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FAIR HOP ROMAN DOMINATING FUNCTION IN GRAPHS

LEOMARICH F. CASINILLO[1]*

Abstract. This paper introduces a new variant of a hop Roman dominating function in graphs called the fair hop Roman dominating function. Moreover, several important combinatorial properties and characterizations of fair hop Roman dominating functions in various graph classes were investigated.

Keywords. Fair domination, hop domination, Roman dominating function, Fair hop Roman dominating function
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1. Introduction and preliminaries

Graph theory is an active area of research that focuses on the study of the structure and properties of vertices and edges. [9]. Domination in graphs is one of the intriguing and growing topics in graph theory, which concerns several applications that include security problems, optimization, networking, and resource allocation, among others [10]. The hop Roman dominating function is one of the interesting restricted parameters of domination in graphs, which was formally introduced by Shabani [13] in the year 2017. Nowadays, the hop Roman dominating function was studied by some graph theorists, and there are now several variants of the said parameter which can be found in [6, 11, 14]. Meanwhile, fair domination in graphs pioneered by Caro et al. [8], has captured the interest of many discrete mathematicians. An important application of fair domination is the technique of equal distribution of resources in the form of resource allocation in economics and networking. With the two interesting parameters of domination on a graph mentioned above, the author is motivated to combine the two ideas, and came up with a stronger parameter involving the Roman dominating function in graphs called the fair hop Roman dominating function. In the scenario of Roman domination, the new parameter ensures that the undefended regions (vertices) have an equal distribution of protection. In view of computer science, an application of fair hop Roman domination is designing a network that finds the optimal placements of routers or communication relays that ensure an equitable allocation of coverage and resilience.

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* Corresponding author

For the definitions and concepts not mentioned here but used in this study, the readers may refer to [4, 5, 6, 9, 10]. Let $G = (V(G), E(G))$ be a graph where $V(G)$ and $E(G)$ are the vertex and edge sets, respectively. A subset D of $V(G)$ is a *dominating set* of G if each vertex outside D is adjacent to some vertex in D [10]. The smallest cardinality of D on graph G is called the *domination number* on G, denoted by $\gamma(G)$. If $|D| = \gamma(G)$, then D is a γ-set on G. Some interesting studies on domination in graphs can be found in [1, 2, 3, 4, 5, 7, 12]. A subset H of $V(G)$ is a *hop dominating set* of G if for each vertex $a \in V(G) \setminus H$, there exists $b \in H$ such that $d_G(a, b) = 2$. The smallest cardinality of H on G is called the *hop domination number* of G, denoted by $\gamma_h(G)$. If $|H| = \gamma_h(G)$, then H is a γ_h-set on G. Let $S \subseteq V(G)$. Then S is called a fair vertex set on G if for every $a, b \in S$, $|N_G(a) \cap (V(G) \setminus S)| = |N_G(b) \cap (V(G) \setminus S)| \geq 1$. Let $x \in V(G)$. The set $N_G^2(x) = \{y \in V(G) : deg_G(x, y) = 2\}$ is called the *open hop-neighborhood* and each element of $N_G^2(x)$ is called hop-neigbor of x. Let F be a subset of $V(G)$. Then F is called a *fair dominating set* of G provided that for every vertex in $V(G) \setminus F$ is adjacent to some vertex in F and the set $V(G) \setminus F$ is a fair vertex set on G, that is, for any $a, b \in V(G) \setminus F$, $|N_G^2(a) \cap F| = |N_G^2(b) \cap F| \geq 1$ [8]. The *fair domination number* of G is defined as the minimum cardinality of F on G and is denoted by $fd(G)$. The set F is called FD-set on G provided that $|D_f| = fd(G)$. Let $\phi : V(G) \to \{0, 1, 2\}$ be a function on any graph G. Then consider the following sets:

$$\begin{aligned} V_0 &= \{x \in V(G) : \phi(x) = 0\}; \\ V_1 &= \{x \in V(G) : \phi(x) = 1\}; and \\ V_2 &= \{x \in V(G) : \phi(x) = 2\}. \end{aligned}$$

Hence, a function ϕ can be written as $\phi = (V_0, V_1, V_2)$. A function $\phi = (V_0, V_1, V_2)$ is a *hop Roman dominating function* (HRDF) on G if for each vertex in V_0 is a hop neighbor of some vertex in V_2 [13]. The *weight* of a function ϕ is denoted by $\omega_G^{hR}(f)$ and is defined as $\omega_G^{hR}(f) = \sum_{z \in V(G)} f(z) = |V_1| + 2|V_2|$. The *hop Roman domination number* of graph G, denoted by $\gamma_{hR}(G)$, is defined as the minimum weight of an HRDF ϕ on G. In that case, we have $\gamma_{hR}(G) = min\{\omega_G^{hR}(\phi) : \phi \textit{ is an } \text{HRDF} \textit{ on } G\}$. Every HRDF ϕ on any graph G with $\omega_G^{hR}(f) = \gamma_{hR}(G)$ is called a γ_{hR}-function on G. A function ϕ is a fair hop Roman dominating function (FHRDF) on G if (i) for every $v \in V_0$, there exists $u \in V_2$ such that $d_G(u, v) = 2$, and (ii) for any $x, y \in V_0$, $|N_G^2(x) \cap V_2| = |N_G^2(y) \cap V_2| \geq 1$. The *weight* of FHRDF ϕ on G is denoted by $\omega_G^{fhR}(\phi)$ and is defined as $\omega_G^{fhR}(\phi) = \sum_{z \in V(G)} \phi(z)$. The minimum weight of an FHRDF ϕ on G, denoted by $\gamma_{fhR}(G)$ is called the *fair hop Roman domination number*, that is, $\gamma_{fhR}(G) = min\{\omega_G^{fhR}(\phi) : \phi \textit{ is an } \text{FHRDF} \textit{ on } G\}$. Every FHRDF ϕ on G with $\omega_G^{fhR}(\phi) = \gamma_{fhR}(G)$ is called a γ_{fhR}-function on G. This study aims to investigate the newly introduced restricted version of a hop Roman domination in graphs called the fair hop Roman dominating function. Moreover, the paper discusses the combinatorial properties of a fair hop Roman dominating function in graphs, bounds of fair hop Roman domination number, and some characterizations.

2. Main results

This section explores the important combinatorial and theoretical properties of the fair hop Roman dominating function in some classes of graphs.

