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AN EFFICIENT EXTRAGRADIENT METHOD FOR SOLVING VARIATIONAL INEQUALITY PROBLEMS WITH PSEUDO-MONOTONE OPERATORS

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Abstract. Variational Inequality Problems (VIPs) provide a strong framework for exhibiting equilibrium problems in a variety of disciplines. Global Lipschitz continuity and strong monotonicity are two restrictive assumptions that are frequently used in traditional extragradient methods for solving VIPs, which limit their applicability to solve pseudo-monotone operators. This paper introduces a novel extragradient-type technique that eliminates the need for a global Lipschitz constant. A relaxation parameter that stabilizes the iterative process by taking a convex combination of the current point and a standard projection step is one of the two main innovations included in the new technique. The second innovation is an adaptive line search strategy that dynamically modifies the step size in response to local operator behaviour. We present a thorough convergence analysis that demonstrates the resulting sequence's weak convergence to a VIP solution. The suggested algorithm is more effective and reliable than previous approaches, especially for large-scale issues with sensitive initial circumstances, as shown by numerical experiments on well-known benchmark problems such as Sun's and Kojima-Shindo problems.

Keywords. Extragradient method, variational inequality problem, pseudo-monotone operator, Lipschitz map
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1. Introduction

The VIPs offer a common approach for examining equilibrium models that come up in network analysis, operations research, engineering, economics, and transportation [1, 2, 5, 8, 10, 11, 13, 19]. The VIP is to find a vector x^* in a closed convex set $C \subset \mathbb{R}^m$ for which:

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C, \tag{1.1}$$

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where $F : C \to \mathbb{R}^m$ is a nonlinear operator.
One of the most fundamental techniques for finding the solution x^* of VIPs is the Projected Gradient Method (PGM): For $x_0 \in C$,

$$x_{n+1} = P_C(x_n - \alpha F(x_n)), \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where $\alpha > 0$ is a step-size parameter. It is a known fact that the PGM converges under the strong monotonicity assumptions. Otherwise, the method (1.2) fails if the operator F is only monotone. In [12], an Extragradient Method (EGM) was introduced to stabilize the iteration (1.2) as follows: For $x_0 \in C$,

$$\begin{cases} y_n = P_C\big(x_n - \gamma F(x_n)\big), \\ x_{n+1} = P_C\big(x_n - \gamma F(y_n)\big), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{1.3}$$

where $\gamma > 0$ is a step-size parameter. A significant drawback of the EGM is its requirement for a known Lipschitz constant L to set the step size $\gamma \in (0, 1/L)$. This means that the knowledge of a Lipschitz constant is required so as to make the method feasible. Recent studies focus on algorithmic development [6, 13, 17, 18] by emphasizing properties like pseudo-monotonicity that generalizes monotonicity and applies to real-world problems. The major goal is to loosen the presumptions and develop techniques for the larger class of pseudo-monotone operators [9, 7]. Adaptive step sizes and line search techniques are another way to do away with the need for the global Lipschitz constant [14, 17].
In this paper, a new method that improves the numerical stability of the extragradient framework is proposed. This study extends the adaptive line-search of [16] by introducing a relaxation parameter that stabilizes the trajectory and blending the current iterate with the standard projection, reducing oscillatory behaviour near constraint boundaries. Analytically, the relaxation parameter introduces a new degree of freedom that allows larger effective step sizes while maintaining convergence, improving numerical robustness especially for ill-conditioned or non-Lipschitz problems. The convergence proof requires careful handling of the interplay between the relaxation parameter and the adaptive step-size rule which is a non-trivial extension. In summary, the primary contributions of this work are:

- A relaxation parameter that forms a convex combination between the current iterate and the standard projection, thereby improving stability.
- An adaptive line search strategy, building on [16], that dynamically selects step sizes without prior knowledge of the Lipschitz constant.
- A weak convergence proof of the sequence generated to a VIP solution under pseudo-monotonicity and weak continuity assumptions.
- Numerical superiority that demonstrates the efficiency and robustness of the present method over the existing methods.

