

MULTIPLICITY RESULT FOR VARIABLE EXPONENTS ANISOTROPIC ELLIPTIC PROBLEMS

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ABSTRACT. We consider a class of nonlinear elliptic problems governed by an anisotropic differential operator. We prove existence and multiplicity results for nontrivial solutions using suitable variational methods and a variant of the Mountain Pass lemma in anisotropic Sobolev spaces.

Keywords. Anisotropic equations, anisotropic Sobolev spaces, variational method, multiple solutions.

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1. INTRODUCTION

The study of anisotropic partial differential equations has received increasing attention in the field of mathematical research. This is partly due to their frequent appearance in various domains of application, including image restoration [1, 9], elastic mechanics [38], porous media [5, 17], electromagnetism [6, 37], nonstandard growth [2, 8, 10, 23, 26, 39], and electrorheological fluids [3, 4, 29, 36]. The natural framework for solving these kinds of problems is that of Sobolev spaces with variable exponents, whose origins date back to the pioneering work of Orlicz [28]. For example, in [12], the authors considered the isotropic $p(x)$ -Laplacian operator and established existence and multiplicity results of solutions for the following problem

$$\begin{cases} -\Delta_{p(x)}(u) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

This previous work has been generalized in [21], where the authors considered the anisotropic $\vec{p}(x)$ -Laplacian operator, which possesses more complicated nonlinearities, mainly due to the fact that it is not homogeneous. In fact, they provided

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an existence and multiplicity result of solutions for the following problem

$$\begin{cases} -\Delta_{\vec{p}(x)}(u) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

In the present work, we extend the results above to a more general case. We consider the problem

$$(\mathcal{P}) \begin{cases} -\sum_{i=1}^N \partial_{x_i} a_i(x, \nabla u) + \lambda(x)|u|^{p_0(x)-2}u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary denoted by $\partial\Omega$ and $\text{meas}(\Omega) > 0$. In (1.3), we assume that for any $i = 1, \dots, N$, $a_i : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function (i.e., $a_i(\cdot, \xi)$ is measurable on Ω for each $\xi \in \mathbb{R}^N$ and $a_i(x, \cdot)$ is continuous on $\mathbb{R}^N$ for almost all $x \in \Omega$) and is the continuous derivative with respect to ξ of the mapping $A_i : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$, $A_i = A_i(x, \xi)$, i.e. $a(x, \xi) = \nabla_{\xi} A(x, \xi)$ such that, for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$:

$$A_i(x, 0) = 0, \quad (1.4)$$

$$\exists C, C' > 0, \quad |a_i(x, \xi)| \leq C|\xi_i|^{p_i(x)-1} + C', \quad (1.5)$$

and

$$|\xi_i|^{p_i(x)} \leq a_i(x, \xi) \cdot \xi_i \leq p_i(x) A_i(x, \xi). \quad (1.6)$$

$f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function satisfying the following conditions

$$\begin{cases} |f(x, t)| \leq \alpha(1 + |t|^{q(x)-1}), \text{ for some } \alpha > 0, \forall t \in \mathbb{R}, \text{ a.e. } x \in \Omega, \quad q \in C(\overline{\Omega}) \\ \text{and } 1 < q(x) < P_{-, \infty}, \forall x \in \overline{\Omega}. \end{cases} \quad (1.7)$$

$$\begin{cases} \text{There exist a constant } M > 0 \text{ and } \theta > p_0^+ \text{ such that} \\ 0 < \theta F(x, t) \leq t f(x, t), \text{ for all } x \in \Omega \text{ and } |t| \geq M, \\ \text{where } F(x, t) = \int_0^t f(x, s) ds. \end{cases} \quad (1.8)$$

$$f(x, t) = o(|t|^{p_0^+-1}) \text{ as } t \rightarrow 0 \text{ and uniformly for } x \in \Omega, \text{ with } q^- > p_0^+. \quad (1.9)$$

$$\lambda \in L^\infty(\Omega) \text{ and there exists } \lambda_0 > 0 \text{ such that } \lambda(x) \geq \lambda_0 \text{ for all } x \in \Omega. \quad (1.10)$$

This special structure of the differential operator and the incorporation of the nonlinearity $\lambda(x)|u|^{p_0(x)-2}u$ make the problem challenging. To overcome these difficulties, we combine tools, techniques, and ideas developed in [18, 19, 20, 27] and references therein. In this direction, we can find many other works (see [13, 25, 30, 31, 32, 33, 34, 35]).

