

## HARDY SPACE OF CERTAIN HARMONIC UNIVALENT FUNCTIONS

JOCELYN JOHNSON<sup>1</sup> AND S. SUNIL VARMA<sup>2\*</sup>

**ABSTRACT.** We determine the Hardy spaces of harmonic mappings whose analytic part satisfies certain conditions, where the associated functions belong to certain subclasses of analytic functions.

*Keywords.* Analytic function, Univalent function, Hardy spaces, Harmonic function, Integral mean, Subordination.

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### 1. INTRODUCTION

One area of study in the field of Complex Analysis is the behaviour of analytic functions near the boundary of its domain. The domain we consider in this paper is the open unit disk  $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$  and let  $h$  be an analytic function in  $\Delta$ . The mean value of  $|h(z)|$  along the boundary of  $\Delta$  offers deep insight into their properties. Hardy [13] studied the growth of  $h$  in  $\Delta$  by examining the maximum modulus

$$M(r) = \sup_{0 < \theta \leq 2\pi} |h(re^{i\theta})|. \quad (1.1)$$

The supremum gives the maximum value that the analytic function  $h$  takes on the boundary  $|z| = r$ ,  $r < 1$  rather than the overall distribution of the function's magnitude on the boundary. F. Riesz [22] extended this analysis by introducing integral means  $M_p(r, h)$ , defined as

$$M_p(r, h) = \left( \frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}, \text{ when } 0 < p < \infty$$

and (1.1) when  $p = \infty$ , which provided a more refined study of boundary behavior of analytic function  $h$ . F. Riesz in honor of Hardy's work introduced the Hardy spaces  $H^p$ , the class of analytic functions  $h$  in  $\Delta$  satisfying  $\lim_{r \rightarrow 1^-} M_p(r, h) < \infty$ . The space  $H^\infty$  is composed of all analytic functions in  $\Delta$  that remain uniformly bounded. The inclusion  $H^p \supset H^q$  holds whenever  $0 < p < q \leq \infty$ .

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\* Corresponding author

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Denote by  $\mathcal{A}$  the class of analytic functions  $h$  on  $\Delta$  having the normalization,  $h(0) = 0$  and  $h'(0) = 1$ . The subclass  $\mathcal{S}$  consists of univalent functions belonging to  $\mathcal{A}$  [25].

A function  $f = u + iv$  that is continuous in a domain  $D \subseteq \mathbb{C}$  is considered harmonic [7] in  $D$  if both  $u$  and  $v$  are real harmonic functions in  $D$ . In particular, if  $D$  is a simply connected domain, such as the given domain  $\Delta$ , the function  $f$  admits a unique decomposition as

$$f = h + \bar{g}$$

where both  $h$  and  $g$  are analytic in  $\Delta$  and are called the “analytic” and the “co-analytic” part of  $f$  respectively. The collection of such functions is denoted by  $\mathcal{H}$ . If  $g$  vanishes identically, then  $\mathcal{H}$  reduces to the class of analytic functions defined in  $\Delta$ .

A harmonic map  $f$  defined on  $\Delta$  is of harmonic Hardy space  $h^p$  ( $0 < p \leq \infty$ ) if  $M_p(r, f)$  is uniformly bounded with respect to  $r$ . More information on Hardy spaces can be found in [8].

The harmonic function  $f = h + \bar{g}$  is locally univalent iff its Jacobian [7], defined as

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2,$$

is nonzero in  $\Delta$  [16]. A univalent harmonic map  $f$  is sense-preserving if  $J_f(z) > 0$  for all  $z \in \Delta$  and sense-reversing if  $J_f(z) < 0$  for all  $z \in \Delta$ . The collection  $\mathcal{S}_{\mathcal{H}}$  consists of functions in  $\mathcal{H}$  which are univalent sense-preserving in  $\Delta$  satisfying  $h(0) = 0, h'(0) = 1$  and  $g(0) = 0$  while  $\mathcal{S}_{\mathcal{H}}^0$ , refers to those members of  $\mathcal{S}_{\mathcal{H}}$  that satisfy,  $g'(0) = 0$ . For more detailed study on harmonic univalent mappings, one may refer the monograph by Duren [9].