The rest of this manuscript is organized as follows: The next section provides some important lemmas and assumptions. Section 3 introduces the new efficient extragradient method, proves the convergence of the sequence generated by this algorithm, and presents numerical experiments demonstrating its efficiency. The last section concludes the paper.

2. Preliminaries and Assumptions

In this section, we present some useful assumptions and lemmas.
Let $\mathbb{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $C \subset \mathbb{H}$ be a nonempty, closed, and convex set. The metric projection $P_C(\cdot)$ is defined as $P_C(x) = \arg\min_{y \in C} \|x - y\|$.
The weak convergence of $\{x_n\}$ to x^* is denoted by $x_n \rightharpoonup x^*$ as $n \to \infty$.
The following assumptions were considered:

- (A_1) Existence of solution: The solution set $S = \{x^* \in C : \langle F(x^*), x - x^* \rangle \geq 0 \ \forall x \in C\}$ is nonempty.
- (A_2) Pseudo-monotonicity: The operator F is pseudo-monotone, i.e., for all $u, v \in \mathbb{H}$,

$$\langle F(u), v - u \rangle \geq 0 \quad \Longrightarrow \quad \langle F(v), v - u \rangle \geq 0.$$

- (A_3) Weak Continuity: F is sequentially weakly continuous on $\mathbb{H}$.
- (A_4) Local Lipschitz: F is locally Lipschitz on $\mathcal{B}$.
- (A_5) Projection: The metric projection $P_C(\cdot)$ is computationally feasible.

The following well-known lemmas are useful in the proof of convergence analysis:

Lemma 2.1. [3] *Let $C \subset \mathbb{H}$ be a nonempty, closed, convex subset. Then, for any $x \in \mathbb{H}$ and $y \in C$:*

$$I. \ \|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|x - P_C(x)\|^2$$
$$II. \ \langle x - P_C(x), y - P_C(x) \rangle \leq 0.$$

Lemma 2.2. [15] *Let $\{x_n\} \subset \mathbb{H}$ satisfies*

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0,$$

and suppose that every weak cluster point of $\{x_n\}$ belongs to a nonempty set $S \subset \mathbb{H}$. Then:

$$x_n \rightharpoonup x^* \in S$$

Lemma 2.3. [16] *Under pseudo-monotonicity, $x^* \in C \iff \langle F(y), y - x^* \rangle \geq 0$ for all $y \in C$.*

3. Main Results

This section presents the main results of the paper.

3.1. **The Proposed Algorithm.**

Here, we propose an efficient extragradient algorithm for solving VIPs with pseudo-monotone operators (see Algorithm 1). The constant $\alpha_{\max} = \max\{\alpha_n\}$ is given in the algorithm so that the sequence $\{\alpha_n\}$ has a bound.

3.2. **Convergence Analysis.**

Lemma 3.1 (Line Search Termination)**.** *The line search in Algorithm 1 terminates after finitely many inner iterations.*

Algorithm 1 The new extragradient algorithm

Initialization: Choose $\sigma \in (0,1)$, $\lambda \in (0,1)$, $\theta \in (0,1)$, $\alpha_0 > 0$, $\alpha_{\max} > 0$, and $x_0 \in C$.
Iteration steps Given the current iterates x_n and α_n, we then calculate the next x_{n+1} and α_{n+1} as follows:
Step 1. Take $j = 0$ and run the following line search:
Take $\beta_n = \theta^j$ and compute:
a. $y_n = (1-\lambda)x_n + \lambda P_C(x_n - \beta_n \alpha_n F(x_n)), \quad \lambda \in (0,1).$
b. Choose largest $\alpha_{n+1} \leq \min\left\{\frac{(1+\beta_n)}{\beta_n}\alpha_n, \alpha_{\max}\right\}$ such that:

$$\|\alpha_{n+1}F(y_n) - \beta_n\alpha_n F(x_n)\| \leq \sigma\|y_n - x_n\|$$

c. Break line search if such α_{n+1} exists. Otherwise, set $j = j+1$ and return to a. above.
If $x_n = y_n$, then stop. Otherwise, proceed to next step.
Step 2. Compute update $x_{n+1} = P_C(x_n - \alpha_{n+1}F(y_n))$.
Set $n = n+1$ and go back to Step 1.