Let us recall that a function $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ is a weak solution to problem (1.3) if the following identity holds

$$\int_{\Omega} \sum_{i=1}^N a_i(x, \nabla u) \partial_{x_i} v dx + \int_{\Omega} \lambda(x) |u|^{p_0(x)-2} u v dx = \int_{\Omega} f(x, u) v dx, \quad (1.11)$$

for all $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Related to the problem (1.3), we define the so-called energy functional $\Phi : W_0^{1,\vec{p}(\cdot)}(\Omega) \rightarrow \mathbb{R}$ by

$$\Phi(u) = \int_{\Omega} \sum_{i=1}^N A_i(x, \nabla u) dx + \int_{\Omega} \frac{\lambda(x)}{p_0(x)} |u|^{p_0(x)} dx - \int_{\Omega} F(x, u) dx. \quad (1.12)$$

It is well known that under (1.7), Φ is well defined and is a $\mathcal{C}^1$ functional with derivative given by

$$\langle \Phi'(u), v \rangle = \int_{\Omega} \sum_{i=1}^N a_i(x, \nabla u) \partial_{x_i} v dx + \int_{\Omega} \lambda(x) |u|^{p_0(x)-2} u v dx - \int_{\Omega} f(x, u) v dx,$$

for all $u, v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Hence, from (1.11), it follows that u is a weak solution of (1.3) if and only if u is a critical point of Φ , i.e.

$$u \in W_0^{1,\vec{p}(\cdot)}(\Omega) \text{ weak solution of (1.3)} \iff \langle \Phi'(u), v \rangle = 0 \quad (1.13)$$

for all $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Hence with the previous definition, we state as follows our main result, namely the existence at least two nontrivial solutions to Problem (1.3):

Theorem 1.1. *Assume that (1.7)–(1.9) hold true and $f(x, 0) = 0$ for a.e. $x \in \Omega$. Then the problem (1.3) has at least two nontrivial solutions, in which one is non-negative and one is non-positive.*

The remaining part of this paper is organized as follows: In Section (2), we recall some notations and preliminaries about the anisotropic variable exponent Sobolev spaces. In Section (3), we give the main steps of our method to establish the proof of our result.

2. VARIATIONAL FRAMEWORK

Let Ω be an open bounded domain in $\mathbb{R}^N$ ($N \geq 2$), with smooth boundary $\partial\Omega$ and $\text{meas}(\Omega) > 0$. We recall some known results on variable exponent Lebesgue and Sobolev spaces. See [7, 11, 14, 15, 22] and reference therein for more detail. Define

$$C_+(\overline{\Omega}) = \{p : \overline{\Omega} \rightarrow \mathbb{R} \text{ measurable, such that } 1 < p^- \leq p^+ < N\},$$

where

$$p^- = \text{ess inf}\{p(x) \mid x \in \overline{\Omega}\} \quad \text{and} \quad p^+ = \text{ess sup}\{p(x) \mid x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space by

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable, such that } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the Luxemburg norm

$$|u|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\},$$

where the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \longrightarrow \mathbb{R}$ is the convex modular of the $L^{p(\cdot)}(\Omega)$ space defined by

$$\rho_{p(\cdot)}(u) := \int_{\Omega} |u(x)|^{p(x)} dx,$$

and satisfies the following inequalities (see [11, 14]): for any $u \in L^{p(\cdot)}(\Omega)$,

$$\min \{ |u|_{p(\cdot)}^{p^-}; |u|_{p(\cdot)}^{p^+} \} \leq \rho_{p(\cdot)}(u) \leq \max \{ |u|_{p(\cdot)}^{p^-}; |u|_{p(\cdot)}^{p^+} \} \leq |u|_{p(\cdot)}^{p^+} + 1 \quad (2.1)$$

and

$$\min \{ \rho_{p(\cdot)}^{1/p^+}(u); \rho_{p(\cdot)}^{1/p^-}(u) \} \leq |u|_{p(\cdot)} \leq \max \{ \rho_{p(\cdot)}^{1/p^+}(u); \rho_{p(\cdot)}^{1/p^-}(u) \} \leq (\rho_{p(\cdot)}(u) + 1)^{1/p^-}. \quad (2.2)$$

