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SEVERAL FIXED POINT RESULTS IN HEXAGONAL METRIC AND HEXAGONAL b-METRIC SPACES

JAYESH TIWARI[1] AND DILEEP KUMAR SHARMA[2*]

Abstract. The aim of this paper is to introduce hexagonal metric space and hexagonal b-metric space. Firstly, we prove existence and uniqueness of several fixed point results for innovative contractions in both metric spaces. finally, we provide some examples to support and validate these new results.

1. Introduction and preliminaries

Banach contraction principle, is given in 1920 by Banach, is the main result of fixed point theory. After that result, many researchers have come across the number of generalizations of metric spaces and Banach contraction principle. In this sequal Bakhtin[3] and Czerwik[6] introduced b-metric spaces as a generalization of metric spaces. They proved contraction principle in b-metric spaces. Since then many scholars have proposed a series of new concepts of contraction mapping and new fixed point theorems for single valued and multivalued functions in b-metric spaces ([4],[5],[7]).

In 2000, Branciari[2] presented the concept of rectangular metric space where the triangular inequality of a metric space was replaced by another inequality the so called rectangular inequality. In this paper the author also extended the celebrated Banach contraction mapping principle in the context of rectangular metric spaces *(RMS)*. Since then many fixed point theorems for various contractions on rectangular metric space appeared ([7],[8]).

In 2015, George et al.[9] introduced the notion of rectangular b-metric space *(RbMS)*. rectangular b-metric space is an extended form of metric space, rectangular metric space and b-metric space. They presented fixed point results for contraction mappings in rectangular b-metric space. After that a lot of fixed point results are made in rectangular b-metric spaces ([9],[10],[11],[12]).

Abba Auwalu and Evren Hincal[1] proved common fixed point for two functions in cone pentagonal metric space.

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* Corresponding author.

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In this paper we provide two new notions, named hexagonal metric space and hexagonal b-metric space as a most extended form of metric spaces. We give some proper and interesting examples to validate our fixed point results in both spaces. Our results generalize many known results in fixed point theory.

2. Main results

Definition 2.1 Let X be a non empty set. A function $d : X \times X \to \mathbb{R}^+$ is called a hexagonal metric on X if it satisfies the following conditions:

(Pm_1) $d(x, y) = 0$ iff $x = y$ for all $x, y \in X$.
(Pm_2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(Pm_3) $d(x, z) \leq d(x, t)+d(t, u)+d(u, v)+d(v, w)+d(w, z)$ for all distinct points $x, z, t, u, v, w \in X$. (hexagonal inequality)
Then the pair (X, d) is called a hexagonal metric space (in short HMS).

Definition 2.2 Let X be a non empty set. A function $d : X \times X \to \mathbb{R}^+$is called a hexagonal b-metric on X if it satisfies the following conditions:

(Pbm_1) $d(x, y) = 0$ iff $x = y$ for all $x, y \in X$.
(Pbm_2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(Pbm_3) $d(x, z) \leq s\Big[d(x, t) + d(t, u) + d(u, v) + d(v, w) + d(w, z)\Big]$ {(hexagonal b-inequality) for some fixed real number $s \geq 1$ *and* for all distinct points $x, z, t, u, v, w \in X$.

Then the pair (X, d) is called a hexagonal b-metric space (in shortHbMS).

Example 2.3 Let $X = \mathbb{N}$, define $d : X \times X \to \mathbb{R}^+$ by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 6\beta, & \text{if } (x, y) \in \{1, 2\} \text{ and } x \neq y; \\ \beta, & \text{if } x \text{ or } y \text{ or both } \notin \{1, 2\} \text{ and } x \neq y. \end{cases}$$

where $\beta > 0$ is a real fixed numbers. Then (X, d) is a hexagonal b-metric space with coefficient $s = \frac{6}{5} > 1$, but (X, d) is not a hexagonal metric space, pentagonal metric space, rectangular metric space and not a metric space, as

$$\begin{aligned} d(1, 2) = 6\beta > 5\beta &= d(1, 3) + d(3, 4) + d(4, 5) + d(5, 6) + d(6, 2) \\ &= \beta + \beta + \beta + \beta + \beta. \end{aligned}$$

$$\begin{aligned} d(1, 2) = 6\beta > 4\beta &= d(1, 3) + d(3, 4) + d(4, 5) + d(5, 2) \\ &= \beta + \beta + \beta + \beta. \end{aligned}$$

similarly

$$\begin{aligned} d(1, 2) = 6\beta > 3\beta &= d(1, 3) + d(3, 4) + d(4, 2) \\ &= \beta + \beta + \beta. \end{aligned}$$

and

$$\begin{aligned} d(1,2) = 6\beta > 2\beta &= d(1,3) + d(3,2) \\ &= \beta + \beta. \end{aligned}$$