Proof. Suppose for contradiction it runs indefinitely for some n. Define:

$$y_n(j) = (1-\lambda)x_n + \lambda(x_n - \beta_n\alpha_n F(x_n)) \text{ and } \beta_n = \theta^j.$$

As $j \to \infty$, $\beta_n \to 0$ and $y_n(j) \to x_n$. By projection non-expansiveness:

$$\|P_c(y_n(j)) - x_n\| \leq \beta_n\alpha_n\|F(x_n)\| \to 0$$

Thus, $y_n \to x_n$. Since $\{x_n\}$ is bounded, $\exists\ \sigma > 0$ and $L > 0$ such that F is L-Lipschitz on $\mathcal{B}(x_n, \sigma)$. For large j:

$$\beta_n < \frac{1}{L\alpha_n} \implies \|\alpha F(y_n) - \beta_n\alpha_n F(x_n)\| \leq \beta_n\alpha_n L\|y_n - x_n\| < \sigma\|y_n - x_n\|$$

where $\alpha = \beta_n\alpha_n$, contradicting the failure condition. □

Lemma 3.2 (Step Size Properties)**.** *If $\{x_n\}$ is bounded, then:*

(1) $\{\alpha_n\}$ *is bounded with* $0 < \alpha_n \leq \alpha_{\max}$
(2) $\limsup_{n\to\infty} \alpha_n > 0$

Proof. The proof of (1) is obvious. For (2), suppose $\liminf \alpha_n = 0$. Then $\exists\ \{\alpha_{n_k}\} \to 0$. By boundedness of $\{x_n\}$, F is L-Lipschitz on a ball containing $\{x_n\}$ with constant L. For large k:

$$\alpha_{n_k} < \frac{\sigma}{L}, \quad \beta_n = 1 \ (j = 0)$$

Then:

$$\|y_n - x_n\| \leq \lambda\alpha_n\|F(x_n)\| \leq \lambda\alpha_n M$$

where $M = \sup_{x\in C}\|F(x)\|$. Now:

$$\|\alpha_n F(y_n) - \alpha_n F(x_n)\| \leq \alpha_n L\|y_n - x_n\| \leq \lambda L M\alpha_n^2$$

Since $\alpha_n < \sigma/L$:

$$\lambda L M\alpha_n^2 \leq \sigma\lambda\alpha_n M = \sigma\|y_n - x_n\|$$

so $\alpha_{n+1} \geq \alpha_n$. Thus $\{\alpha_{n_k}\}$ is nondecreasing and cannot converge to 0. □

Lemma 3.3 (Weak Cluster Points are Solutions)**.** *If $\{x_n\}$ is bounded and $\|x_n - y_n\| \to 0$, then every weak cluster point of $\{x_n\}$ is in C.*

Proof. Let $x_{n_k} \rightharpoonup x^*$. Then $y_{n_k} \rightharpoonup x^*$ and $x^* \in C$ (closed convex). From line search:

$$\|\alpha_{n_k+1} F(y_{n_k}) - \beta_{n_k}\alpha_{n_k} F(x_{n_k})\| \leq \sigma \|y_{n_k} - x_{n_k}\| \to 0$$