In 2015, Ponnusamy *et.al* [21] introduced the subclass  $\mathcal{S}_{\mathcal{H}}^0(\mathcal{S})$  of  $\mathcal{S}_{\mathcal{H}}^0$  defined by

$$\mathcal{S}_{\mathcal{H}}^0(\mathcal{S}) = \{h + \bar{g} \in \mathcal{S}_{\mathcal{H}}^0 : \Phi = h + e^{i\theta}g \in \mathcal{S} \text{ for some } \theta \in \mathbb{R}\} \quad (1.2)$$

to explore harmonic analogue [7] of the classical Bieberbach conjecture [3].

If  $f \in \mathcal{S}_{\mathcal{H}}^0(\mathcal{S})$  [14], then

$$\left| \frac{\Phi'(z)}{h'(z)} - 1 \right| < 1, \text{ i.e., } \operatorname{Re} \left( \frac{h'(z)}{\Phi'(z)} \right) > \frac{1}{2}, z \in \Delta. \quad (1.3)$$

In 2018, Kayumov *et. al* [14] introduced class  $\mathcal{T}$ , consisting of functions in  $\mathcal{S}_{\mathcal{H}}^0$  that satisfy (1.3).

Let  $\mathcal{H}_{\phi}^{\alpha}$  consists of analytic functions  $h$  in  $\Delta$  that satisfy

$$\left| \frac{z\Phi'}{(1-\alpha)h(z) + \alpha zh'(z)} - 1 \right| < 1 \quad (1.4)$$

for some  $\phi \in \mathcal{S}$  with  $0 \leq \alpha \leq 1$ . When  $\alpha = 1$ , (1.4) reduces to (1.3).

For  $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}^0$ , depending on the modulus value of the second Taylor's coefficient of  $h$  (refer [1]), for  $p > 0$   $h, g \in H^p$  and  $f \in h^p$ . In the special case of harmonic Koebe function  $K = H + \bar{G}$  where,

$$H = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} \text{ and } G = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}$$

it was established that  $H, G \in H^p$  for  $p < \frac{1}{3}$  while  $H, G \notin H^{\frac{1}{3}}$  (refer [9]). Similar results of harmonic mappings associated with convex, starlike and close-to-convex subclasses can be found in [2, 6, 18]. Kayumov et al. [14] further demonstrated that when  $f = h + \bar{g}$  is an element of  $\mathcal{T}$  and its analytic part satisfies (1.3) for some  $\Phi \in \mathcal{S}$ , then for all  $p < \frac{1}{3}$   $h, g \in H^p$  and  $f \in h^p$ .

Integral mean estimates play an important role in understanding function behavior within Hardy spaces (see, e.g., Wanjohi *et al.* [26]). In this paper, we employ integral mean techniques to determine Hardy spaces of analytic and co-analytic part of the harmonic mapping  $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}^0$ , and consequently the Hardy space of  $f$ , whose analytic part  $h$  satisfy conditions (say (1.3) or (1.4)) where  $\Phi$  is varied over different subclasses of  $\mathcal{S}$ .

The results obtained in this paper have several potential applications in computer science and related computational fields such as geometric modeling, computer graphics and image processing, where harmonic mappings are employed for surface parameterization and texture distortion control. Furthermore, the integral mean techniques used in this study are closely related to techniques used in numerical analysis and signal processing, where they help analyze the stability and convergence of algorithms in computational harmonic analysis.

## 2. PRELIMINARIES

### 2.1. Preliminary Lemmas.

**Lemma 2.1.** [8] *If  $f' \in H^p$  then  $f \in H^{\frac{p}{1-p}}$ , if  $p < 1$ .*

**Lemma 2.2.** [8] *If  $f$  is analytic and univalent in  $|z| < 1$ , then  $f \in H^p \forall p < \frac{1}{2}$ .*

**Lemma 2.3.** [8] *Every analytic function  $f(z)$  with positive real part in  $|z| < 1$  is of class  $H^p \forall p < 1$ .*

**Lemma 2.4.** [15] *If  $f \in \mathcal{A}$  satisfies  $z^\gamma f(z) \in H^p, (0 < p < \infty)$  for some real  $\gamma$ , then  $f \in H^p$ .*

### 2.2. Some Classes of Analytic Functions.

**Definition 2.5.** [4, 5, 20] An analytic function  $p$  defined in  $\Delta$  is in class  $\mathcal{P}$  if  $p(0) = 1$  and  $Re(p(z)) > 0$  for  $z \in \Delta$

We now recall certain standard subclasses of  $\mathcal{S}$ .