The space $(L^{p(\cdot)}(\Omega), |u|_{p(\cdot)})$ is a separable reflexive Banach space (see [22]), and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, for all $x \in \Omega$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the following Hölder type inequality holds true:

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(\cdot)} |v|_{p'(\cdot)}. \quad (2.3)$$

Let $\vec{p}(\cdot) = (p_0(\cdot), p_1(\cdot), \dots, p_N(\cdot))$ such that for any $i = 0, 1, \dots, N$, $p_i(\cdot) \in C_+(\Omega)$ with

$$p_0(x) := \max\{p_i(x)\} \text{ for all } x \in \Omega \text{ and } i = 1, \dots, N,$$

and put

$$1 < p_i^- := \operatorname{ess\,inf}_{x \in \Omega} p_i(x) \leq \operatorname{ess\,sup}_{x \in \Omega} p_i(x) := p_i^+ < \infty,$$

$$p_m(x) := \min\{p_1(x), \dots, p_N(x)\} \text{ for all } x \in \Omega,$$

$$p_m^- := \min\{p_1^-, \dots, p_N^-\}, \quad p_m^+ := \max\{p_1^-, \dots, p_N^-\}, \quad p_0^+ := \max\{p_1^+, \dots, p_N^+\}.$$

We define the anisotropic variable exponent Sobolev space by

$$W^{1, \vec{p}(\cdot)}(\Omega) := \{ u \in L^{p_0}(\Omega) \text{ and } \partial_{x_i} u \in L^{p_i(\cdot)}(\Omega), \forall i = 1, \dots, N \}$$

and $W_0^{1, \vec{p}(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1, \vec{p}(\cdot)}(\Omega)$. Equipped with the following norm

$$||u||_{\vec{p}(\cdot)} = |u|_{p_0(\cdot)} + \sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}, \quad (2.4)$$

the spaces $W^{1, \vec{p}(\cdot)}(\Omega)$ and $W_0^{1, \vec{p}(\cdot)}(\Omega)$ are separable, reflexive and uniform convex Banach spaces (see [24, 16]). We note that if $q \in C(\overline{\Omega})$ and $q(x) < P_{-, \infty}, \forall x \in \overline{\Omega}$, then the embedding $W_0^{1, \vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous.

3. PROOF OF MAIN RESULTS

Let us denote by $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$ the positive and negative parts of u . Before going further, we recall the following technical and useful lemma:

Lemma 3.1 (See [21]).

(1) If $u \in W_0^{1, \vec{p}(\cdot)}(\Omega)$, then $u^+, u^- \in W_0^{1, \vec{p}(\cdot)}(\Omega)$ and

$$\nabla u^+ = \begin{cases} \nabla u & \text{if } u > 0, \\ 0 & \text{if } u \leq 0, \end{cases} \quad \nabla u^- = \begin{cases} 0 & \text{if } u \geq 0, \\ \nabla u & \text{if } u < 0. \end{cases} \quad (3.1)$$

(2) The mappings $u \mapsto u^\pm$ are continuous on $W_0^{1, \vec{p}(\cdot)}(\Omega)$.

Next, we consider the following truncated problem

$$(\mathcal{P}_\pm) \begin{cases} -\sum_{i=1}^N \partial_{x_i} a_i(x, \nabla u) + \lambda(x) |u|^{p_0(x)-2} u = f_\pm(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.2)$$

where

$$f_\pm(x, t) = \begin{cases} f(x, t) & \text{if } \pm t \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (3.3)$$

Lemma 3.2. All solutions of $(\mathcal{P}_+)$ (resp. $(\mathcal{P}_-)$) are non-negative (resp. non-positive) solutions of $(\mathcal{P})$.

Proof. Let us define $\Phi_\pm : W_0^{1, \vec{p}(\cdot)}(\Omega) \longrightarrow \mathbb{R}$, by

$$\begin{aligned} \Phi_\pm(u) &= \int_\Omega \sum_{i=1}^N A_i(x, \nabla u) dx + \int_\Omega \frac{\lambda(x)}{p_0(x)} |u|^{p_0(x)} dx - \int_\Omega F_\pm(x, u) dx \\ &= \int_\Omega \sum_{i=1}^N A_i(x, \nabla u) dx + \int_\Omega \frac{\lambda(x)}{p_0(x)} |u|^{p_0(x)} dx - \int_\Omega F(x, u^\pm) dx, \end{aligned} \quad (3.4)$$

where $F_\pm(x, s) = \int_0^s f_\pm(x, t) dt$.