Example 2.4 Let $X = \mathbb{N}$, define $d : X \times X \to \mathbb{R}^+$ such that $d(x,y) = d(y,x)$ for all $x, y \in X$ and

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 17\beta, & \text{if } x = 1, y = 2; \\ \beta, & \text{if } x \in \{1,2\} \text{ and } y = 3; \\ 2\beta, & \text{if } x \in \{1,2,3\} \text{ and } y = 4; \\ 3\beta, & \text{if } x \in \{1,2,3,4\} \text{ and } y = 5; \\ 4\beta, & \text{if } x \in \{1,2,3,4,5\} \text{ and } y = 6; \\ 5\beta, & \text{if } x \text{ or } y \notin \{1,2,3,4,5,6\} \text{ and } x \neq y. \end{cases}$$

where $\beta > 0$ is a fixed real number. Then (X,d) is a hexagonal b-metric space with coefficient $s = \frac{17}{14} > 1$, but (X,d) is not a hexagonal and pentagonal metric space and hence neither a rectangular metric space nor a metric space as

$$\begin{aligned} d(1,2) = 17\beta > 14\beta &= d(1,3) + d(3,4) + d(4,5) + d(5,6) + d(6,2) \\ &= \beta + 2\beta + 3\beta + 4\beta + 4\beta. \end{aligned}$$

$$\begin{aligned} d(1,2) = 17\beta > 11\beta &= d(1,3) + d(3,4) + d(4,6) + d(6,2) \\ &= \beta + 2\beta + 4\beta + 4\beta. \end{aligned}$$

Similarly

$$\begin{aligned} d(1,2) = 17\beta > 9\beta &= d(1,3) + d(3,6) + d(6,2) \\ &= \beta + 4\beta + 4\beta. \end{aligned}$$

and

$$\begin{aligned} d(1,2) = 17\beta > 2\beta &= d(1,3) + d(3,2)) \\ &= \beta + \beta. \end{aligned}$$

Definition 2.5 Let (X,d) be a hexagonal b-metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be any sequence in X and $x \in X$. Then,

(a) The sequence $\{x_n\}$ is said to be hexagonal b-convergent in (X,d) and converges to x, if for every $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq n_0$ i.e. $\lim_{n\to\infty} x_n = x$.

(b) The sequence $\{x_n\}$ is said to be hexagonal b-Cauchy sequence in (X,d) if for every $\epsilon > 0$ there exist $m, n \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$ i.e. $\lim_{n,m\to\infty} d(x_n, x_m) = d(x,x)$ i.e. limit exist and finite.

(c) Pair (X,d) is said to be a complete hexagonal b-metric space if every hexagonal b-Cauchy sequence in X converges to some $x \in X$.

(d) Let (X,d) be a non empty hexagonal metric space. Let ρ be the set of increasing continuous functions $\rho : X \to X$ such that

i. $0 < \rho(t) < t$ for all $t \in X \setminus \{0\}$ *and* $\rho(0) = 0$

ii. The series $\sum_{n\geq 0} \rho^n(t)$ converges for all $t \in X \setminus \{0\}$. Therefore for all $t \in X \setminus \{0\}$ we have

$$\lim_{n\to 0} \rho^n(t) = 0.$$

We now state our main results.

Firstly we prove existence and uniqueness of a fixed point in hexagonal b-metric spaces.