Geometric function theory extensively studies several subclasses of  $\mathcal{S}$ , with two notable classes being starlike and convex functions, defined as follows.

A function  $h \in \mathcal{S}$  is in the class of starlike functions  $S^*$  iff  $\frac{zh'}{h} \in \mathcal{P}$  and it is said to be in the class of convex functions  $C$  iff  $1 + \frac{zh''}{h'} \in \mathcal{P}$  [20, 25].

*Remark 2.6.* [20]  $h \in \mathcal{C} \iff zh' \in \mathcal{S}^*$ .

**Definition 2.7.** [25] A function  $h \in \mathcal{A}$  is said to be in class  $\mathcal{K}$  of close to convex functions, if there exists a  $g \in \mathcal{S}^*$  such that

$$\operatorname{Re}\left(\frac{zh'(z)}{g(z)}\right) > 0, |z| < 1.$$

**Definition 2.8.** [19] A function  $h \in \mathcal{A}$  is said to be in class  $M(\beta)$  if

$$\operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\} < \beta, \text{ for } z \in \Delta, \beta > 1.$$

**Definition 2.9.** ([19]) A function  $h \in \mathcal{A}$  is said to be in class  $N(\beta)$  if

$$\operatorname{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} < \beta, \text{ for } z \in \Delta, \beta > 1.$$

*Remark 2.10.* [12, 19]  $h \in N(\beta) \iff zh' \in M(\beta)$ .

Let [21]  $\mathcal{B} = \{\omega \in \mathcal{A} : |\omega(z)| < 1 \text{ for } z \in \Delta\}$  and  $\mathcal{B}_0$  consists of functions in  $\mathcal{B}$  for which  $\omega(0) = 0$ .

### 3. MAIN THEOREMS AND PROOFS

**Theorem 3.1.** Suppose  $f = h + \bar{g} \in \mathcal{S}_H^0$  and  $h$  satisfy condition (1.4) for some  $\Phi \in \mathcal{S}$ . Then for all  $p < \frac{1}{3}$ ,  $g, h \in H^p$  and  $f \in h^p$ .

*Proof.* Starting from condition (1.4) we have

$$\frac{\Phi'(z)}{(1 - \alpha)\frac{h(z)}{z} + \alpha h'(z)} = 1 - \phi(z), \quad \phi \in \mathcal{B}_0.$$

Hence

$$\frac{\Phi'(z)}{1 - \phi(z)} = (1 - \alpha)\frac{h(z)}{z} + \alpha h'(z).$$

Rewriting the right-hand side, observe that

$$\frac{h(z)}{z} + \alpha \left(z \left(\frac{h(z)}{z}\right)'\right) = \alpha z^{1-\frac{1}{\alpha}} \left(z^{\frac{1}{\alpha}} \left(\frac{h(z)}{z}\right)\right)'.$$

Therefore

$$\frac{\Phi'(z)}{1 - \phi(z)} = \alpha z^{1-\frac{1}{\alpha}} \left(z^{\frac{1}{\alpha}} \left(\frac{h(z)}{z}\right)\right)'. \quad (3.1)$$

Take the integral mean on both sides. Using Lemma 3.1 of [14] we know that for every  $p \in (\frac{2}{7}, 2)$  there exists  $C > 0$  such that

$$M_p\left(r, \frac{\Phi'(re^{i\theta})}{1 - \phi(re^{i\theta})}\right) \leq \frac{C |\log(1 - r)|^{1/2}}{(1 - r)^{\frac{4p-1}{p}}}.$$

Combining this with (3.1) and using Lemma 2.4, we obtain

$$M_p\left(r, (re^{i\theta})^{\frac{1}{\alpha}-1}h(re^{i\theta})\right) \leq \frac{C}{\alpha} \int_0^r \frac{|\log(1-\rho)|^{1/2}}{(1-\rho)^{\frac{4p-1}{p}}} d\rho. \quad (3.2)$$