By Lemma 3.2, $\exists\ \alpha > 0$ and $\beta \geq 0$ such that (sub-sequentially) $\alpha_{n_k} \to \alpha$, $\beta_{n_k} \to \beta$. By weak continuity:

$$F(x_{n_k}) \rightharpoonup F(x^*), \quad F(y_{n_k}) \rightharpoonup F(x^*)$$

Using the projection property (Lemma 2.1(II)):

$$\langle x_{n_k} - \beta_{n_k}\alpha_{n_k} F(x_{n_k}) - w_{n_k}, z - w_{n_k}\rangle \leq 0, \quad \forall z \in C$$

where $w_{n_k} = P_c(x_{n_k} - \beta_{n_k}\alpha_{n_k} F(x_{n_k}))$. Substitute:

$$w_{n_k} = x_{n_k} + \frac{y_{n_k} - x_{n_k}}{\lambda}$$

As $k \to \infty$, $\|x_{n_k} - w_{n_k}\| = \frac{1}{\lambda}\|x_{n_k} - y_{n_k}\| \to 0$. Thus:

$$\liminf_{k\to\infty} \langle F(x_{n_k}), z - x_{n_k}\rangle \geq 0, \quad \forall z \in C$$

By pseudo-monotonicity and Lemma 2.3, $x^* \in S$. □

Lemma 3.4 (Descent Inequality)**.** *For any $u \in C$:*

$$\|x_{n+1} - u\|^2 \leq \|x_n - u\|^2 - (1 - \sigma^2)\|x_n - y_n\|^2$$

Proof. From Lemma 2.1(I):

$$\begin{aligned}
\|x_{n+1} - u\|^2 &\leq \|x_n - \alpha_{n+1}F(y_n) - u\|^2 - \|x_n - \alpha_{n+1}F(y_n) - x_{n+1}\|^2 \\
&= \|x_n - u\|^2 - 2\alpha_{n+1}\langle F(y_n), (x_n - u)\rangle + \alpha_{n+1}^2\|F(y_n)\|^2 \\
&\quad - \|x_n - x_{n+1}\|^2 + 2\alpha_{n+1}\langle F(y_n), (x_n - x_{n+1})\rangle - \alpha_{n+1}^2\|F(y_n)\|^2 \\
&= \|x_n - u\|^2 - \|x_n - x_{n+1}\|^2 + 2\alpha_{n+1}\langle F(y_n), (x_{n+1} - u)\rangle.
\end{aligned}$$

By pseudo-monotonicity and Lemma 2.3:

$$\langle F(y_n), y_n - u\rangle \geq 0 \implies \langle F(y_n), x_{n+1} - u\rangle \leq \langle F(y_n), x_{n+1} - y_n\rangle$$

The line search condition gives:

$$2\alpha_{n+1}\langle F(y_n), x_{n+1} - y_n\rangle \leq \sigma^2\|x_n - y_n\|^2$$

Combining these yields the result. □

3.3. **Convergence Theorem.**

Theorem 3.5 (Weak Convergence)**.** *The sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to some $x^* \in C$.*

Proof. Step 1: Boundedness and cost descent. Fix $u \in C$. By Lemma 3.4:

$$\|x_{n+1} - u\|^2 \leq \|x_n - u\|^2$$

So $\{\|x_n - u\|\}$ converges, and $\{x_n\}$ is bounded. From Lemma 3.4:

$$(1 - \sigma^2)\|x_n - y_n\|^2 \to 0 \implies \|x_n - y_n\| \to 0$$

Step 2: Consecutive iterate convergence.