Theorem 2.1. *Let G be a graph and $\phi = (V_0, V_1, V_2)$ be a γ_{fhR}-function on G. Then the following holds:*

(*i*) *$V_1 \cup V_2$ is a fair hop dominating set on G; and*
(*ii*) *$V_0 = \varnothing$ if and only if $V_2 = \varnothing$.*

Proof. Assume that $\phi = (V_0, V_1, V_2)$ be a γ_{fhR}-function on G. Then it follows that ϕ is an FHRDF on G. Let $v \in V(G) \setminus (V_1 \cup V_2)$. Then $v \in V_0$. Since f is an FHRDF on G, there exists $u \in V_2$ such that $d_G(u, v) = 2$ and for any $x, y \in V_0$, $|N^2_G(x) \cap V_2| = |N^2_G(y) \cap V_2| \geq 1$. Hence, it implies that $V_1 \cup V_2$ is a fair hop dominating set on G and so, (i) holds. Next, let $V_0 = \varnothing$. Assume for a moment that $V_2 \neq \varnothing$. Then we let $u \in V_2$. Set $U_0 = V_0$, $U_1 = V_1 \cup \{u\}$, and $U_2 = V_2 \setminus \{u\}$. This implies that $\lambda = (U_0, U_1, U_2)$ is an FHRDF on G. So, we have $\omega^{fhR}_G(g) = |U_1| + 2|U_2| = (|V_1| + 1) + 2(|V_2| - 1) = |V_1| + 2|V_2| - 1 < \omega^{fhR}_G(f) = \gamma_{fhR}(G)$. This is a contradiction to the assumption that ϕ is a γ_{fhR}-function on G. Thus, it implies that $V_2 = \varnothing$. As for the converse, we suppose $V_2 = \varnothing$. Since ϕ is a γ_{fhR}-function on G, the assumption that $V_2 = \varnothing$ forces $V_0 = \varnothing$. This shows that (ii) holds. □

Corollary 2.2. *Let $G = K_n$ where $n \geq 1$. Then $\gamma_{fhR}(G) = n$*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a γ_{fhR}-function on $G = K_n$. By Theorem 2.1(i), $V_1 \cup V_2$ is a fair hop dominating set on G. Since G is a complete graph, the set of hop-dominated vertices must be empty. Hence, $V_1 \cup V_2 = V(G)$ and $V_0 = \varnothing$. Since ϕ is a γ_{fhR}-function on G, by Theorem 2.1(ii), we get $V_2 = \varnothing$. Therefore, we end up with $\gamma_{fhR}(G) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. □

Theorem 2.3. *Let G be a graph with $|V(G)| = n \geq 1$ and let $\phi = (V_0, V_1, V_2)$ be a γ_{fhR}-function on G. Then the following statements hold:*

(*i*) *$|V_0| = |V_2| \geq 0$ if and only if $\gamma_{fhR}(G) = n$; and*
(*ii*) *$1 \leq |V_2| < |V_0|$ if and only if $\gamma_{fhR}(G) < n$.*

Proof. Assume that $\phi = (V_0, V_1, V_2)$ is a γ_{fhR}-function on G of order n where n is a positive integer. Suppose $|V_0| = |V_2| \geq 0$. If $|V_0| = |V_2| = 0$, then it follows that $\gamma_{fhR}(G) = \omega^{fhR}_G(\phi) = |V_1| = |V(G)| = n$. Now, if $|V_0| = |V_2| \geq 1$, then it implies that $\gamma_{fhR}(G) = \omega^{fhR}_G(\phi) = |V_1| + 2|V_2| = |V_0| + |V_1| + |V_2| = |V(G)| = n$. As for the converse, we suppose $\gamma_{fhR}(G) = n$. Assume for a moment that $|V_0| \neq |V_2|$. Then it implies that either $|V_0| > |V_2|$ or $|V_0| < |V_2|$. If $\gamma_{fhR}(G) = \omega^{fhR}_G(\phi) = |V_1| + 2|V_2| < |V_0| + |V_1| + |V_2| = |V(G)| = n$, a contradiction. On the other hand, if $|V_0| < |V_2|$, then $\gamma_{fhR}(G) = \omega^{fhR}_G(\phi) = |V_1| + 2|V_2| > |V_0| + |V_1| + |V_2| = |V(G)| = n$. This is again a contradiction to our assumption. Therefore, we obtain $|V_0| = |V_2| \geq 0$ and so, (i) holds.

Next, we suppose that $\gamma_{fhR}(G) < n$. Then it implies that $V_0 \neq \varnothing$. By definition of γ_{fhR}-function, we have $|V_2| \geq 1$. And by (i), we get $|V_0| \neq |V_2|$. Thus, either

$|V_0| > |V_2|$ or $|V_0| < |V_2|$. If $|V_2| > |V_0|$, then $\gamma_{fhR}(G) = \omega_G^{fhR}(\phi) = |V_1| + 2|V_2| > |V_1| + |V_2| + |V_0| = |V(G)| = n$. This is a contradiction since ϕ is a γ_{fhR}-function on G. Accordingly, we have $1 \leq |V_2| < |V_0|$. Conversely, we suppose that $1 \leq |V_2| < |V_0|$. Then it implies that $\gamma_{fhR}(G) = \omega_G^{fhR}(\phi) = |V_1| + 2|V_2| < |V_1| + |V_2| + |V_0| = |V(G)| = n$. Therefore, we have $\gamma_{fhR}(G) < n$ and hence, (ii) holds. □

Corollary 2.4. *Let G be a graph with $|V(G)| = n \geq 1$. If $\gamma_{fhR}(G) < n$ and $|V_2| = 1$, then $|V_1| \neq 0$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a γ_{fhR}-function on G. Suppose $\gamma_{fhR}(G) < n$ and $|V_2| = 1$. By Theorem 2.3, we have $1 = |V_2| < |V_0|$. Then for every $x \in V_0$, $|N_G^2(x) \cap V_2| = 1$. Assume for a moment that $|V_1| = 0$. Let $V_2 = \{u\}$ and $v \in V_0$. Then $d_G(u, v) = 2$. This implies that there exists $w \in N_G(u) \cap N_G(v)$. Since $|V_2| = 1$ and $|V_1| = 0$, it follows that $w \in V_0 \setminus \{v\}$. And since $w \in N_G(u)$, it means that $|N_G^2(w) \cap V_2| = 0$, implying that $V_2 = \{u\}$ is not a fair hop dominating set on G, a contradiction. Therefore, we conclude that $|V_1| \neq 0$. □