Theorem 2.6 Let (X, d) be a complete hexagonal b-metric space. Let T be a map define onto X itself and satisfying follow inequality for any $x, y \in X$

$$d(Tx, Ty) \leq a\{d(x, Tx) + d(y, Ty)\} \tag{2.1}$$

where $a \in [0, \frac{1}{s+1}]$ *such that* $sa \leq 1$ *for* $s > 1$. *Then* T *has a unique fixed point.*

Proof: Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. We prove that $\{x_n\}$ is a hexagonal b-Cauchy sequence.

case(i): If $x_n = x_{n+1}$ then $x_{n+1} = Tx_n = x_n$. Hence x_n is a fixed point of T.

case(ii): If $x_n \neq x_{n+1}$ for all $n \geq 0$. Setting $d\big(x_n, x_{n+1}\big) = d_n$. Then from inequality (2.1), we get

$$\begin{aligned}
d\big(x_n, x_{n+1}\big) &= d\big(Tx_{n-1}, Tx_n\big) \\
d_n &\leq a\Big\{d\big(x_{n-1}, Tx_{n-1}\big) + d\big(x_n, Tx_n\big)\big)\Big\} \\
&= a\Big\{d\big(x_{n-1}, x_n\big) + d\big(x_n, x_{n+1}\big)\Big\} \\
&= a\{d_{n-1} + d_n\}. \\
(1-a)d_n &\leq ad_{n-1} \\
d_n &\leq \frac{a}{1-a}d_{n-1} = \delta d_{n-1}\big(\delta = \frac{a}{1-a} < 1\big),
\end{aligned}$$

Continuing above process, we get

$$d_n \leq \delta^n d_0. \tag{2.2}$$

Let x_0 is not a periodic point of T. If $x_0 = x_n$ then using (2.2), for any $n \geq 2$, we have

$$\begin{aligned}
d(x_0, Tx_0) &= d(x_n, Tx_n) \\
d(x_0, x_1) &= d(x_n, x_{n+1}) \\
d_0 &= d_n \\
d_0 &\leq \delta^n d_0.
\end{aligned}$$

which is a contradiction. Thus $d_0 = 0$ implies that $x_0 = x_1$ and so x_0 is a fixed point of T.
Now we assume $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$
Now setting $d(x_n, x_{n+2}) = d_n^*$. From inequality (2.1), we obtain

$$\begin{aligned}
d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \\
d_n^* &\leq a[d(x_{n-1}, Tx_{n-1}) + d(x_{n+1}, Tx_{n+1})] \\
&= a[d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})] \\
d_n^* &\leq a(\delta^{n-1} d_0 + \delta^{n+1} d_0) \\
&= a(\delta^{n-1} + \delta^{n+1}) d_0 \\
&= a\delta^{n-1}(1 + \delta^2) d_0. \qquad (2.3)
\end{aligned}$$

Now setting $d(x_n, x_{n+3}) = d_n^{**}$. Then from inequality (2.1), we obtain

$$\begin{aligned}
d(x_n, x_{n+3}) &= d(Tx_{n-1}, Tx_{n+2}) \\
d^{**} &\leq a[d(x_{n-1}, Tx_{n-1}) + d(x_{n+2}, Tx_{n+2})] \\
&= a[d(x_{n-1}, x_n) + d(x_{n+2}, x_{n+3})] \\
d^{**} &\leq a(\delta^{n-1} d_0 + \delta^{n+2} d_0) \\
&= a(\delta^{n-1} + \delta^{n+2}) d_0 \\
&= a\delta^{n-1}(1 + \delta^3) d_0. \qquad (2.4)
\end{aligned}$$

Now setting $d(x_n, x_{n+4}) = d_n^{***}$. Then from inequality (2.1), we obtain

$$\begin{aligned}
d(x_n, x_{n+4}) &= d(Tx_{n-1}, Tx_{n+3}) \\
d^{***} &\leq a[d(x_{n-1}, Tx_{n-1}) + d(x_{n+3}, Tx_{n+3})] \\
&= a[d(x_{n-1}, x_n) + d(x_{n+3}, x_{n+4})] \\
d^{***} &\leq a(\delta^{n-1} d_0 + \delta^{n+3} d_0) \\
&= a(\delta^{n-1} + \delta^{n+3}) d_0 \\
&= a\delta^{n-1}(1 + \delta^4) d_0. \qquad (2.5)
\end{aligned}$$