In view of Lemma 3.1 and Condition (1.7), $\Phi_\pm$ is well defined on $W_0^{1, \vec{p}(\cdot)}(\Omega)$, weakly lower semi-continuous and $\mathcal{C}^1$ -functionals. From (1.13), we derive the following equivalence:

$$u \in W_0^{1, \vec{p}(\cdot)}(\Omega) \text{ weak solution of } (\mathcal{P}_+) \iff \langle \Phi'_+(u), v \rangle = 0 \quad (3.5)$$

for all $v \in W_0^{1, \vec{p}(\cdot)}(\Omega)$.

Replacing v by u^- in (3.5), we obtain

$$\int_\Omega \sum_{i=1}^N a_i(x, \nabla u) \partial_{x_i} u^- dx + \int_\Omega \lambda(x) |u|^{p_0(x)-2} u u^- dx - \int_\Omega f_+(x, u) u^- dx = 0.$$

Next, using (1.6), it follows that

$$\int_{\Omega} \sum_{i=1}^N |\partial_{x_i} u^-|^{p_i(x)} dx + \int_{\Omega} \lambda(x) |u^-|^{p_0(x)} dx = 0.$$

From (1.10), we have

$$0 \leq \min(1; \lambda_0) \int_{\Omega} \sum_{i=0}^N |\partial_{x_i} u^-|^{p_i(x)} dx \leq \int_{\Omega} \sum_{i=1}^N |\partial_{x_i} u^-|^{p_i(x)} dx + \int_{\Omega} \lambda(x) |u^-|^{p_0(x)} dx = 0.$$

Which imply that

$$\int_{\Omega} \sum_{i=0}^N |\partial_{x_i} u^-|^{p_i(x)} dx = 0.$$

Thanks to (2.1), we have $\|u^-\|_{\vec{p}(\cdot)} = 0$. Then, we deduce that $u^- = 0$ and $u = u^+$. Similarly, nontrivial critical points of Φ_- are non-positive solutions of $(\mathcal{P})$. $\square$

We prove the following result on the required properties of the energy functional Φ related to problem (1.3).

Lemma 3.3. *Let conditions (1.7), (1.8) and (1.9) be satisfied. Then, the energy functional Φ_+ given in (3.4) satisfies the (PS)-condition.*

Proof. Let $\{u_n\} \subseteq W_0^{1, \vec{p}(\cdot)}(\Omega)$ be a (PS) sequence for the functional Φ_+ : $\Phi_+(u_n)$ is bounded and $\Phi'_+(u_n) \rightarrow 0$. Let us show that $(u_n)_n$ is bounded in $W_0^{1, \vec{p}(\cdot)}(\Omega)$. Using (1.8) and (1.6), since $\Phi_+(u_n)$ is bounded, we observe that:

$$\begin{aligned} C_1 &\geq \int_{\Omega} \sum_{i=1}^N A_i(x, \nabla u_n) dx + \int_{\Omega} \frac{\lambda(x)}{p_0(x)} |u_n|^{p_0(x)} dx - \int_{\Omega} F(x, u_n^+) dx \\ &\geq \frac{1}{p_0^+} \int_{\Omega} \sum_{i=1}^N a_i(x, \nabla u_n) \partial_{x_i} u_n dx + \frac{1}{p_0^+} \int_{\Omega} \lambda(x) |u_n|^{p_0(x)} dx - \int_{\Omega} \frac{u_n^+}{\theta} f(x, u_n^+) dx + C_2, \end{aligned}$$

where C_1 and C_2 are two constants. Note that

$$\begin{aligned} \langle \Phi'_+(u_n), u_n \rangle &= \int_{\Omega} \sum_{i=1}^N a_i(x, \nabla u_n) \partial_{x_i} u_n dx + \int_{\Omega} \lambda(x) |u_n|^{p_0(x)} dx - \int_{\Omega} f(x, u_n^+) u_n dx \\ &= \int_{\Omega} \sum_{i=1}^N a_i(x, \nabla u_n) \partial_{x_i} u_n dx + \int_{\Omega} \lambda(x) |u_n|^{p_0(x)} dx - \int_{\Omega} f(x, u_n^+) u_n^+ dx, \end{aligned}$$

which implies, by using (1.6)

$$C_1 \geq \left(\frac{1}{p_0^+} - \frac{1}{\theta} \right) \left(\int_{\Omega} \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} dx + \int_{\Omega} \lambda(x) |u_n|^{p_0(x)} dx \right) + \frac{1}{\theta} \langle \Phi'_+(u_n), u_n \rangle + C_2.$$