Theorem 2.5. *Let G be a graph and let $\phi = (V_0, V_1, V_2)$ be an FHRDF on G with $|V_2| \geq 2$. Then for every distinct vertices $u, v \in V_2$, $N_G^2(u) \cap N_G^2(v) \neq \varnothing$ if and only if for any $y \in V_0$, $|N_G^2(y) \cap V_2| = k > 1$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be an FHRDF on G for which $|V_2| \geq 2$. Suppose for every distinct vertices $u, v \in V_2$, $N_G^2(u) \cap N_G^2(v) \neq \varnothing$. Assume for a moment that for any $y \in V_0$, $|N_G^2(y) \cap V_2| = k \leq 1$. If for any $y \in V_0$, $|N_G^2(y) \cap V_2| = k = 0$, then it follows that $V_1 \cup V_2$ is not a fair hop dominating set on G, a contradiction. Now, if for any $y \in V_0$, $|N_G^2(y) \cap V_2| = k = 1$, then it implies that there exists $a, b \in V_2$ such that $N_G^2(a) \cap N_G^2(b) = \varnothing$, a contradiction to our assumption. Therefore, for any $y \in V_0$, $|N_G^2(y) \cap V_2| = k > 1$. Conversely, we suppose that for any $y \in V_0$, $|N_G^2(y) \cap V_2| = k > 1$. Assume for a moment that there exists $u \in V_2$ such that $N_G^2(u) \cap N_G^2(v) = \varnothing$ for all $v \in V_2 \setminus \{u\}$. Then it follows that there exists $z \in V_0$ such that $N_G^2(z) \cap V_2 = \{u\}$ and it implies that $|N_G^2(z) \cap V_2| = 1$, indicating that V_0 is not a fair vertex on G, a contradiction. Accordingly, for every distinct vertices $u, v \in V_2$, $N_G^2(u) \cap N_G^2(v) \neq \varnothing$. □

Corollary 2.6. *Let G be a graph and let $\phi = (V_0, V_1, V_2)$ be a γ_{fhR}-function on G. If $V_2 = \{u, v\}$ and $N_G^2(u) \cap N_G^2(v) = \varnothing$, then for any $x \in V_0$, $|N_G^2(x) \cap V_2| = 1$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a γ_{fhR}-function on G. Then by Theorem 2.1(i), $V_1 \cup V_2$ is a fair hop dominating set on G. Suppose that $V_2 = \{u, v\}$ and $N_G^2(u) \cap N_G^2(v) = \varnothing$. Seeking a contradiction. Assume for a moment that for any $x \in V_0$, $|N_G^2(x) \cap V_2| \neq 1$. Then it implies that either $|N_G^2(x) \cap V_2| = 0$ or $|N_G^2(x) \cap V_2| = 2$. If for any $x \in V_0$, $|N_G^2(x) \cap V_2| = 0$, then $V_1 \cup V_2$ is not a fair hop dominating set on G, a contradiction. If for any $x \in V_0$, $|N_G^2(x) \cap V_2| = 2$, then by Theorem 2.5, it follows that $N_G^2(u) \cap N_G^2(v) \neq \varnothing$. This is again a contradiction. Therefore, we conclude that for any $x \in V_0$, $|N_G^2(x) \cap V_2| = 1$. □

Proposition 2.7. *If $G = K_{m,n}$ where $m, n \geq 2$, then $\gamma_{fhR}(G) = 4$.*

Proof. Let $G = K_{m,n} = (U, V, E)$ denote a bipartite graph whose partition has the parts of vertex sets U and V, that is, $|U| = m$ and $|V| = n$, and E denotes the edges of G. Let $\phi = (V_0, V_1, V_2)$ be a γ_{fhR}-function on G. Now, let $U = \{u_1, u_2, ..., u_m\}$ and $V = \{v_1, v_2, ..., v_n\}$. Note that if we arbitrarily pick $u_j \in V_2$ where $j \in \{1, 2, 3, ..., m\}$, then $d_G(u_j, u_i) = 2$ for all $i \neq j \in \{1, 2, 3, ..., m\}$. This implies that $U \subseteq N_G^2[u_j]$ for each $j \in \{1, 2, 3, ..., m\}$. Similarly, $V \subseteq N_G^2[v_k]$ for each $k \in \{1, 2, 3, ..., n\}$. Hence, we set $V_2 = \{u_j, v_k\}$ for some $j \in \{1, 2, 3, ..., m\}$ and $k \in \{1, 2, 3, ..., n\}$. This follows that $V(G) \subseteq N_G^2[u_j, v_k]$, and so, $V_1 = \varnothing$ and $V_0 = V(G) \setminus V_2$. In addition, it follows that $N_G^2(u_j) \cap N_G^2(v_k) = \varnothing$. By Corollary 2.6, it implies that $x \in V_0$, $|N_G^2(x) \cap V_2| = 1$, indicating that V_0 is a fair vertex set on G. Thus, $V_1 \cup V_2 = V_2$ is a fair hop Roman dominating set on G. Accordingly, by construction, we conclude that $\gamma_{fhR}(G) = \omega_G^{fhR}(\phi) = |V_1| + 2|V_2| = 2(2) = 4$. □

Theorem 2.8. *Let G be a graph of order n. Then the following inequality holds:*

$$max\{\gamma_R(G), \gamma_{fh}(G)\} \leq \gamma_{fhR}(G) \leq min\{2\gamma_{fh}(G), n\}.$$

Proof. Let $\phi = (V_0, V_1, V_2)$ be a γ_{fhR}-function on G. By Theorem 2.1(i), $V_1 \cup V_2$ is a fair hop dominating set on G. This follows that $\gamma_{fh}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \omega_G^{fhR}(\phi) = \gamma_{fhR}(G)$. Now, since every fair hop Roman dominating function is a Roman dominating function on G, it implies that $\gamma_R(G) \leq \gamma_{fhR}(G)$. Consequently, $max\{\gamma_{fh}(G), \gamma_R(G)\} \leq \gamma_{fhR}(G)$. Next, we let $V_0 = \varnothing$. Since ϕ is a γ_{fhR}-function on G, by Theorem 2.1(ii), we have $V_2 = \varnothing$. Hence, $\phi = (V_0 = \varnothing, V_1 = V(G), V_2 = \varnothing)$ is an FHRDF on G. Accordingly, we obtain $\gamma_{fhR}(G) \leq \omega_G^{fhR}(\phi) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. Suppose that ϕ is a γ_{fhR}-function on G with $V_1 = \varnothing$. By Theorem 2.1(i), it implies that $V_1 \cup V_2 = V_2$ is a fair hop dominating set on G. Since ϕ is a γ_{fhR}-function on G, it follows that $\gamma_{fh}(G) = |V_2|$. Thus, it implies that $\gamma_{fhR}(G) \leq \omega_G^{fhR}(\phi) = 2|V_2| = 2\gamma_{fh}(G)$. Consequently, $\gamma_{fhR}(G) \leq min\{n, 2\gamma_{fh}(G)\}$. Therefore, the conclusion follows. □