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_{n+1} - y_n\| + \|y_n - x_n\| \\ &\leq \|x_n - \alpha_{n+1}F(y_n) - y_n\| + \|y_n - x_n\| \\ &\leq 2\|y_n - x_n\| + \alpha_{n+1}\|F(y_n)\| \end{aligned}$$

Since $\{y_n\}$ (as $\{x_n\}$ is bounded) and $\{\alpha_n\}$ are bounded, then $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Step 3: Weak convergence to solution. By Lemma 3.3, every weak cluster point of $\{x_n\}$ is in C. Suppose two subsequences:

$$x_{n_k} \rightharpoonup x^*, \quad x_{n_j} \rightharpoonup \tilde{x}, \quad x^*, \tilde{x} \in C$$

The identity:

$$\langle x_n, \tilde{x} - x^* \rangle = \frac{1}{2}\|x_n - x^*\|^2 - \frac{1}{2}\|x_n - \tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}\|^2 - \frac{1}{2}\|x^*\|^2$$

Taking limits along $\{n_k\}$ and $\{n_j\}$:

$$\langle x^*, \tilde{x} - x^* \rangle = \langle \tilde{x}, \tilde{x} - x^* \rangle$$

Simplifying gives $\|x^* - \tilde{x}\|^2 = 0$, so $x^* = \tilde{x}$. Thus $\{x_n\}$ converges weakly to $x^* \in C$. □

3.4. **Numerical Experiments.** All algorithms were implemented in Python 3.10 on a standard Laptop Intel Corei5-4200M, 4GB RAM. Performances were measured by iteration count (Iter.) and CPU time in seconds (Time) using the stopping criterion $D_n = \|x_{n+1} - y_n\| + \|y_n - x_n\| \leq \epsilon = 10^{-6}$. We compared:

- Algorithm 1: The proposed method ($\sigma = 0.7, \theta = 0.9, \lambda = 0.5$).
- Algorithm 2: [16]'s method (same σ, θ, no relaxation parameter).
- Algorithm 3: [12]'s classical EGM ($\gamma = 0.1$).

The initial step size α_0 for Algorithm 1 and Algorithm 2 was set adaptively using a perturbation of the initial point, that is,

$$\alpha_0 = \frac{\sigma\|\tilde{x} - x_0\|}{\|F(\tilde{x}) - F(x_0)\|}.$$

3.5. **Benchmark Problems.** To fully evaluate the performance and robustness of the proposed method, two credible academic test problems were considered from the literature. These problems were chosen since they test several aspects of the algorithms, including their scalability, initial condition sensitivity, and capacity to handle pseudo-monotonicity.

Example 3.6. (Sun's Problem) To evaluate the efficiency and scalability of algorithms in high-dimensional areas, this extensive challenge was created [16]. Although, it permits controlled tests, its structure is sufficiently detailed to be challenging.

- Feasible Set: The standard simplex in $\mathbb{R}^m$ scaled by m:
$$C = \left\{ x \in \mathbb{R}^m_+ : \sum_{i=1}^{m} x_i = m \right\}. \tag{3.1}$$
- Operator: The operator $F : \mathbb{R}^m \to \mathbb{R}^m$ is defined as the sum of a nonlinear component and an affine component:
$$F(x) = F_1(x) + F_2(x), \tag{3.2}$$
where the nonlinear part $F_1(x)$ is given by:
$$F_1(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix}, \quad \text{with} \quad f_i(x) = x_{i-1}^2 + x_i^2 + x_{i-1}x_i + x_i x_{i+1}, \quad \text{for } i = 1, 2, \ldots, m.$$
The boundary conditions are set as $x_0 = x_{m+1} = 0$.
The affine part $F_2(x)$ is defined as:
$$F_2(x) = Dx + c,$$
where $D = (d_{ij})$ is an $m \times m$ band matrix with entries:
$$d_{ij} = \begin{cases} 4, & \text{if } i = j, \\ 1, & \text{if } i - j = 1, \\ -2, & \text{if } i - j = -1, \\ 0, & \text{otherwise,} \end{cases}$$
and the vector $c \in \mathbb{R}^m$ is given by $c = (-1, -1, \ldots, -1, 1)^T$.
- Tests: The problem is solved for increasing dimensions $m = 10^3$, $m = 10^4$, and $m = 10^5$ to thoroughly analyse the algorithmic scalability and computational efficiency.