It remains to check when the integral on the right-hand side remains bounded as  $r \rightarrow 1^-$ . Set  $t = 1 - \rho$ . Put

$$a = \frac{4p-1}{p} = 4 - \frac{1}{p}.$$

Then the improper integral

$$\int_0^\varepsilon t^{-a} |\log t|^{1/2} dt$$

converges iff  $a < 1$  (By the  $p$ -test for improper integrals (see Rudin [23, p. 124]), the integral

$$\int_0^\varepsilon t^{-a} dt$$

converges if and only if  $a < 1$ . Consequently,  $\int_0^\varepsilon t^{-a} |\log t|^\beta dt$  also converges for  $a < 1$ . Thus we require

$$4 - \frac{1}{p} < 1 \iff \frac{1}{p} > 3 \iff p < \frac{1}{3}.$$

Therefore the integral in (3.2) is bounded uniformly in  $r$  whenever  $p < \frac{1}{3}$ . Hence  $h \in H^p$ ,  $\forall p < \frac{1}{3}$ . As  $f \in \mathcal{S}_H^0$ ,  $g \in H^p$  and consequently  $f \in h^p$  for all  $p < \frac{1}{3}$  (refer [1], Theorem 2).  $\square$

**Theorem 3.2.** Suppose  $f = h + \bar{g} \in \mathcal{S}_H^0$  and  $h$  satisfy condition (1.3) for some  $\Phi \in \mathcal{S}^*$ . Then for all  $p < \frac{1}{3}$ ,  $h, g \in H^p$  and  $f \in h^p$ .

*Proof.* By the hypothesis (1.3), for some  $\phi \in \mathcal{B}_0$

$$h'(z) = \frac{\Phi'(z)}{1 - \phi(z)}. \quad (3.3)$$

There is some  $P \in \mathcal{P}$  for which

$$\frac{1 + \phi(z)}{1 - \phi(z)} = P(z). \quad (3.4)$$

Using (3.4) and (3.3) gives

$$h'(z) = \frac{\Phi'(z)P(z)}{2} + \frac{\Phi'(z)}{2},$$

which leads to

$$|h'(z)| \leq \frac{1}{2}|\Phi'(z)P(z)| + \frac{1}{2}|\Phi'(z)|.$$

Applying  $(a+b)^p \leq 2^p(a^p + b^p)$  for some positive  $a, b$  and  $p > 0$ , we obtain

$$|h'(z)|^p \leq |\Phi'(z)P(z)|^p + |\Phi'(z)|^p.$$

$$\begin{aligned} \therefore M_p^p(r, h') &\leq \int_0^{2\pi} |\Phi'(re^{i\theta})P(re^{i\theta})|^p d\theta + \int_0^{2\pi} |\Phi'(re^{i\theta})|^p d\theta \\ &= I_1(r) + I_2(r). \end{aligned}$$

Using Hölder's inequality,

$$I_1(r) \leq \left( \int_0^{2\pi} |P(re^{i\theta})|^{pm} d\theta \right)^{\frac{1}{m}} \left( \int_0^{2\pi} |\Phi'(re^{i\theta})|^{pn} d\theta \right)^{\frac{1}{n}},$$

where  $\frac{1}{m} + \frac{1}{n} = 1$ . From [8], for  $pm < 1$ ,  $J_1(r) < \infty$  as  $r \rightarrow 1^-$  and from [10], for  $pn < \frac{1}{3}$ ,  $J_2(r) < \infty$  as  $r \rightarrow 1^-$ .

Therefore,  $I_1(r)$  is finite for  $p < \frac{1}{4}$ , and  $I_2(r)$  is finite for  $p < \frac{1}{3}$ . Consequently,  $M_p^p(r, h')$  is bounded as  $r \rightarrow 1^-$  when  $p < \frac{1}{4}$ , by Lemma 2.1,  $h \in H^p$  for  $p < \frac{1}{3}$ . By Theorem 2 in [1], the sense-preserving nature of  $f$  ensures  $g \in H^p$  and thereby  $f$  to be in  $h^p$  for  $p < \frac{1}{3}$ .  $\square$

**Theorem 3.3.** Suppose  $f = h + \bar{g} \in \mathcal{S}_H^0$  and  $h$  satisfy condition (1.3) for some  $\Phi \in \mathcal{C}$ . Then for all  $p < \frac{1}{2}$ ,  $g, h \in H^p$  and  $f \in h^p$ .