Theorem 2.9. *Let $G = G_1 \cup G_2 \cup \ldots \cup G_k$ be a graph where G_j is a component of G for each $j \in \{1, 2, ...k\}$. Then $\phi = (V_0, V_1, V_2)$ is an FHRDF on G if and only if the restriction function $\phi|_{G_j} = (V_0^j, V_1^j, V_2^j)$ is an FHRDF on G_j for each $j \in \{1, 2, ...k\}$.*

Proof. Let $G_1, G_2, \ldots, G_k$ be the components of a graph G. Suppose $\phi = (V_0, V_1, V_2)$ is an FHRDF on G. Then for each $j \in \{1, 2, ..., k\}$, let $V_0^j = V_0 \cap V(G_j)$, $V_1^j = V_1 \cap V(G_j)$ and $V_2^j = V_2 \cap V(G_j)$. So, we have $\phi|_{G_j} = (V_0^j, V_1^j, V_2^j)$ for all $j \in \{1, 2, ..., k\}$. Now, let $j \in \{1, 2, ..., k\}$ and let $v \in V_0^j$. This follows that $v \in V_0$. Since ϕ is a fair hop Roman dominating function on G, it implies that there exists $u \in V_2$ such that $d_G(u, v) = 2$ and for any $x, y \in V_0$, $|N_G^2(x) \cap V_2| = |N_G^2(y) \cap V_2| \geq 1$. Since $d_G(a, b) = \infty$ for any $a \in G_i$ and for any $b \in G_l$ where $i \neq l$, it means $u \in V_2^j$ and $d_{G_j}(u, v) = 2$, and for any $x, y \in V_0^j$, $|N_G^2(x) \cap V_2^j| = |N_G^2(y) \cap V_2^j| \geq 1$. Consequently, $\phi|_{G_j} = (V_0^j, V_1^j, V_2^j)$ is a FHRDF on G_j for each $j \in \{1, 2, ..., k\}$.

Conversely, suppose $\phi|_{G_j} = (V_0^j, V_1^j, V_2^j)$ is an FHRDF on G_j for each $j \in \{1, 2, ...k\}$. Set $V_0 = \bigcup_{j=1}^k V_0^j$, $V_1 = \bigcup_{j=1}^k V_1^j$ and $V_2 = \bigcup_{j=1}^k V_2^j$. Let $\overline{v} \in V_0$. Since

$V_0 = \bigcup_{j=1}^{k} V_0^j$, it implies that $\overline{v} \in V_0^j$ for some $j \in \{1, 2, ...k\}$. Now, since $\phi|_{G_j}$ is a fair hop Roman dominating function on G, it implies that there exists $\overline{u} \in V_2^j$ such that $d_{G_j}(\overline{u}, \overline{v}) = 2$, and for any $x, y \in V_0^j$, $|N_G^2(x) \cap V_2^j| = |N_G^2(y) \cap V_2^j| \geq 1$. Since $V_2 = \bigcup_{j=1}^{k} V_2^j$, it follows that $\overline{u} \in V_2$ and $d_G(\overline{u}, \overline{v}) = 2$ and for any $x, y \in V_0$, $|N_G^2(x) \cap V_2| = |N_G^2(y) \cap V_2| \geq 1$. Therefore, $\phi = (V_0, V_1, V_2)$ is an FHRDF on G. □

The next corollary is a consequence of Theorem 2.9.

Corollary 2.10. *Let $G_1, G_2, \ldots, G_k$ be the components of a graph G. Then, $\gamma_{fhR}(G) = \sum_{j=1}^{k} \gamma_{fhR}(G_j)$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a γ_{fhR}-function on G. For each $j \in \{1, 2, ..., k\}$, set $V_i^j = V_i \cap V(G_j)$ for all $i \in \{0, 1, 2\}$. By Theorem 2.9, it follows that $\phi|_{G_j} = (V_0^j, V_1^j, V_2^j)$ is an FHRDF on G_j for each $j \in \{1, 2, ..., k\}$. Thus

$$\begin{aligned}
\gamma_{fhR}(G) = \omega_G^{fhR}(\phi) &= |V_1| + 2|V_2| \\
&= \sum_{j=1}^{k} |V_1^j| + 2\sum_{j=1}^{k} |V_2^j| \\
&= \sum_{j=1}^{k} \left(|V_1^j| + 2|V_2^j|\right) \\
&\geq \sum_{j=1}^{k} \gamma_{fhR}(G_j).
\end{aligned}$$

On the other hand, let $\phi|_{G_j} = (W_0^j, W_1^j, W_2^j)$ be an FHRDF on G_j for each $j \in \{1, 2, ...k\}$. Set $W_i = \bigcup_{j=1}^{k} W_i^j$ for all $i \in \{0, 1, 2\}$. Then by Theorem 2.9, it implies that $\phi = (W_0, W_1, W_2)$ is an FHRDF on G. So, we have

$$\begin{aligned}
\sum_{j=1}^{k} \gamma_{fhR}(G_j) = \sum_{j=1}^{k} \omega_G^{fhR}(\phi|_{G_j}) &= \sum_{j=1}^{k} \left(|W_1^j| + 2|W_2^j|\right) \\
&= \sum_{j=1}^{k} |W_1^j| + 2\sum_{i=1}^{k} |W_2^j| \\
&= |W_1| + 2|W_2| \\
&\geq \gamma_{fhR}(G).
\end{aligned}$$

Consequently, we end up with $\gamma_{fhR}(G) = \sum_{j=1}^{k} \gamma_{fhR}(G_j)$. □

The following result is immediate from Corollary 2.2 and Corollary 2.10.

Proposition 2.11. *Let G be a graph with $|V(G)| = n$. If each of the components of graph G is complete, then $\gamma_{fhR}(G) = n$. In particular, $\gamma_{shR}(\overline{K}_n) = \gamma_{shR}(K_n) = n$.*

Proof. The proof is immediate from Corollary 2.2 and Corollary 2.10. □

Next is the characterization of the fair hop Roman domination number involving small values.