For the sequence $\{x_n\}$ we consider $d(x_n, x_{n+p})$ in two cases.
If p is odd say $2m+1$, then using hexagonal b-inequality, we obtain

$$\begin{aligned}
d(x_n, x_{n+2m+1}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})] \\
&\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}]) + s^2[d(x_{n+4}, x_{n+5}) \\
&+ d(x_{n+5}, x_{n+6}) + d(x_{n+6}, x_{n+7}) + d(x_{n+7}, x_{n+8}) + d(x_{n+8}, x_{n+2m+1})] \\
&\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] + s^2[d(x_{n+4}, x_{n+5}) \\
&+ d(x_{n+5}, x_{n+6}) + d(x_{n+6}, x_{n+7}) + d(x_{n+7}, x_{n+8})] + s^3 d(x_{n+8}, x_{n+9}) + ... \\
&+ s^{m/3}[d(x_{n+2m}, x_{n+2m+1}) \\
&\leq s[\delta^n d_0 + \delta^{n+1} d_0 + \delta^{n+2} d_0 + \delta^{n+3} d_0] + s^2[\delta^{n+4} d_0 + \delta^{n+5} d_0 + \delta^{n+6} d_0 + \delta^{n+7} d_0] \\
&+ s^3 \delta^{n+8} d_0 + ... + s^{m/3} \delta^{n+2m} d_0 \\
&= s\delta^n[1 + s\delta^4 + s^2\delta^8 + ...]d_0 + s\delta^{n+1}[1 + s\delta^4 + s^2\delta^8 + ...]d_0 \\
&+ s\delta^{n+2}[1 + s\delta^4 + s^2\delta^8 + ...]d_0 \\
&= (1 + \delta + \delta^2)(1 - s\delta^4)^{-1} s\delta^n d_0.
\end{aligned}$$

$$d(x_n, x_{n+2m+1}) \leq \frac{1 + \delta + \delta^2}{1 - s\delta^4} s\delta^n d_0 (s\delta^4 < 1). \tag{2.6}$$

Similarly if p is even say $2m$ then using hexagonal b-inequality, we get

$$d(x_n, x_{n+2m+1}) \leq \frac{1 + \delta + \delta^2}{1 - s\delta^4} s\delta^n d_0 + (\frac{1}{s} - \delta)^{-1} d_0^*. \tag{2.7}$$

On taking $lim_{n\to\infty}$ in inequalities (2.6) and (2.7), we get for all $p > 0$

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0.$$

Thus $\{x_n\}$ is a hexagonal b-Cauchy sequence in X. By completeness of (X, d) there exists $u \in X$ such that

$$lim_{n\to\infty} x_n = u.$$

Now we show that u is a fixed point of T. Again

$$\begin{aligned}
d(u, Tu) &\leq s[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, Tu)] \\
&= s[d(u, x_n) + d_n + d_{n+1} + d_{n+2} + d(Tx_{n+2}, Tu)] \\
&\leq s[d(u, x_n) + d_n + d_{n+1} + d_{n+2} + a\{d(x_{n+2}, Tx_{n+2}) + d(u, Tu)\}] \\
(1 - sa)d(u, Tu) &\leq s[d(u, x_n) + d_n + d_{n+1} + ad(x_{n+2}, x_{n+3})] \\
(1 - sa)d(u, Tu) &\leq s[d(u, x_n) + \delta^n d_0 + \delta^{n+1} d_0 + a\delta^{n+2} d_0].
\end{aligned}$$

On taking $lim_{n\to\infty}$ and since $1 - sa \geq 0$, we have

$$d(u, Tu) = 0.$$

implies that

$$Tu = u.$$

Thus u is a fixed point of T in X.
Next, on the contrary assume u and v be two fixed points of T.
Then $Tu = u$ and $Tv = v$.