*Proof.* From the given assumption (1.3) we get (3.3) and taking the integral mean, we obtain

$$M_p(r, h') = \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\Phi'(re^{i\theta})}{1 - \phi(re^{i\theta})} \right|^p d\theta \right)^{\frac{1}{p}}. \quad (3.5)$$

Using Hölder's inequality, we obtain

$$\begin{aligned} M_p(r, h') &\leq \frac{1}{r^p} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{r\Phi'(re^{i\theta})}{\Phi(re^{i\theta})} \right|^2 d\theta \right)^{\frac{p}{2}} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\Phi(re^{i\theta})}{1 - \phi(re^{i\theta})} \right|^{\frac{2p}{2-p}} d\theta \right)^{\frac{2-p}{2}} \\ &= \frac{1}{r^p} I_1^{\frac{p}{2}}(r) I_2^{\frac{2-p}{2}}(r). \end{aligned}$$

where the first integral  $I_1$  is estimated using Parseval's inequality (cf. [17], p. 87), leading to

$$I_1^{\frac{p}{2}}(r) \leq 2^{\frac{p}{2}} + \frac{2^p}{e^{\frac{p}{2}}} \frac{|\log(1 - \sqrt{r})|^{\frac{p}{2}}}{(1 - r)^{\frac{p}{2}}}.$$

For  $I_2$ , bounding  $\Phi$  using a standard growth estimate for functions in  $\mathcal{C}$ , we derive

$$I_2^{\frac{2-p}{2}}(r) \leq \frac{C_1 r^p}{(1 - r)^{\frac{5p-2}{2}}}.$$

Substituting these estimates into our expression for  $M_p(r, h')$ , we obtain

$$M_p(r, h') \leq 2^{\frac{1}{p}} \left( \frac{2^{\frac{1}{2}} C_1}{(1 - r)^{\frac{(5/2)p-1}{p}}} + \frac{2C_1}{e^{\frac{1}{2}}} \frac{|\log(1 - \sqrt{r})|^{\frac{1}{2}}}{(1 - r)^{\frac{3p-1}{p}}} \right).$$

By integrating this bound (cf. [8], p. 80), we find that  $M_p(r, h) < \infty$  as  $r \rightarrow 1^-$  for  $p < \frac{1}{2}$ . Finally, applying Theorem 2 from [1], we conclude that  $g \in H^p$  and  $f \in h^p$  for all  $p < \frac{1}{2}$ .  $\square$

**Theorem 3.4.** Suppose  $f = h + \bar{g} \in \mathcal{S}_H^0$  and  $h$  satisfy (1.3) for some  $\Phi \in \mathcal{K}$ . Then, for all  $p < \frac{1}{3}$ ,  $g, h \in H^p$  and  $f \in h^p$ .

*Proof.* From the hypothesis (see (1.3) and (3.5)) we have

$$M_p(r, h') = \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\Phi'(re^{i\theta})}{1 - \phi(re^{i\theta})} \right|^p d\theta \right)^{1/p}.$$

By Theorem 2 of [11], there exists a constant  $C > 0$  such that for all  $r \in (0, 1)$  and  $\theta \in [0, 2\pi)$ ,

$$|\Phi'(re^{i\theta})| \leq C |1 - re^{i\theta}|^{-3}.$$

Hence

$$\left| \frac{\Phi'(re^{i\theta})}{1 - \phi(re^{i\theta})} \right|^p \leq C^p \frac{1}{|1 - re^{i\theta}|^{3p}} \cdot \left| \frac{1}{1 - \phi(re^{i\theta})} \right|^p.$$

Next, from assumption  $\phi \in \mathcal{B}_0$ ,  $|\phi(z)| < 1$  in the unit disk, we have

$$|1 - \phi(z)| \geq 1 - |\phi(z)|.$$

Fix  $r < 1$ , then by the maximum modulus principle,  $|\phi(z)|$  attains its maximum on  $|z| = r$ . Let  $M_r = \max_{|z|=r} |\phi(z)| < 1$ . Then for all  $|z| \leq r$ ,

$$|1 - \phi(z)| \geq 1 - M_r =: \delta_r > 0.$$

Thus, the reciprocal  $1/(1 - \phi(z))$  is bounded on the disk  $|z| \leq r$ .