Proposition 2.12. *Let G be a graph of order $n \geq 1$. Then*

(i) $\gamma_{fhR}(G) = 1$ *if and only if* $G = K_1$*; and*

(ii) $\gamma_{fhR}(G) = 2$ *if and only if* $G \in \{K_2, \overline{K_2}\}$.

Proof. Let $\phi = (V_0, V_1, V_2)$ be a γ_{fhR}-function on G.

(i) Suppose $\gamma_{fhR}(G) = 1$. Then we have $|V_1| + 2|V_2| = 1$. This follows that $|V_2| = 0$. By Theorem 2.1(ii), we get $|V_0| = 0$. Therefore, $\gamma_{fhR}(G) = \omega_G^{fhR} = |V_1| = |V(G)| = 1$ and so, $G = K_1$. Conversely, by Corollary 2.2, we have $\gamma_{fhR}(K_1) = 1$.

(ii) Suppose $\gamma_{shR}(G) = 2$. Then we obtain $|V_1| + 2|V_2| = 2$ and thus, $|V_2| \leq 1$. If $|V_2| = 1$, then $|V_1| = 0$ and $|V_0| \neq 0$. Now, let $V_2 = \{u\}$ and let $v \in V_0$. Then $d_G(u, v) = 2$. Let $w \in N_G(u) \cap N_G(v)$. Since $|V_1| = 0$, it follows that $w \in V_0$. However, it means that $N_G^2(w) \cap V_2 = \varnothing$. This is a contradiction. Hence, $|V_2| = 0$ and by Theorem 2.1(ii), it follows that $|V_0| = 0$. This implies that $\gamma_{fhR}(G) = \omega_G^{fhR} = |V_1| = |V(G)| = 2$. Thus, $G \in \{K_2, \overline{K_2}\}$. Conversely, suppose $G \in \{K_2, \overline{K_2}\}$. Then by Corollary 2.2, we therefore have $\gamma_{fhR}(G) = 2$.

This completes the proof. □

Theorem 2.13. *Let G be a graph. Then $\gamma_{fhR}(G) = 3$ if and only if it satisfies one of the following:*

(i) $|V(G)| = 3$*; or*

(ii) $|V(G)| \geq 4$ *for which there exists $a, b \in V(G)$ such that $N_G(a) = \{b\}$, $N_G^2(a) = V(G) \setminus \{a, b\}$, and $N_G(b) = V(G) \setminus \{a, b\}$ where $|V(G) \setminus \{a, b\}| \geq 2$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a γ_{fhR}-function on G. Suppose $\gamma_{shR}(G) = 3$. Then we have $|V_1| + 2|V_2| = 3$. Thus, $|V_2| \leq 1$. If $|V_2| = 0$, then by Theorem 2.1(ii), it follows that $|V_0| = 0$. Hence, we have $\gamma_{fhR}(G) = \omega_G^{fhR}(\phi) = |V_1| = |V(G)| = 3$ and so, (i) holds. Next, if $|V_2| = 1$, then $|V_1| = 1$ and $|V_0| \geq 1$. Let $V_2 = \{a\}$ and $V_1 = \{b\}$. Since $\phi = (V_0, V_1, V_2)$ is an FHRDF on G, it follows that $N_G^2(a) = V(G) \setminus \{a, b\}$. In addition, this implies that $N_G(a) = \{b\}$ and $N_G(b) = V(G) \setminus \{a, b\}$. Moreover, it means that for any $y \in V_0$, $|N_G^2(y) \cap V_2| = 1$ which indicates that V_0 is a fair vertex set on G. Since $|V_0| \geq 1$, it follows that either $|V_0| = 1$ or $|V_0| \geq 2$. If $|V_0| = 1$, then (i) holds. If $|V_0| \geq 2$, then $|V(G) \setminus \{a, b\}| \geq 2$. Consequently, $|V(G)| \geq 4$ and it is concluded that there exists $a, b \in V(G)$ such that $N_G(a) = \{b\}$, $N_G^2(a) = V(G) \setminus \{a, b\}$, and $N_G(b) = V(G) \setminus \{a, b\}$ where $|V(G) \setminus \{a, b\}| \geq 2$. Thus, (ii) holds.

Conversely, suppose that $|V(G)| = 3$, then $G \in \{K_3, P_3, \overline{K_3}, K_2 \cup K_1\}$. Hence, we have $\gamma_{shR}(G) = 3$. Suppose that $|V(G)| \geq 4$ for which there exists $a, b \in V(G)$ such that $N_G(a) = \{b\}$, $N_G^2(a) = V(G) \setminus \{a, b\}$, and $N_G(b) = V(G) \setminus \{a, b\}$ where $|V(G) \setminus \{a, b\}| \geq 2$. Then set $V_2 = \{a\}$, $V_1 = \{b\}$ and $V_0 = V(G) \setminus (V_1 \cup V_2)$. In that case, $V(G) = N_G^2[V_1 \cup V_2]$ and for all $z \in V_0$, $|N_G^2(z) \cap V_2| = 1$, implying that V_0 is a fair vertex set. Therefore, we conclude that $\gamma_{fhR}(G) = \omega_G^{fhR}(\phi) = |V_1| + 2|V_2| = 1 + 2(1) = 3$. This proves the assertion. $\square$

Proposition 2.14. *Let $G = P_n$ with $n \geq 1$. Then*

$$\gamma_{fhR}(G) = \begin{cases} 1, & \text{if } n = 1, \\ 4, & \text{if } n = 6, \\ 5, & \text{if } n = 7, \\ \frac{2n+3}{3}, & \text{if } 6 \neq n \equiv 0 \pmod 3, \\ \frac{2n+4}{3}, & \text{if } 1, 7 \neq n \equiv 1 \pmod 3, \\ \frac{2n+2}{3}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. If $G = P_1 = K_1$, then by Corollary 2.2, $\gamma_{fhR}(G) = 1$. Let $G = P_n = [x_1, x_2, ..., x_n]$ where $n \geq 2$ and let $\phi = (V_0, V_1, V_2)$ be a γ_{fhR}-function on G. If $G = P_6$, then we can have $V_2 = \{x_3, x_4\}$, $V_0 = V(G) \setminus V_2$ and $V_1 = \varnothing$. This follows that $V(G) = N_G^2[V_2]$ and for each $a \in V_0$, $|N_G^2(a) \cap V_2| = 1$, and thus ϕ is an FHRDF on G. By construction, we have $\gamma_{fhR}(G) = \omega_G^{fhR}(\phi) = |V_1| + 2|V_2| = 2|\{x_3, x_4\}| = 4$. If $G = P_7$, then we can have $V_2 = \{x_3, x_4\}$, $V_0 = V(G) \setminus V_2$ and $V_1 = \{x_7\}$. Again, this implies that $V(G) = N_G^2[V_1 \cup V_2]$ and for each $b \in V_0$, $|N_G^2(b) \cap V_2| = 1$, indicating that ϕ is an FHRDF on G. Hence, by construction, we obtain $\gamma_{fhR}(G) = \omega_G^{fhR}(\phi) = |V_1| + 2|V_2| = |\{x_7\}| + 2|\{x_3, x_4\}| = 5$. For $n \in \mathbb{N} \setminus \{1, 6, 7\}$, we consider the three cases below:

Case 1: $6 \neq n \equiv 0 \pmod 3$

Set $V_2 = \{x_3, x_6, ..., x_n\}$, $V_1 = \{x_2\}$, and $V_0 = V(G) \setminus (V_1 \cup V_2)$. This implies that $V(G) = N_G^2[V_1 \cup V_2]$ and for every $y \in V_0$, $|N_G^2(y) \cap V_2| = 1$, indicating that ϕ is an FHRDF on G. By construction, we have $\gamma_{fhR}(G) = \omega_G^{fhR}(\phi) = |V_1| + 2|V_2| = |\{x_2\}| + 2|\{x_3, x_6, ..., x_n\}| = 1 + 2\left(\frac{n}{3}\right) = \frac{2n+3}{3}$.

Case 2: $1 \neq n \equiv 1 \pmod 3$

This follows that $n - 1 \equiv 0 \pmod 3$. Then we set $V_2 = \{x_3, x_6, ..., x_{n-1}\}$, $V_1 = \{x_2, x_n\}$, and $V_0 = V(G) \setminus (V_1 \cup V_2)$. In this case, we obtain $V(G) = N_G^2[V_1 \cup V_2]$ and for every $w \in V_0$, $|N_G^2(w) \cap V_2| = 1$, implying that ϕ is an FHRDF on G. And by construction, we therefore have $\gamma_{fhR}(G) = \omega_G^{fhR}(\phi) = |V_1| + 2|V_2| = |\{x_2, x_n\}| + 2|\{x_3, x_6, ..., x_{n-1}\}| = 2 + 2\left(\frac{n-1}{3}\right) = \frac{2n+4}{3}$.

Case 3: $n \equiv 2 \pmod 3$

This implies that $n - 2 \equiv 0 \pmod 3$. Hence, we set $V_2 = \{x_3, x_6, ..., x_{n-2}\}$, $V_1 = \{x_2, x_{n-1}\}$, and $V_0 = V(G) \setminus (V_1 \cup V_2)$. This follows that $V(G) = N_G^2[V_1 \cup V_2]$ and for every $z \in V_0$, $|N_G^2(z) \cap V_2| = 1$. Thus, it means that ϕ is an FHRDF on G. By construction, we end up with $\gamma_{fhR}(G) = \omega_G^{fhR}(\phi) = |V_1| + 2|V_2| = |\{x_2, x_{n-1}\}| + 2|\{x_3, x_6, ..., x_{n-2}\}| = 2 + 2\left(\frac{n-2}{3}\right) = \frac{2n+2}{3}$.

This completes the proof. $\square$

Proposition 2.15. *Let $G = C_n$ with $n \geq 3$. Then,*

$$\gamma_{fhR}(G) = \begin{cases} 4, & \text{if } n = 4, 6 \\ 5, & \text{if } n = 7, \\ \frac{2n+3}{3}, & \text{if } 6 \neq n \equiv 0 \pmod 3, \\ \frac{2n+4}{3}, & \text{if } 4, 7 \neq n \equiv 1 \pmod 3, \\ \frac{2n+2}{3}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. The proof is similar to that of Proposition 2.14. □

Theorem 2.16. *Let G be any graph with $|V(G)| = n \geq 3$ and $\gamma_{fh}(G) < \gamma_{fhR}(G)$. Then $\gamma_{fhR}(G) = \gamma_{fh}(G) + 1$ if and only if there exist a set $F \subseteq V(G)$ and vertex $u \in V(G)$ such that $F \cup \{u\}$ is a γ_{fh}-set on G and $V(G) \setminus (F \cup \{u\}) \subseteq N_G^2(u)$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a γ_{fhR}-function on G with $|V(G)| = n \geq 3$ and $\gamma_{fh}(G) < \gamma_{fhR}(G)$. Then we have $\gamma_{fh}(G) \leq |V_1| + |V_2|$. Suppose $\gamma_{fhR}(G) = \gamma_{fh}(G) + 1$. First, we consider $\gamma_{fh}(G) = |V_1| + |V_2|$, that is, $V_1 \cup V_2$ is a γ_{fh}-set on G. It implies that $|V_1| + |V_2| + 1 = |V_1| + 2|V_2|$. Hence, we obtain $|V_2| = 1$ and $|V_1| = \gamma_{fh}(G) - 1$. Let $V_2 = \{u\}$ and $F = V_1$. This follows that $V_0 = V(G) \setminus (V_1 \cup V_2) = V(G) \setminus (F \cup \{u\})$. Let $v \in V_0$ be arbitrary. Since ϕ is an FHRDF on G, we get $d_G(u, v) = 2$ and for any $x, y \in V_0$, $|N_G^2(x) \cap V_2| = |N_G^2(y) \cap V_2| > 0$. Accordingly, we obtain $V(G) \setminus (F \cup \{u\}) \subseteq N_G^2(u)$. Secondly, we suppose $\gamma_{fh}(G) < |V_1| + |V_2|$. This follows that $\gamma_{fh}(G) + 1 \leq |V_1| + |V_2|$. Now, since $\gamma_{fhR}(G) = \gamma_{fh}(G) + 1$, we get $|V_1| + 2|V_2| \leq |V_1| + |V_2|$. Thus, $|V_2| = 0$. By Theorem 2.1(ii), it implies that $|V_0| = 0$. Accordingly, we have $\gamma_{fhR}(G) = \omega_G^{fhR}(f) = |V_1| = |V(G)| = n$. Thus, $\gamma_{fh}(G) = n - 1$. Let $E = V(G) \setminus \{v\}$ be a γ_{fh}-set on G. This follows that there exists $u \in E$ such that $d_G(u, v) = 2$. Let $F = E \setminus \{u\}$. Therefore, it suffices to say that $F \cup \{u\} = E$ is γ_{fh}-set on G and $V(G) \setminus (F \cup \{u\}) = \{v\} \subseteq N_G^2(u)$.