Suppose, by contradiction that $(u_n)_n$ unbounded in $\in W_0^{1,\vec{p}(\cdot)}(\Omega)$, so $\|u_n\|_{\vec{p}(\cdot)} \geq 1$ for rather large values of n . Using relation (2.1) we have

$$\int_{\Omega} \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} dx + \int_{\Omega} \lambda(x) |u_n|^{p_0(x)} dx \geq \sum_{i=1}^N |\partial_{x_i} u_n|_{p_i(\cdot)}^{p_m^-} - N + \lambda_0 \int_{\Omega} |u_n|^{p_0(x)} dx,$$

then, according to the convexity of the application $t \in \mathbb{R}^+ \mapsto t^{p_m^-}$, $p_m^- > 1$, we have

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} dx + \int_{\Omega} \lambda(x) |u_n|^{p_0(x)} dx \\ & \geq \frac{1}{N^{p_m^- - 1}} \left(\sum_{i=1}^N |\partial_{x_i} u_n|_{p_i(\cdot)} \right)^{p_m^-} + \lambda_0 \int_{\Omega} |u_n|^{p_0(x)} dx - N \\ & \geq \frac{1}{2^{p_m^- - 1}} \min \left\{ \frac{1}{N^{p_m^- - 1}}, \lambda_0 \right\} \|u_n\|_{\vec{p}(\cdot)}^{p_m^-} - N - \lambda_0. \end{aligned} \quad (3.6)$$

Moreover, $\Phi'_+(u_n) \rightarrow 0$ assure that there exists $C_3 > 0$ such that

$$-C_3 \|u_n\|_{\vec{p}(\cdot)} \leq \langle \Phi'_+(u_n), u_n \rangle \leq C_3 \|u_n\|_{\vec{p}(\cdot)}$$

for rather large values of n . Consequently,

$$C_1 \geq \left(\frac{1}{p_0^+} - \frac{1}{\theta} \right) \left(\frac{1}{2^{p_m^- - 1}} \min \left\{ \frac{1}{N^{p_m^- - 1}}, \lambda_0 \right\} \|u_n\|_{\vec{p}(\cdot)}^{p_m^-} - N - \lambda_0 \right) - \frac{C_3}{\theta} \|u_n\|_{\vec{p}(\cdot)} + C_2.$$

Since $p_m^- > 1$ and $\left(\frac{1}{p_0^+} - \frac{1}{\theta} \right) > 0$, we end up with

$$\left(\frac{1}{p_0^+} - \frac{1}{\theta} \right) \left(\frac{1}{2^{p_m^- - 1}} \min \left\{ \frac{1}{N^{p_m^- - 1}}, \lambda_0 \right\} \|u_n\|_{\vec{p}(\cdot)}^{p_m^-} - N - \lambda_0 \right) - \frac{C_3}{\theta} \|u_n\|_{\vec{p}(\cdot)} + C_2 \rightarrow +\infty$$

as $\|u_n\|_{\vec{p}(\cdot)} \rightarrow +\infty$, what is a contradiction. So $(u_n)_n$ is a bounded sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. $\square$

Lemma 3.4. *There exist $r > 0$ and $\alpha > 0$ such that $\Phi_+(u) \geq \alpha$, for all $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ with $\|u\|_{\vec{p}(\cdot)} = r$.*

Proof. Combining Conditions (1.7) and (1.9), we deduce the following inequality

$$|F(x, t)| \leq \varepsilon |t|^{p_0^+} + C(\varepsilon) |t|^{q(x)} \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}. \quad (3.7)$$

Taking $\|u\|_{\vec{p}(\cdot)}$ small enough and using (1.6), we have

$$\Phi_+(u) \geq \frac{1}{p_0^+} \int_{\Omega} \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} dx + \frac{1}{p_0^+} \int_{\Omega} \lambda(x) |u|^{p_0(x)} dx - \int_{\Omega} F(x, u^+) dx.$$