For each fixed  $r < 1$ , the bound

$$\left| \frac{1}{1 - \phi(re^{i\theta})} \right| \leq K_r,$$

is positive for some finite  $K_r = 1/\delta_r$ . Thus

$$\begin{aligned} \left| \frac{\Phi'(re^{i\theta})}{1 - \phi(re^{i\theta})} \right|^p &\leq C^p \frac{1}{|1 - re^{i\theta}|^{3p}} \cdot K_r^p \\ \implies \left| \frac{\Phi'(re^{i\theta})}{1 - \phi(re^{i\theta})} \right|^p &\leq C_2 |1 - re^{i\theta}|^{-3p}, \quad C_2 = (C \cdot K_r)^p. \end{aligned}$$

Therefore

$$M_p(r, h')^p \leq \frac{1}{2\pi} \int_0^{2\pi} C_2 |1 - re^{i\theta}|^{-3p} d\theta = \frac{C_2}{2\pi} \int_0^{2\pi} |1 - re^{i\theta}|^{-3p} d\theta.$$

Using the estimate (see Theorem 2 in [11]) for  $\alpha > 0$  and  $r \in (0, 1)$ ,

$$\int_0^{2\pi} |1 - re^{i\theta}|^{-\alpha} d\theta \leq C_\alpha (1 - r)^{1-\alpha}, \quad (3.6)$$

where  $C_\alpha$  is a constant depending only on  $\alpha$ . Applying (3.6) with  $\alpha = 3p$  yields

$$\int_0^{2\pi} |1 - re^{i\theta}|^{-3p} d\theta \leq C_{3p} (1 - r)^{1-3p},$$

and hence

$$M_p(r, h')^p \leq \frac{C_2 C_{3p}}{2\pi} (1 - r)^{1-3p}.$$

Taking the  $p$ -th root gives

$$M_p(r, h') \leq \frac{C_1}{(1-r)^{\frac{4p-1}{p}}}, \quad C_1 = \left( \frac{C_2 C_{3p}}{2\pi} \right)^{1/p}.$$

Finally, by a result from [8] (p. 80), for  $r \in (0, 1)$ , we have

$$M_p(r, h) \leq \int_0^r M_p(s, h') ds \leq C_1 \int_0^r \frac{ds}{(1-s)^{\frac{4p-1}{p}}}.$$

The improper integral on the right remains finite as  $r \rightarrow 1^-$  when

$$\frac{4p-1}{p} < 1 \iff p < \frac{1}{3}.$$

Therefore  $M_p(r, h)$  is uniformly bounded in  $r$  for all  $p < \frac{1}{3}$ , i.e.  $h \in H^p$  for every  $p < \frac{1}{3}$ . Since  $f \in \mathcal{S}_{\mathcal{H}}^0$ , it follows from [1] that  $g \in H^p$  and hence  $f$  belongs to  $h^p$  for all  $p < \frac{1}{3}$ .  $\square$

**Theorem 3.5.** Suppose  $f = h + \bar{g} \in \mathcal{S}_H^0$  and  $h$  satisfy condition (1.3) for some  $\Phi \in \mathcal{M}(\beta)$ . Then, for all  $p < \frac{1}{4\beta-3}$ , we have  $g, h \in H^p$  and  $f \in h^p$ .

*Proof.* We have the hypothesis (1.3). Hence, there exists a function  $P \in \mathcal{P}$  such that

$$\begin{aligned} h'(z) &= \frac{\Phi'(z)}{2} [P(z) + 1]. \\ \implies |h'(z)|^p &\leq 2^{p-1} (|\Phi'(z)P(z)|^p + |\Phi'(z)|^p). \end{aligned}$$

Therefore,

$$\begin{aligned} M_p^p(r, h') &= \frac{1}{2\pi} \int_0^{2\pi} |h'(re^{i\theta})|^p d\theta \\ &\leq \frac{2^{p-1}}{2\pi} \int_0^{2\pi} |\Phi'(re^{i\theta})P(re^{i\theta})|^p d\theta + \frac{2^{p-1}}{2\pi} \int_0^{2\pi} |\Phi'(re^{i\theta})|^p d\theta. \end{aligned}$$

Applying Hölder's inequality. Then

$$\int_0^{2\pi} |\Phi'(re^{i\theta})P(re^{i\theta})|^p d\theta \leq \left( \int_0^{2\pi} |P(re^{i\theta})|^{pm} d\theta \right)^{1/m} \left( \int_0^{2\pi} |\Phi'(re^{i\theta})|^{pn} d\theta \right)^{1/n}, \quad \frac{1}{m} + \frac{1}{n} = 1.$$