As for the converse, we suppose that there exist a set $F \subseteq V(G)$ and vertex $u \in V(G)$ such that $F \cup \{u\}$ is a γ_{fh}-set on G and $V(G) \setminus (F \cup \{u\}) \subseteq N_G^2(u)$. Accordingly, we set $V_0 = V(G) \setminus (F \cup \{u\})$, $V_1 = F$, and $V_2 = \{u\}$. Consequently, we have $\phi' = (V_0, V_1, V_2)$ is an FHRDF on G. Thus, $\gamma_{fhR}(G) \leq \omega_G^{fhR}(\phi') = |V_1| + 2|V_2| = |F| + 2(1) = (\gamma_{fh}(G) - 1) + 2 = \gamma_{fh}(G) + 1$. To this end, since $\gamma_{fh}(G) < \gamma_{fhR}(G)$, we conclude that $\gamma_{fhR}(G) = \gamma_{fh}(G) + 1$. □

The *join* of graphs G and H is the graph $G + H$ with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \in \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Theorem 2.17. *Let G be a graph and let $\phi = (V_0, V_1, V_2)$ is a γ_{fhR}-function on G. If $G = K_1 + H$ where H is a non-complete graph, then $|V_2| = 1$ and $|V_1| \geq 1$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ is a γ_{fhR}-function on G. Assume that $G = K_1 + H$ where H is a non-complete graph. Then it follows that there exists $u \in V(H)$ such that $d_H(u, v) \geq 2$ for some $v \in V(H)$ where $u \neq v$. Hence, $d_G(u, x) = 2$ for all $x \in V(H)$ for which $d_H(u, x) \geq 2$. In that case, we let $u \in V_2$. Now, if we let $S = \{x \in V(G) : d_G(u, x) = 2\}$, then it implies that $S \subseteq N_G^2[u]$ since $G = K_1 + H$. Hence, it suffices to say that $V_2 = \{u\}$ and so, $|V_2| = 1$. Meanwhile,

$z \in V(K_1) \subseteq V(G)$ implies that $z \in N_G(u)$. Since ϕ is a γ_{fhR}-function on G, it follows that $z \in V_1$. Moreover, if there exists $a \in V(H)$ such that $d_G(a,u) = 1$, then $a \in V_1$. Thus, we therefore have $|V_1| \geq 1$. $\square$

The next corollary is immediate from Theorem 2.17.

Corollary 2.18. *Let n be a positive integer. Then the following hold:*

(i) $\gamma_{fhR}(S_n) = 3$ *if* $n \geq 2$;
(ii) $\gamma_{fhR}(F_n) = 4$ *if* $n \geq 3$; *and*
(iii) $\gamma_{fhR}(W_n) = 5$ *if* $n \geq 4$.

The following theorem is a characterization of a fair hop Roman dominating function under the join of two non-complete graphs.

Theorem 2.19. *Let G and H be non-complete graphs. Then, $\phi = (V_0, V_1, V_2)$ is an FHRDF on $G + H$ if and only if $V_1 \cup V_2 = (V_1^G \cup V_2^G) \cup (V_1^H \cup V_2^H)$ where $\phi|_G = (V_0^G, V_1^G, V_2^G)$ and $\phi|_H = (V_0^H, V_1^H, V_2^H)$ are FHRDF on G and H, respectively, for which $V_j^G = V_j \cap V(G)$ and $V_j^H = V_j \cap V(H)$ for each $j \in \{0,1,2\}$.*

Proof. Suppose $\phi = (V_0, V_1, V_2)$ is an FHRDF on $G + H$. Then for each $j \in \{0,1,2\}$, let $V_j^G = V_j \cap V(G)$ and $V_j^H = V_j \cap V(H)$. This follows that $f|_G = (V_0^G, V_1^G, V_2^G)$ and $f|_H = (V_0^H, V_1^H, V_2^H)$. Let $v \in V_0^G$ be arbitrary. Then we have $v \in V_0$. Since ϕ is an FHRDF on $G + H$, there exists $u \in V_2$ such that $d_{G+H}(u,v) = 2$ and for any $x, y \in V_0$, $|N^2_{G+H}(x) \cap V_2| = |N^2_{G+H}(x) \cap V_2| \geq 1$. Since $d_G(v,a) = 1$ for all $a \in V(H)$, it means that $u \in V_2^G$ and $x, y \in V_0^G$, $|N^2_G(x) \cap V_2^G| = |N^2_G(x) \cap V_2^G| \geq 1$. Hence, $\phi|_G$ is an FHRDF on G. In a similar manner, we also conclude that $\phi|_H$ is an FHRDF on H. Therefore, we obtain $V_1 \cup V_2 = (V_1^G \cup V_2^G) \cup (V_1^H \cup V_2^H)$.

As for the converse, we suppose that $V_1 \cup V_2 = (V_1^G \cup V_2^G) \cup (V_1^H \cup V_2^H)$ where $\phi|_G = (V_0^G, V_1^G, V_2^G)$ and $\phi|_H = (V_0^H, V_1^H, V_2^H)$ are FHRDF on G and H, respectively, for which $V_j^G = V_j \cap V(G)$ and $V_j^H = V_j \cap V(H)$ for each $j \in \{0,1,2\}$. Then $\phi = (V_0, V_1, V_2)$ is a function on $G + H$. Let $v \in V_0$. Then, it implies that either $v \in V_0^G$ or $v \in V_0^H$. Suppose $v \in V_0^G$. Since $\phi|_G$ is an FHRDF on G, there exists $u \in V_2^G$ such that $d_G(u,v) = 2$ and for any $x, y \in V_0^G$, $|N^2_G(x) \cap V_2^G| = |N^2_G(x) \cap V_2^G| \geq 1$. Since $V_j^G \subseteq V_j$ for all $j \in \{0,1,2\}$, it implies that $u \in V_2$ for which $d_{G+H}(u,v) = 2$ and for any $x, y \in V_0$, $|N^2_{G+H}(x) \cap V_2| = |N^2_{G+H}(x) \cap V_2| \geq 1$. On the other hand, suppose $v \in V_0^H$. Then, using a similar argument, we arrive at the same conclusion. Therefore, in any case, $\phi = (V_0, V_1, V_2)$ is an FHRDF on $G + H$. $\square$

The corollary below is an immediate consequence of Theorem 2.19.