Example 3.7. (Kojima-Shindo Problem) This classical nonlinear complementarity problem (NCP) is a standard benchmark for testing algorithms on non-monotone or complex operator behaviour [16]. It is known for having multiple solutions, making the choice of initial point crucial.

- Feasible Set: The problem is defined on the simplex in $\mathbb{R}^4$:
$$C = \left\{ x \in \mathbb{R}^4_+ : x_1 + x_2 + x_3 + x_4 = 4 \right\}. \tag{3.3}$$

- Operator: The operator $F : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by its component functions:

$$F(x) = \begin{bmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{bmatrix}. \tag{3.4}$$

- Initial Points: To rigorously test robustness and sensitivity to initial conditions, the following two distinct points are used: $x_0^{(1)} = (1,1,1,1)^T$ and $x_0^{(2)} = (4,0,0,0)^T$.

The numerical results are presented in tables and are further substantiated using some figures.

Table 1. The numerical results for Example 3.6.

	$m = 10^3$		$m = 10^4$		$m = 10^5$	
Algorithms	Iter.	Time	Iter.	Time	Iter.	Time
Algorithm 1	55	0.03	60	0.12	63	1.34
Algorithm 2	92	0.05	92	0.16	98	1.66
Algorithm 3	inf	NA	inf	NA	inf	NA

In Table 1, Algorithm 1 outperforms Algorithm 2 with fewer iterations and less time. While Algorithm 3 diverges in all the cases. Figures 1, 2, and 3 show the

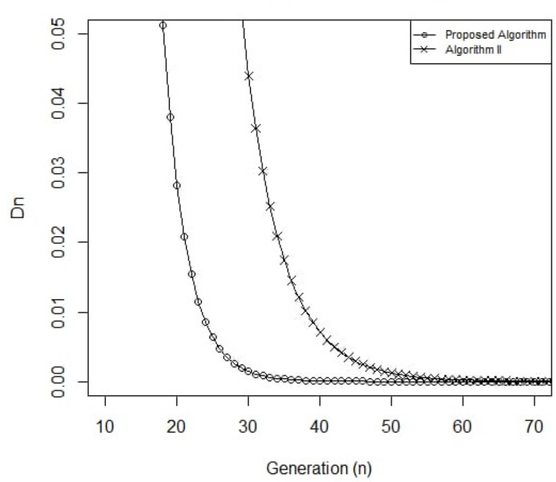

Figure 1. Convergence Plot of Example 3.6 with m=1000.

convergence plots with faster descent for Algorithm 1.

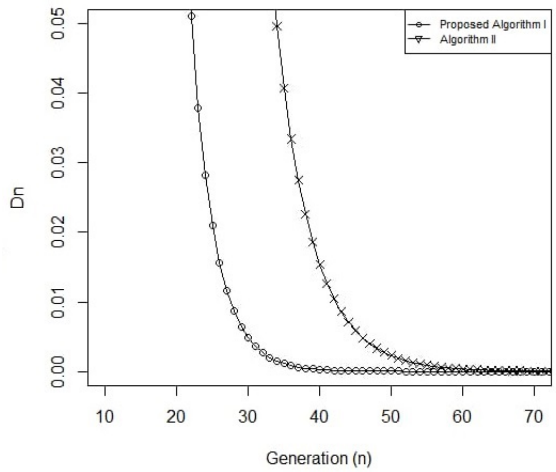

FIGURE 2. Convergence Plot of Example 3.6 with m=10000.