Hence,

$$\begin{aligned} M_p^p(r, h') &\leq C_p \left[ \left( \int_0^{2\pi} |P(re^{i\theta})|^{pm} d\theta \right)^{1/m} \left( \int_0^{2\pi} |\Phi'(re^{i\theta})|^{pn} d\theta \right)^{1/n} + \int_0^{2\pi} |\Phi'(re^{i\theta})|^p d\theta \right] \\ &= C_p \left[ (J_1(r))^{\frac{1}{m}} \cdot (J_2(r))^{\frac{1}{n}} + \int_0^{2\pi} |\Phi'(re^{i\theta})|^p d\theta \right], \quad C_p = \frac{2^{p-1}}{2\pi}. \end{aligned}$$

Since  $\Phi \in \mathcal{M}(\beta)$ , we have from Theorem 4.1 of [24],

$$\int_0^{2\pi} |\Phi'(re^{i\theta})|^\lambda d\theta \leq C(\beta, \lambda), \quad \lambda < \frac{1}{4\beta-3},$$

where  $C(\beta, \lambda)$  is finite and independent of  $r$ . Thus, the integral

$$J_2(r) = \int_0^{2\pi} |\Phi'(re^{i\theta})|^{pn} d\theta$$

remains bounded as  $r \rightarrow 1^-$  for  $pn < \frac{1}{4\beta-3}$ . Since  $P \in \mathcal{P}$ , by Lemma 2.3 we have

$$J_1(r) = \int_0^{2\pi} |P(re^{i\theta})|^{pm} d\theta$$

bounded as  $r \rightarrow 1^-$  for  $pm < 1$ .

Combining these,  $M_p(r, h')$  remains finite for

$$p < \frac{1}{4\beta-2}$$

as  $r \rightarrow 1^-$ . By Lemma 2.1, this implies  $h \in H^p$  for  $p < \frac{1}{4\beta-3}$ . Since  $f$  is sense-preserving, Theorem 2 of [1] ensures that  $g \in H^p$ , and thus  $f \in h^p$  for the same range of  $p$ .  $\square$

**Theorem 3.6.** *Suppose  $f = h + \bar{g} \in \mathcal{S}_H^0$ , where  $h$  satisfies condition (1.3) for some  $\Phi \in \mathcal{N}(\beta)$ . Then for all  $p < \frac{1}{4(\beta-1)}$   $g, h \in H^p$  and  $f \in h^p$ .*

*Proof.* The proof is in similar lines with the proof of Theorem 3.5 with  $M_p(r, h')$  finite for  $p < \frac{1}{4\beta-3}$  as  $r \rightarrow 1^-$ .  $\square$

**Theorem 3.7.** *Suppose that  $f = h + \bar{g} \in \mathcal{S}_H^0$  and that there exists  $\Phi \in \mathcal{A} \cap H^p$  for some  $p$  with  $p < 1$ . Additionally, if*

$$\operatorname{Re}\left(\frac{h(z)}{\Phi(z)}\right) > \alpha, \quad 0 \leq \alpha \leq 1, \quad (3.7)$$

*then for all  $p < \frac{1}{2}$ ,  $h, g \in H^p$  and  $f \in h^p$ .*

*Proof.* By condition (3.7), we have,

$$\frac{\frac{h(z)}{\Phi(z)} - \alpha}{1 - \alpha} \in \mathcal{P}.$$

Thus, for some  $P \in \mathcal{P}$ ,

$$\begin{aligned} h(z) &= (1 - \alpha)P(z)\Phi(z) + \alpha\Phi(z) \\ \implies |h(z)| &\leq (1 - \alpha)|P(z)\Phi(z)| + \alpha|\Phi(z)|. \end{aligned}$$

$$\therefore M_p^p(r, h) \leq \frac{(2(1 - \alpha))^p}{2\pi} \int_0^{2\pi} |P(re^{i\theta})\Phi(re^{i\theta})|^p d\theta + \frac{(2\alpha)^p}{2\pi} \int_0^{2\pi} |\Phi(re^{i\theta})|^p d\theta.$$

Using Hölder's inequality,

$$\int_0^{2\pi} |P(re^{i\theta})\Phi(re^{i\theta})|^p d\theta \leq \left( \int_0^{2\pi} |P(re^{i\theta})|^{pm} d\theta \right)^{\frac{1}{m}} \left( \int_0^{2\pi} |\Phi(re^{i\theta})|^{pn} d\theta \right)^{\frac{1}{n}},$$

where  $m$  and  $n$  are conjugate exponents.