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &\leq a[d(u,Tu) + d(v,Tv)] \\ &= a[d(u,u) + d(v,v)] \\ d(u,v) &\leq 0. \end{aligned}$$

hence,

$$d(u,v) = 0 \Rightarrow u = v.$$

Thus u is a unique fixed point of T. □

Example 2.7. Let $X = A \cup B$ where $A = \left\{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \right\}$ and $B = [1,4]$. Define $d : X \times X \to [0,\infty)$ such that $d(x,y) = d(y,x)\ \forall\, x,y \in X$ and

$$\begin{cases} d\left(\frac{1}{2}, \frac{3}{4}\right) & = d\left(\frac{5}{6}, \frac{7}{8}\right) = 0.03 \\ d\left(\frac{1}{2}, \frac{7}{8}\right) & = d\left(\frac{3}{4}, \frac{5}{6}\right) = 0.02 \\ d\left(\frac{1}{2}, \frac{5}{6}\right) & = d\left(\frac{7}{8}, \frac{3}{4}\right) = 0.6 \\ d\left(\frac{1}{2}, \frac{1}{2}\right) & = d\left(\frac{3}{4}, \frac{3}{4}\right) = 0 \\ d\left(\frac{5}{6}, \frac{5}{6}\right) & = d\left(\frac{7}{8}, \frac{7}{8}\right) = 0. \end{cases}$$

and $d\big(x,y\big) = |x-y|^2$ if $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$.
since

$$\begin{aligned} & d\Big(2, \frac{1}{2}\Big) > d\Big(2, \frac{3}{4}\Big) + d\Big(\frac{3}{4}, \frac{5}{6}\Big) + d\Big(\frac{5}{6}, \frac{7}{8}\Big) + d\Big(\frac{7}{8}, 1\Big) + d\Big(1, \frac{1}{2}\Big) \\ & \quad 2.25 > 1.5625 + 0.02 + 0.03 + 0.015625 + 0.25 \\ & \quad 2.25 > 1.8781. \end{aligned}$$

Here d does not hold the hexagonal inequality.

But d holds the hexagonal b-inequality. Hence d is a hexagonal b-metric with coefficient $s = 1.198 > 1$.

Let $T : X \to X$ be defined as

$$Tx = \begin{cases} \frac{7}{8} & \text{if } \ x \in [1,4] \\ \\ \frac{5}{6} & \text{if } \ x \in [\frac{1}{2}, \frac{3}{4}, \frac{5}{6}] \\ \\ \frac{3}{4} & \text{if } \ x \in [\frac{7}{8}]. \end{cases}$$

Let $x = \frac{7}{8}$ and $y = \frac{1}{2}$, then by condition (3.1), we have

$$\begin{aligned} d(T_{\frac{7}{8}}, T_{\frac{1}{2}}) &\leq a[d(\frac{7}{8}, T_{\frac{7}{8}}) + d(\frac{1}{2}, T_{\frac{1}{2}})] \\ d(\frac{3}{4}, \frac{5}{6}) &\leq a[d(\frac{7}{8}, \frac{3}{4}) + d(\frac{1}{2}, \frac{5}{6})] \\ 0.02 &\leq a(0.6 + 0.6) \\ 0.02 &\leq a(1.2) \\ 0.0166 &\leq a \\ a &\geq 0.0166. \end{aligned}$$

Thus $a \in [0, \frac{1}{s+1}]$, where $s = 1.198 > 1$ then

$$sa = (1.198)(0.0166) = 0.0198868 \leq 1.$$

Hence T satisfies all the conditions of theorem (2.6) and has a unique fixed point $x = \frac{5}{6}$.

Now we prove the following fixed point theorem for two mappings in hexagonal metric space.

Theorem 2.8 Let (X, d) be a complete hexagonal metric space and let f, g be two self maps define onto X itself such that

$$d(fx, gy) \leq \alpha\{d(x, fx) + d(y, gy)\}. \tag{2.8}$$

for all $x, y \in X$, where $\alpha \in [0, \frac{1}{2})$. Then f and g have a unique common fixed point.

Proof: Firstly we prove the existence of fixed point of f and g.

Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\}$ in X as $x_{n+1} = fx_n$ and $x_{n+2} = gx_{n+1}$ for $n \geq 1$.

If $x_n = x_{n+1}$ for all $n \geq 1$. Then $gx_n = x_{n+1} = x_n = fx_n$. Thus f and g have a unique common fixed point x_n.