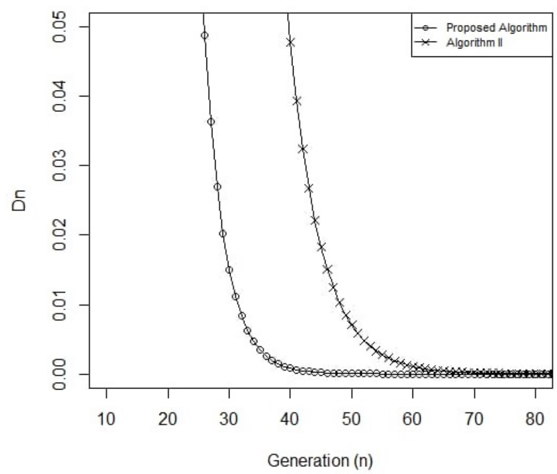

FIGURE 3. Convergence Plot of Example 3.6 with m=100000.

TABLE 2. The numerical results for Example 3.7.

	$x_0 = (1,1,1,1)$		$x_0 = (4,0,0,0)$	
Algorithms	Iter.	Time	Iter.	Time
Algorithm 1	64	0.1876	64	0.1509
Algorithm 2	76	0.2654	155	0.2903
Algorithm 3	inf	NA	inf	NA

Also in Table 2, Algorithm 1 shows a high level of robustness for the two initial guesses $(1,1,1,1)$ and $(4,0,0,0)$ with 64 iterations each; while Algorithm 2 maintains level of sensitivity with 76 and 155 iterations, respectively; and Algorithm

3 diverges. The convergence plots for the two distinct points in Example 3.7 are

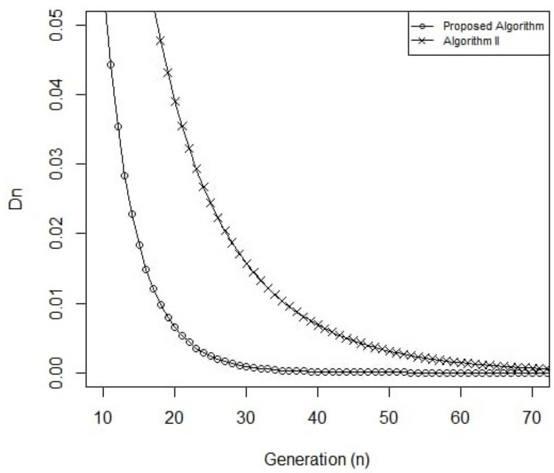

Figure 4. Convergence Plot of Example 3.7 at x_0 =(1,1,1,1).

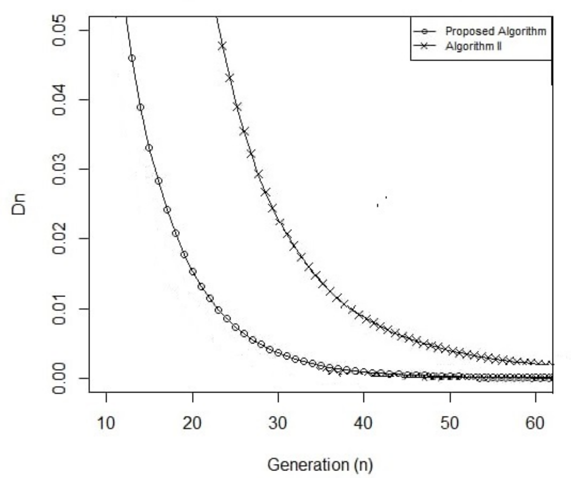

Figure 5. Convergence Plot of Example 3.7 at x_0 =(4,0,0,0).

shown in Figures 4 and 5.

4. Conclusion and Remarks

This study developed and analysed a new relaxation-based extragradient method for solving variational inequality problems in Hilbert spaces. The algorithm incorporates a refined projection strategy and adaptive step-size rule to enhance convergence behaviour without imposing restrictive assumptions such as global Lipschitz continuity. Numerical experiments conducted on two benchmark problems confirmed the efficiency and robustness of the proposed method. Compared with existing extragradient-type schemes, the new approach achieved faster convergence and reduced computational effort, demonstrating its potential as a reliable tool for solving pseudo-monotone and non-Lipschitz variational inequalities.

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