measure. Hence

$$\int_A |\nabla u|^p d\mu = \int_{A_{1/e, r_0}} |\nabla u_1|^p d\mu + \int_{A_{r_0, 1}} |\nabla u_2|^p d\mu > \int_A |\nabla u_2|^p d\mu = Q_2 I_v,$$

where the last inequality follows from the fact that u_2 has the maximal p -energy allowed by Q_2 on A . Since $Q_1 \leq Q_2$ then this concludes the proof of the Theorem 1.3. $\square$

Theorem 1.3 shows in general that there is an increase in the quasisuperminimizing constant when taking the minimum of two quasiminimizers but does not give any quantitative estimate of the increase. Next, we give an estimate for the lower bound for this increase in the quasisuperminimizing constant.

Given $Q_1, Q_2 > 1$ and $p = n + \beta > 1$, let $0 < 1 - \frac{1}{n+\beta} < \alpha_2 < 1 < \alpha_1$ be such that $Q_1 = Q_{\alpha_1, n, \beta}$ and $Q_2 = Q_{\alpha_2, n, \beta}$ and u_1 and u_2 are their corresponding quasiminimizers as (3.1) and (3.2) respectively. Let $r_0 \in A$ be as in (3.4). Since from Theorems 2.7 and 2.8, $u_1 = u_{Q_1}$ and $u_2 = \bar{u}_{Q_1}$ have the maximal p -energies allowed by Q_1 on $A_{1/e, r_0}$ and Q_2 on $A_{r_0, 1}$ respectively then the p -energy of $u = \min\{u_1, u_2\}$ is given by,

$$\begin{aligned} \int_A |\nabla u|^p d\mu &= \int_{A_{1/e, r_0}} |\nabla u_1|^p d\mu + \int_{A_{r_0, 1}} |\nabla u_2|^p d\mu \\ &= c_{n-1} (Q_1 (\log e r_0)^{(n+\beta)(\alpha_1-1)+1} + Q_2 (-\log r_0)^{(n+\beta)(\alpha_2-1)+1}) \\ &:= c_{n-1} \hat{Q}. \end{aligned} \tag{3.5}$$

Comparing the value in (3.5) with the energy $I_v = c_{n-1}$ given by (3.3) of the minimizer v with the same boundary values as u on A we see that u is not a Q -quasisuperminimizer for any $Q < \hat{Q}$. We note that r_0 is uniquely determined by Q_1 and Q_2 (through α_1 and α_2) and thus $\hat{Q}$ depends only on Q_1 and Q_2 (and on n, β). The lower bounds in Table 2 have been obtained by letting Maple 18 evaluate $\hat{Q}$ for some values of $Q := Q_1 = Q_2$. Since calculating $\hat{Q}$ involves first solving equation (3.4) twice for α , so that $Q_1 = Q_1(\alpha_1)$ and $Q_2 = Q_2(\alpha_2)$, then finding $\frac{1}{e} < r_0 < 1$ such that (3.4) is satisfied, and finally evaluating $\hat{Q}$ for the obtained values of α_1, α_2 and r_0 then this can be done numerically but not analytically.

Remark 3.2. Table 2 and Table 2 in Björn–Björn–Korte [8] have similar columns and rows for $p = 1.2$ and $p = 100$ with their corresponding values of Q . This suggests that the lower bound to the increase in Q when u is a minimum of

Q	$p = 1.2$	$p = 2.5$	$p = 10$	$p = 100$	$\bar{Q} = \frac{2Q}{Q+1}$
1.001	1.001480628	1.001480662	1.001480664	1.001480665	1.001500250
1.01	1.014821935	1.014825359	1.014825583	1.014825593	1.015024876
1.125	1.187625011	1.188130758	1.188164386	1.188165836	1.191176471
2	2.599606519	2.620539063	2.622093879	2.622161265	2.666666667
10	17.45294063	17.695180523	17.72058231	17.72170691	18.18181818
100	195.7168148	196.48153668	196.5905036	196.5955633	198.0198020

TABLE 2. $\widehat{Q}$ for certain values of $p = n + \beta$ with $Q_1 = Q_2 = Q$, as well as $\bar{Q}$ from Theorem 1.1.

logarithmic functions u_1 and u_2 is similar to that described in equation (3.8) in Björn–Björn–Korte [8] for the power functions that is:

$$\widehat{Q} - Q_2 > (Q_1 - 1)\alpha_2^{p+\frac{1}{\alpha_1-1}}. \quad (3.6)$$

We now show that indeed (3.6) is true for the weighted case. It is enough to show that $\widetilde{Q} = \widehat{Q}$. Let $p = n + \beta > 1$, $Q_1, Q_2 > 1$ be given, and choose $1 - 1/(n + \beta) < \alpha_2 < 1 < \alpha_1$ such that $Q_1 = Q_{\alpha_1, n, \beta}$ and $Q_2 = Q_{\alpha_2, n, \beta}$ as given in (3.1). With $p = n + \beta > 1$ this choice of α_1 and α_2 also satisfy (3.2) in Björn–Björn–Korte [8] since $x \mapsto \log(er)$ implies $A \mapsto (0, 1)$. Next choose $r_0 \in A$ to be the unique solution of (3.4) then $\widehat{Q}$ is given by (3.5). To relate this to $\widetilde{Q}$ in [8], we let $x_0 = \log er_0 \in (0, 1)$. It then follows from (3.4) that

$$1 = (1 - x_0)^{\alpha_2} + x_0^{\alpha_1},$$

i.e. x_0 is the unique solution of this equation, which is the same as equation (3.4) in [8]. (That the solution is unique was shown in [8], but also follows from the uniqueness of $r_0 \in A$. Using (3.6) in [8] (with $p = n + \beta$) we see that

$$\widetilde{Q} = Q_1 x_0^{(n+\beta)(\alpha_1-1)+1} + Q_2 (1 - x_0)^{(n+\beta)(\alpha_2-1)+1} \quad (3.7)$$

$$= Q_1 (\log er_0)^{(n+\beta)(\alpha_1-1)+1} + Q_2 (-\log r_0)^{(n+\beta)(\alpha_2-1)+1} = \widehat{Q}. \quad (3.8)$$

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