By assumption,  $pn < 1$  and by Lemma 2.3,  $M_p^p(r, h) < \infty$  as  $r \rightarrow 1^-$ , provided that  $p < \frac{1}{2}$ . Since  $f \in \mathcal{S}_H^0[1]$ , it follows that  $g$  is in  $H^p$  for  $p < \frac{1}{2}$ , and consequently,  $f$  belongs to  $h^p$  for  $p < \frac{1}{2}$ .  $\square$

**Theorem 3.8.** *Suppose  $f = h + \bar{g} \in \mathcal{S}_H^0$  and satisfies condition (3.7) where  $\Phi \in \mathcal{S}$ . Then for all  $p < \frac{1}{3}$ ,  $h, g \in H^p$  and  $f \in h^p$ .*

*Proof.* From the condition (3.7), we have,

$$\frac{\frac{h(z)}{\Phi(z)} - \alpha}{1 - \alpha} \in \mathcal{P}.$$

Thus, for some  $P \in \mathcal{P}$ ,

$$\begin{aligned} h(z) &= (1 - \alpha)P(z)\Phi(z) + \alpha\Phi(z) \\ \implies |h(z)| &\leq (1 - \alpha)|P(z)\Phi(z)| + \alpha|\Phi(z)|. \end{aligned}$$

$$\therefore M_p^p(r, h) \leq \frac{(2(1 - \alpha))^p}{2\pi} \int_0^{2\pi} |P(re^{i\theta})\Phi(re^{i\theta})|^p d\theta + \frac{(2\alpha)^p}{2\pi} \int_0^{2\pi} |\Phi(re^{i\theta})|^p d\theta.$$

Using Hölder's inequality,

$$\int_0^{2\pi} |P(re^{i\theta})\Phi(re^{i\theta})|^p d\theta \leq \left( \int_0^{2\pi} |P(re^{i\theta})|^{pm} d\theta \right)^{\frac{1}{m}} \left( \int_0^{2\pi} |\Phi(re^{i\theta})|^{pn} d\theta \right)^{\frac{1}{n}},$$

where  $m$  and  $n$  are conjugate exponents.

By Lemma 2.2,  $pn < \frac{1}{2}$  and by Lemma 2.3,  $pm < 1$ ,  $M_p^p(r, h) < \infty$  as  $r \rightarrow 1^-$ , provided that  $p < \frac{1}{3}$ . Since  $f \in \mathcal{S}_H^0[1]$ , it follows that  $g$  is in  $H^p$  for  $p < \frac{1}{2}$ , and consequently,  $f$  belongs to  $h^p$  for  $p < \frac{1}{2}$ .  $\square$

The proof of the following theorems are in similar lines as the previous theorem.

**Theorem 3.9.** *Suppose  $f = h + \bar{g} \in \mathcal{S}_H^0$  and satisfies condition (3.7) where  $\Phi \in \mathcal{S}^*$ . Then for all  $p < \frac{1}{3}$ ,  $h, g \in H^p$ , and  $f \in h^p$ .*

**Theorem 3.10.** *Suppose  $f = h + \bar{g} \in \mathcal{S}_H^0$  and satisfies condition (3.7) where  $\Phi \in \mathcal{C}$ . Then for all  $p < \frac{1}{2}$ ,  $h, g \in H^p$  and  $f \in h^p$ .*

#### 4. CONCLUSION

In this study, we have identified the Hardy spaces of harmonic mappings whose analytic part satisfies specific differential inequalities, with the associated functions belonging to certain geometric subclasses of  $\mathcal{S}$ .

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<sup>1</sup> DEPARTMENT OF MATHEMATICS, MADRAS CHRISTIAN COLLEGE(AUTONOMOUS), TAMBARAM, CHENNAI - 600059, TAMIL NADU, INDIA.

*Email address:* [jocelynjohnson1010@gmail.com](mailto:jocelynjohnson1010@gmail.com)

<sup>2</sup> DEPARTMENT OF MATHEMATICS, MADRAS CHRISTIAN COLLEGE(AUTONOMOUS), TAMBARAM, CHENNAI - 600059, TAMIL NADU, INDIA.

*Email address:* [sunilvarma@mcc.edu.in](mailto:sunilvarma@mcc.edu.in)