Corollary 2.20. *Let G and H be non-complete graphs. Then $\gamma_{fhR}(G + H) = \gamma_{fhR}(G) + \gamma_{fhR}(H)$.*

Proof. Let $\phi' = (V_0^G, V_1^G, V_2^G)$ and $\phi'' = (V_0^H, V_1^H, V_2^H)$ be γ_{fhR}-functions on G and H, respectively. Then let $V_i = V_i^G \cup V_i^H$ for each $i \in \{0,1,2\}$. This implies that $\phi' = \phi|_G$ and $\phi'' = \phi|_H$ where $\phi = (V_0, V_1, V_2)$ is a function on $G + H$. In view of Theorem 2.19, it follows that ϕ is an FHRDF on $G+H$. Thus, we get

$$\begin{aligned}\gamma_{fhR}(G+H) \leq \omega_{G+H}^{fhR}(f) &= |V_1| + 2|V_2| \\ &= |V_1^G \cup V_1^H| + 2|V_2^G \cup V_2^H| \\ &= (|V_1^G| + 2|V_2^G|) + (|V_1^H| + 2|V_2^H|) \\ &= \omega_G^{fhR}(\phi') + \omega_H^{fhR}(\phi'') \\ &= \gamma_{fhR}(G) + \gamma_{fhR}(H).\end{aligned}$$

Meanwhile, we let $\lambda = (U_0, U_1, U_2)$ be a γ_{fhR}-function on $G+H$. Then by Theorem 2.19, it follows that $\lambda|_G = (U_0 \cap V(G), U_1 \cap V(G), U_2 \cap V(G))$ and $\lambda|_H = (U_0 \cap V(H), U_1 \cap V(H), U_2 \cap V(H))$ are FHRDF on G and H, respectively. To this end,

$$\begin{aligned}\gamma_{fhR}(G+H) = \omega_{G+H}^{fhR}(\lambda) &= |U_1| + 2|U_2| \\ &= |(U_1 \cap V(G)) \cup (U_1 \cap V(H))| + 2|(U_2 \cap V(G)) \cup (U_2 \cap V(H))| \\ &= (|U_1 \cap V(G)| + 2|U_2 \cap V(G)|) + (|U_1 \cap V(H)| + 2|U_2 \cap V(H)|) \\ &= \omega_G^{fhR}(\lambda|_G) + \omega_H^{fhR}(\lambda|_H) \\ &\geq \gamma_{fhR}(G) + \gamma_{fhR}(H).\end{aligned}$$

Therefore, the conclusion follows. □

The following remark is immediate from Corollary 2.2 and Corollary 2.20.

Remark 2.21. Let G and H be any complete graphs with $|V(G)| = n$ and $|V(H)| = m$. Then $\gamma_{fhR}(G+H) = n + m$.

3. Conclusion

This paper introduced a new restricted variation of hop Roman domination called the fair hop Roman dominating function in graphs. The study determined exact values of the fair hop Roman domination number of some graphs and also obtained lower and upper bounds for any graph. Moreover, a characterization for a fair hop Roman dominating function in the join of two graphs was constructed. For future studies, the newly defined parameter can be studied for some product operations in graphs.

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References

1. Alqesmah, A., & Gangabylaiah, D. (2025). On the roman domination polynomial of the commuting and non-commuting graphs associated to the dihedral groups. Annals of Mathematics and Computer Science, 27, 44–54. https://doi.org/10.56947/amcs.v27.482
2. Anand, B. S., Dayap, J. A., Casinillo, L. F., Pepper, R., & Nair, R. S. (2025). On distance k-domination number of graphs under product operations. *Gulf Journal of Mathematics*, 19(1), 117–124. https://doi.org/10.56947/gjom.v19i1.2032

3. Casinillo, L. F. (2018). A note on Fibonacci and Lucas number of domination in path, *Electronic Journal of Graph Theory and Applications*, 6(2), 317–325. http://dx.doi.org/10.5614/ejgta.2018.6.2.11
4. Casinillo, L. F. (2020). Odd and even repetition sequences of independent domination number. *Notes on Number Theory and Discrete Mathematics*, 26(1), 8–20. https://doi.org/10.7546/nntdm.2020.26.1.8-20
5. Casinillo, L. F. (2023). A closer look at a path domination number in grid graphs. *Journal of Fundamental Mathematics and Applications*, 6(1), 18–26. https://doi.org/10.14710/jfma.v6i1.16608
6. Casinillo, L., & Canoy Jr, S. (2025). Super Hop Roman Domination in Graphs. *European Journal of Pure and Applied Mathematics*, 18(4), 1–15. https://doi.org/10.29020/nybg.ejpam.v18i4.7078
7. Cockayne, E. J., Dawes, R. M., & Hedetniemi, S. T. (1980). Total domination in graphs. *Networks*, 10(3), 211–219. https://doi.org/10.1002/net.3230100304
8. Caro, Y., Hansberg, A., & Henning, M. (2012). Fair domination in graphs. *Discrete Mathematics*, 312(19), 2905–2914. https://doi.org/10.1016/j.disc.2012.05.006
9. Gross, J. L., Yellen, J., & Anderson, M. (2018). *Graph theory and its applications*. Chapman and Hall/CRC. https://doi.org/10.1201/9780429425134
10. Haynes, T. W., Hedetniemi, S., & Slater, P. (2013). *Fundamentals of domination in graphs*, CRC press. https://doi.org/10.1201/9781482246582
11. Jafari Rad, N., & Poureidi, A. (2019). On hop Roman domination in trees. *Communications in Combinatorics and Optimization*, 4(2), 201–208. https://doi.org/10.22049/cco.2019.26469.1116
12. Labendia, M. A., & Pornia, C. R. (2023). Integer Programming Formulations and Probabilistic Bounds for Some Domination Parameters. Annals of Mathematics and Computer Science, 30, 1–15. https://doi.org/10.56947/amcs.v30.618
13. Shabani, E. (2017). Hop Roman domination in graphs, *Manuscript*.
14. Shabani, E., Nader, J. R., & Poureidi, A. (2019). Graphs with large hop Roman domination number. *Computer Science Journal of Moldova*, 79(1), 3–22. https://ibn.idsi.md/vizualizare_articol/78366

[1] Department of Mathematics, Visayas State University, Philippines
Email address: leomarichcasinillo02011990@gmail.com