For such an element u we have $|\partial_{x_i} u|_{p_i(\cdot)} < 1$ and, by relation (2.1), we obtain

$$\begin{aligned} \int_{\Omega} \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} dx + \int_{\Omega} \lambda(x) |u|^{p_0(x)} dx &\geq \sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}^{p_0^+} + \lambda_0 \int_{\Omega} |u|^{p_0(x)} dx \\ &\geq \frac{1}{N^{p_0^+-1}} \left(\sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)} \right)^{p_0^+} + \lambda_0 \|u\|_{p_0^+}^{p_0^+} \\ &\geq \frac{1}{2^{p_0^+-1}} \min \left\{ \frac{1}{N^{p_0^+-1}}, \lambda_0 \right\} \|u\|_{\vec{p}(\cdot)}^{p_0^+}. \end{aligned} \quad (3.8)$$

Relations (3.7) and (3.8) imply

$$\begin{aligned} \Phi_+(u) &\geq \frac{1}{p_0^+ 2^{p_0^+-1}} \min \left\{ \frac{1}{N^{p_0^+-1}}, \lambda_0 \right\} \|u\|_{\vec{p}(\cdot)}^{p_0^+} - \varepsilon \int_{\Omega} |u^+|^{p_0^+} dx - C(\varepsilon) \int_{\Omega} |u^+|^{q(x)} dx \\ &\geq \frac{1}{p_0^+ 2^{p_0^+-1}} \min \left\{ \frac{1}{N^{p_0^+-1}}, \lambda_0 \right\} \|u\|_{\vec{p}(\cdot)}^{p_0^+} - \varepsilon \int_{\Omega} |u|^{p_0^+} dx - C(\varepsilon) \int_{\Omega} |u|^{q(x)} dx. \end{aligned} \quad (3.9)$$

From Condition (1.7), it follows $p_0^+ < q^- \leq q(x) < P_{-, \infty} \forall x \in \overline{\Omega}$, then we have the following continuous and compact embedding:

$$W_0^{1, \vec{p}(\cdot)}(\Omega) \subset L^{p_0^+} \quad \text{and} \quad W_0^{1, \vec{p}(\cdot)}(\Omega) \subset L^{q(x)},$$

which implies the existence of $C_4, C_5 > 0$ such that

$$\|u\|_{p_0^+} \leq C_4 \|u\|_{\vec{p}(\cdot)} \quad \text{and} \quad \|u\|_{q(x)} \leq C_5 \|u\|_{\vec{p}(\cdot)} \quad \text{for all } u \in W_0^{1, \vec{p}(\cdot)}(\Omega).$$

Since $\|u\|_{\vec{p}(\cdot)}$ is small enough, we deduce that

$$\int_{\Omega} |u|^{q(x)} dx \leq \|u\|_{q(x)}^{q^-} \leq C_6 \|u\|_{\vec{p}(\cdot)}^{q^-}. \quad (3.10)$$

Combining (3.9) and (3.10) it results that

$$\Phi_+(u) \geq \frac{1}{p_0^+ 2^{p_0^+-1}} \min \left\{ \frac{1}{N^{p_0^+-1}}, \lambda_0 \right\} \|u\|_{\vec{p}(\cdot)}^{p_0^+} - \varepsilon C_4^{p_0^+} \|u\|_{\vec{p}(\cdot)}^{p_0^+} - C_7(\varepsilon) \|u\|_{\vec{p}(\cdot)}^{q^-}$$

where C_i are positives constants. Let us choose $\varepsilon > 0$ such that

$$\varepsilon C_4^{p_0^+} \leq \frac{1}{p_0^+ 2^{p_0^+}} \min \left\{ \frac{1}{N^{p_0^+-1}}, \lambda_0 \right\},$$

we obtain

$$\begin{aligned} \Phi_+(u) &\geq \frac{1}{p_0^+ 2^{p_0^+}} \min \left\{ \frac{1}{N^{p_0^+-1}}, \lambda_0 \right\} \|u\|_{\vec{p}(\cdot)}^{p_0^+} - C_7(\varepsilon) \|u\|_{\vec{p}(\cdot)}^{q^-} \\ &\geq \|u\|_{\vec{p}(\cdot)}^{p_0^+} \left(\frac{1}{p_0^+ 2^{p_0^+}} \min \left\{ \frac{1}{N^{p_0^+-1}}, \lambda_0 \right\} - C_7 \|u\|_{\vec{p}(\cdot)}^{q^- - p_0^+} \right). \end{aligned} \quad (3.11)$$