Now, suppose that $x_n \neq x_{n+1}$ for all $n \geq 1$, then from (3.8)

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, gx_n) \\ &\leq \alpha\{d(x_{n-1}, fx_{n-1}) + d(x_n, gx_n)\} \\ &= \alpha\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} \\ (1-\alpha)d(x_n, x_{n+1}) &\leq \alpha d(x_{n-1}, x_n) \\ d(x_n, x_{n+1}) &\leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n) \\ &= \lambda d(x_{n-1}, x_n)(\lambda = \frac{\alpha}{1-\alpha} < 1). \end{aligned}$$

Continuing this process, we find

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \leq \lambda^2 d(x_{n-2}, x_{n-1}) ... \leq \lambda^n d(x_0, x_1)$$
$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1). \tag{2.9}$$

In similar manner, it again follows that

$$d(x_n, x_{n+2}) \leq \lambda^n d(x_0, x_2).$$

$$d(x_n, x_{n+3}) \leq \lambda^n d(x_0, x_3). \tag{2.10}$$

$$d(x_n, x_{n+4}) \leq \lambda^n d(x_0, x_4). \tag{2.11}$$

Now for $k = 1, 2, 3, ...$ it further follows that

$$d(x_n, x_{n+4k+1}) \leq \lambda^n d(x_0, x_{4k+1}). \tag{2.12}$$

$$d(x_n, x_{n+4k+2}) \leq \lambda^n d(x_0, x_{4k+2}). \tag{2.13}$$

$$d(x_n, x_{n+4k+3}) \leq \lambda^n d(x_0, x_{4k+3}). \tag{2.14}$$

Applying hexagonal inequality and (2.9)

$$\begin{aligned} d(x_0, x_5) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_5) \\ &\leq d(x_0, x_1) + \lambda d(x_0, x_1) + \lambda^2 d(x_0, x_1) + \lambda^3 d(x_0, x_1) + \lambda^4 d(x_0, x_1) \\ &\leq \sum_{j=0}^{4} \lambda^j d(x_0, x_1). \end{aligned}$$

and

$$\begin{aligned} d(x_0, x_9) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_5) + d(x_5, x_6) + d(x_6, x_7) \\ &+ d(x_7, x_8) + d(x_8, x_9) \\ &\leq d(x_0, x_1) + \lambda d(x_0, x_1) + \lambda^2 d(x_0, x_1) + \lambda^3 d(x_0, x_1) + \lambda^4 d(x_0, x_1) + \lambda^5 d(x_0, x_1) \\ &+ \lambda^6 d(x_0, x_1) + \lambda^7 d(x_0, x_1) + \lambda^8 d(x_0, x_1) \\ &\leq \sum_{j=0}^{8} \lambda^j d(x_0, x_1). \end{aligned}$$

By induction method, we have for $k = 1, 2, 3, ...$

$$d(x_0, x_{4k+1}) \leq \sum_{j=0}^{4k} \lambda^j d(x_0, x_1). \tag{2.15}$$

Applying hexagonal inequality, (2.9) and (2.10)

$$\begin{aligned} d(x_0, x_6) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_6) \\ &\leq d(x_0, x_1) + \lambda d(x_0, x_1) + \lambda^2 d(x_0, x_1) + \lambda^3 d(x_0, x_1) + \lambda^4 d(x_0, x_2) \\ &\leq \sum_{j=0}^{3} \lambda^j d(x_0, x_1) + \lambda^4 d(x_0, x_2). \end{aligned}$$

and

$$\begin{aligned}
d(x_0, x_{10}) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_{10}) \\
&\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_5) + d(x_5, x_6) + d(x_6, x_7) \\
&+ d(x_7, x_8) + d(x_8, x_{10}) \\
&\leq d(x_0, x_1) + \lambda d(x_0, x_1) + \lambda^2 d(x_0, x_1) + \lambda^3 d(x_0, x_1) + \lambda^4 d(x_0, x_1) + \lambda^5 d(x_0, x_1) \\
&+ \lambda^6 d(x_0, x_1) + \lambda^7 d(x_0, x_1) + \lambda^8 d(x_0, x_2) \\
&\leq \sum_{j=0}^{7} \lambda^j d(x_0, x_1) + \lambda^8 d(x_0, x_2).
\end{aligned}$$

Again by induction method, we have for $k = 1, 2, 3, ...$

$$d(x_0, x_{4k+2}) \leq \sum_{j=0}^{4k-1} \lambda^j d(x_0, x_1) + \lambda^{4k} d(x_0, x_2). \tag{2.16}$$

Applying hexagonal inequality, (2.9) and (2.11)

$$d(x_0, x_7) \leq \sum_{j=0}^{3} \lambda^j d(x_0, x_1) + \lambda^4 d(x_0, x_3)$$

and

$$d(x_0, x_{11}) \leq \sum_{j=0}^{7} \lambda^j d(x_0, x_1) + \lambda^8 d(x_0, x_3).$$

Again using induction method, we have for $k = 1, 2, 3, ...$

$$d(x_0, x_{4k+3}) \leq \sum_{j=0}^{4k-1} \lambda^j d(x_0, x_1) + \lambda^{4k} d(x_0, x_3). \tag{2.17}$$

Now by inequalities (2.12) and (2.15), we get

$$\begin{aligned}
d(x_n, x_{n+4k+1}) &\leq \lambda^n \sum_{j=0}^{4k} \lambda^j d(x_0, x_1) \\
&\leq \lambda^n [\sum_{j=0}^{4k} \lambda^j \{d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\}] \\
&\leq \lambda^n [\sum_{j=0}^{\infty} \lambda^j \{d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\}]. \qquad (2.18)
\end{aligned}$$

Now by inequalities (2.13)and (2.16), we get

$$
\begin{aligned}
d(x_n, x_{n+4k+2}) &\leq \lambda^n \sum_{j=0}^{4k-1} \lambda^j d(x_0, x_1) + \lambda^{4k} d(x_0, x_2) \\
&\leq \lambda^n [\sum_{j=0}^{4k-1} \lambda^j \{d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\}] \\
&\leq \lambda^n [\sum_{j=0}^{\infty} \lambda^j \{d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\}]. \qquad (2.19)
\end{aligned}
$$

Now by inequalities (2.14)and (2.17), we get

$$d(x_n, x_{n+4k+3}) \leq \lambda^n [\sum_{j=0}^{\infty} \lambda^j \{d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\}]. \qquad (2.20)$$

Therefore by (2.18),(2.19) and (2.20), we find

$$d(x_n, x_m) \leq \lambda^n [\sum_{j=0}^{\infty} \lambda^j \{d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\}]. \qquad (2.21)$$

We know that by definition (2.5)

$$\sum_{j=0}^{\infty} \lambda^j \{d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\}$$

converges.
Thus

$$lim_{n\to\infty} \delta^n [\sum_{j=0}^{\infty} \lambda^j \{d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\}] = 0.$$

Hence,

$$lim_{n\to\infty} d(x_n, x_m) = 0$$

. Hence $\{x_n\}$ is a hexagonal Cauchy sequence in (X, d).

By completeness of (X, d), there exists $r \in X$ such that
$x_n = fx_{n-1} \to r$ as $n \to \infty$

Now we show that r is a fixed point of f i.e. $fr = r$.

Suppose $fr \neq r$. In hexagonal metric space, we have

$$\begin{aligned}
d(fr,r) &\leq d(fr,x_{n-1}) + d(x_{n-1},x_n) + d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2})) + d(x_{n+2},r)) \\
&\leq d(fr,gx_n) + \lambda^{n-1}d(x_0,x_1) + \lambda^n d(x_0,x_1) + \lambda^{n+1}d(x_0,x_1) + d(x_{n+2},r)) \\
&\leq \alpha[d(fr,r) + d(x_n,gx_n)] + \lambda^{n-1}d(x_0,x_1) + \lambda^n d(x_0,x_1) + \lambda^{n+1}d(x_0,x_1) + d(x_{n+2},r).
\end{aligned}$$

Therefore

$$(1-\alpha)d(fr,r) \leq \alpha d(x_n,x_{n+1}) + \lambda^{n-1}d(x_0,x_1) + \lambda^n d(x_0,x_1) + \lambda^{n+1}d(x_0,x_1) + d(x_{n+2},r)$$

$$(1-\alpha)d(fr,r) \leq \alpha\lambda^n d(x_0,x_1) + \lambda^{n-1}d(x_0,x_1) + \lambda^n d(x_0,x_1) + \lambda^{n+1}d(x_0,x_1) + d(x_{n+2},r).$$

On taking $lim_{n\to\infty}$ and since $\alpha \in [0, \frac{1}{2})$, we find

$$d(fr,r) = 0.$$

Therefore

$$fr = r.$$

Hence r is a fixed point of f.