Since $p_0^+ < q^-$, the function $t \mapsto \left(\frac{1}{p_0^+ 2^{p_0^+}} \min \left\{ \frac{1}{N^{p_0^+-1}}, \lambda_0 \right\} - C_7 t^{q^- - p_0^+} \right)$ is strictly positive in a neighborhood of zero. It follows that there exist $r > 0$ and $\alpha > 0$ such that

$$\Phi_+(u) \geq \alpha, \quad \forall u \in W_0^{1, \vec{p}(\cdot)}(\Omega) : \|u\|_{\vec{p}(\cdot)} = r. \quad (3.12)$$

□

In order to apply the Mountain Pass Theorem, it remains to show the following property on the energy functional Φ_+ :

Lemma 3.5. *There exists $e \in W_0^{1, \vec{p}(\cdot)}(\Omega)$ such that $\|e\| > r$ and $\Phi_+(e) < 0$.*

Proof. We prove that

$$\Phi_+(su) \longrightarrow -\infty \text{ as } s \rightarrow +\infty,$$

for a certain $u \in W_0^{1, \vec{p}(\cdot)}(\Omega)$. From Condition (1.8), we obtain

$$F(x, t) \geq c|t|^\theta \text{ for all } (x, t) \in \overline{\Omega} \times \mathbb{R}.$$

Let $u \in W_0^{1, \vec{p}(\cdot)}(\Omega)$ and $s > 1$, we have

$$\Phi_+(su) = \int_{\Omega} \sum_{i=1}^N A_i(x, s \nabla u) dx + \int_{\Omega} \frac{\lambda(x) |s|^{p_0(x)}}{p_0(x)} |u|^{p_0(x)} dx - \int_{\Omega} F(x, (su)^+) dx.$$

Thanks to (1.4) and (1.5), A_i satisfies the upper growth condition:

$$A_i(x, \xi) \leq C(1 + |\xi_i|^{p_i(x)})$$

with C a positive constant. Then, we have

$$\begin{aligned} \Phi_+(su) &\leq C \int_{\Omega} \sum_{i=1}^N \frac{[1 + |s|^{p_i(x)} |\partial_{x_i} u|^{p_i(x)}]}{p_i(x)} dx + \int_{\Omega} \frac{\lambda(x) |s|^{p_0(x)}}{p_0(x)} |u|^{p_0(x)} dx - \int_{\Omega} F(x, (su)^+) dx \\ &\leq s^{p_0^+} \int_{\Omega} \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx + s^{p_0^+} \int_{\Omega} \frac{\lambda(x)}{p_0(x)} |u|^{p_0(x)} dx - cs^\theta \int_{\Omega} |u^+|^\theta dx + C. \end{aligned}$$

The fact $\theta > p_0^+$, gives that

$$\Phi_+(su) \longrightarrow -\infty \text{ as } s \rightarrow +\infty.$$

This implies that there exists $e \in W_0^{1, \vec{p}(\cdot)}(\Omega)$ such that $\|e\| > r$ and $\Phi_+(e) < 0$. □

Proof of Theorem 1.1. In view of Lemma 3.3, Lemma 3.4, Lemma 3.5 and the Mountain Pass Theorem, Φ_+ admits a critical value $\mu \geq \alpha$ which is characterized by

$$\mu = \inf_{h \in \Lambda} \sup_{t \in [0,1]} \Phi_+(h(t)) \quad (3.13)$$

where

$$\Lambda = \{h \in C([0, 1], W_0^{1, \vec{p}(\cdot)}(\Omega)) : h(0) = 0 \text{ and } h(1) = e\}. \quad (3.14)$$

Then, the functional Φ_+ has a critical point u^+ with $\Phi_+(u^+) \geq \alpha$. But, $\Phi_+(0) = 0$, that is, $u^+ \neq 0$. Therefore, the problem $(\mathcal{P}_+)$ has a nontrivial solution which, by lemma 3.2, is a non-negative solution of the problem $(\mathcal{P})$.

Similarly, using Φ_- , we show that there exists furthermore a non-positive solution. □

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