Now we show that r is a fixed point of g. i.e. $gr = r$.
Suppose $gr \neq r$. Then by inequality (2.8)

$$\begin{aligned}
d(r,gr) &= d(fr,gr) \\
&\leq \alpha\{d(r,fr) + d(r,gr)\} \\
&\leq \alpha\{d(r,r) + d(r,\mathrm{g}r)\} \\
(1-\alpha)d(r,gr) &\leq 0.
\end{aligned}$$

but $1-\alpha \neq 0$, since $\alpha \in [0, \frac{1}{2})$. Thus,

$$d(r,gr) = 0.$$

$$gr = r.$$

Hence

$$fr = gr = r.$$

Therefore f and g have a unique common fixed point.

Uniqueness of Fixed Point-

On the contrary suppose p and q are two common fixed points of f and g.
Then clearly $p = fp$ and $q = gq$

$$\begin{aligned}
d(p,q) &= d(fp,gq) \\
&\leq \alpha\{d(p,fp) + d(q,gq)\} \\
&= \alpha\{d(p,p) + d(q,q)\} \\
d(p,q) &\leq 0.
\end{aligned}$$

Thus,

$$d(p,q) = 0 \Rightarrow p = q.$$

Hence this completes the proof. □

Example 2.9 Let $X = \{p,q,r,s,t,u\}$. Define $d : X \times X \to R^2$ such that $d(x,y) = d(y,x)\ \forall\ x,y \in X$ and for a and $b \in R^+$

$d(p,q) = d(q,p) = (6a, 6b)$.

$d(p,r) = d(r,p) = d(r,s) = d(s,r) = d(q,r) = d(r,q) = d(q,s) = d(s,q) = d(p,s) = d(s,p) = (a,b)$.

$d(p,t) = d(t,p) = d(q,t) = d(t,q) = d(r,t) = d(t,r) = d(s,t) = d(t,s) = (3a, 3b)$.

$d(p,u) = d(q,u) = d(r,u) = d(s,u) = d(t,u) = (4a, 4b)$.

Here (X,d) a complete hexagonal metric space.

Now let $f, g : X \to X$ be defined as

$$fx = \begin{cases} s, & \text{if } x \neq t \\ q, & \text{if } x = t \end{cases}$$

and

$$gx = \begin{cases} r, & \text{if } x = p \\ p, & \text{if } x = q \\ q, & \text{if } x = r \\ s, & \text{if } x = s \\ t, & \text{if } x = t \\ u, & \text{if } x = u \end{cases}$$

Let $x = t$ and $y = p$, then by condition (2.8),we have

$$\begin{aligned} d(ft, gp) &\leq \alpha[d(t, ft) + d(p, gp)] \\ d(q,r) &\leq \alpha[d(t,q) + d(p,r)] \\ (a,b) &\leq \alpha[(3a,3b) + (a,b)] \\ (a,b) &\leq \alpha[4(a,b)] \\ 1 &\leq 4\alpha \\ 0.25 &\leq \alpha. \end{aligned}$$

Thus $\alpha \geq 0.25$, which validate our condition that $\alpha \in [0, \frac{1}{2})$. Therefore f and g satisfy all the conditions of theorem (2.8) and have a unique fixed point s of X.

3. Conclusion :

In this article we introduced the hexagonal metric space and hexagonal b-metric space. we proposed and proved some fixed point results in both spaces and verified this obtained results with new suitable examples. These fixed point theorems generalized the Abba Auwalu and Hincle theorems[1].

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[1] Department of Computer Science, Shri Vaishnav Institute of Management, Devi Ahilya University, Indore, M.P.-452009, India.
Email address: jayesh_tiwari2005@yahoo.com

[2] Department of Mathematics, Govt. Polytechnic College Mandsaur, M.P.-458001, India
Email address: dileepk.sharmadec1972@gmail.com