

OUTER-CONNECTED FAIR ROMAN DOMINATING FUNCTION IN GRAPHS

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ABSTRACT. In this paper, we introduce and investigate a new variant of a Roman dominating function in graphs called outer-connected fair Roman dominating function. In addition, we presented some theoretical properties of outer-connected fair Roman dominating functions in some classes of connected graphs and discussed some important results.

Keywords. Outer-connected domination, Fair domination, Roman dominating function, Outer-connected fair Roman dominating function

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1. INTRODUCTION AND PRELIMINARIES

One of the interesting topics in domination theory is the Roman dominating function, which is considered a research motivating variation of dominating sets in graphs. Roman domination was initiated in 2004 by Cockayne et al. [1], and nowadays, there are several papers in the literature that involve Roman dominating function, which can be found in [2, 3, 4, 5]. In the year 2007, Cyman [6] initiated outer-connected domination number in graphs, and Caro et al. [7] introduced the concept of fair dominating sets in graphs. Currently, Duhilag et al. [8] in the year 2025, combined the idea of outer-connected and fair domination, and came up with a new parameter called outer-connected fair domination in graphs. In this study, the authors are motivated to join the concept of outer-connected fair domination and Roman dominating function to come up with a new interesting parameter and a stronger version of network security in graphs. This new restricted parameter is a combination of defensive strategy, fairness of undefended vertices, and their connectivity, which emphasizes a better security of a Roman domination scheme. Hence, this paper is realized. The results of this paper may be useful in network analysis in computer science applications that investigate the behavior and relationship of interconnectedness of entities. For the definitions and concepts in graph theory involved in this study that are not presented here, the readers are advised to refer to [9, 12, 13].

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Let $G = (V(G), E(G))$ be a connected graph where $V(G)$ is the vertex set and $E(G)$ is the edge set of G . Let $D \subseteq V(G)$. Then D is a *dominating set* of G if for every $x \in V(G) \setminus D$, there exists $y \in D$ such that $xy \in E(G)$ [9]. The *domination number* of G , denoted by $\gamma(G)$, is the smallest cardinality of a dominating set D on G . The dominating set D with $|D| = \gamma(G)$ is called γ -set on G . For various parameters and rigorous studies in domination in graphs, the readers may refer to the following references [10, 11, 12, 13, 14]. A set $O \subseteq V(G)$ is called an *outer-connected dominating set* of G provided that for every $a \in V(G) \setminus O$, there exists $b \in O$ such that $ab \in E(G)$ and the subgraph $\langle V(G) \setminus O \rangle$ is connected [6]. The *outer-connected domination number* of G is defined as the minimum cardinality of a dominating set O on G and is denoted by $\tilde{\gamma}_c(G)$. The outer-connected dominating set O that satisfies with $|O| = \tilde{\gamma}_c(G)$ is called $\tilde{\gamma}_c$ -set on G . A proper subset F of $V(G)$ is called a fair vertex set on G if for every $x, y \in F$, $|N_G(x) \cap (V(G) \setminus F)| = |N_G(y) \cap (V(G) \setminus F)| \geq 1$. A set $F_d \subseteq V(G)$ is a *fair dominating set* of G provided that for every $x \in V(G) \setminus F_d$, there exists $y \in F_d$ such that $xy \in E(G)$ and the set $V(G) \setminus F$ is a fair vertex set [7]. The *fair domination number* of G , denoted by $fd(G)$, is the minimum cardinality of a fair dominating set F_d on G . The fair dominating set F_d with $|F_d| = fd(G)$ is called FD -set on G . Let $C_f \subseteq V(G)$. Then C_f is called an *outer-connected fair dominating set* of G provided that for every $v \in V(G) \setminus C_f$, there exists $u \in C_f$ such that $uv \in E(G)$, the set $V(G) \setminus C_f$ is a fair vertex set, and the subgraph $\langle V(G) \setminus C_f \rangle$ is connected [8]. The *outer-connected fair domination number* of G , denoted by $\tilde{\gamma}_{cF}(G)$, is the smallest cardinality of a dominating set C_f on G . Moreover, the dominating set C_f which satisfies $|C_f| = \tilde{\gamma}_{cF}(G)$ is called $\tilde{\gamma}_{cF}$ -set on G .

Let $\lambda : V(G) \rightarrow \{0, 1, 2\}$ be a function on G . Set $V_i = \{x \in V(G) : \lambda(x) = i\}$, for each $i \in \{0, 1, 2\}$. Then we obtain $\lambda = (V_0, V_1, V_2)$. A function $\lambda = (V_0, V_1, V_2)$ is a *Roman dominating function* (RDF) on G if for every vertex $v \in V_0$ there exists $u \in V_2$ such that $uv \in E(G)$ [1]. The *weight* of function λ on G is defined by $\omega_G^R(\lambda) = \sum_{w \in V(G)} \lambda(w) = |V_1| + 2|V_2|$. The *Roman domination number* of G is defined as the smallest weight of an RDF λ on G , that is, $\gamma_R(G) = \min\{\omega_G^R(\lambda) : \lambda \text{ is an RDF on } G\}$ and denoted by $\gamma_R(G)$. Every RDF λ on G satisfying the condition $\omega_G^R(\lambda) = \gamma_R(G)$ is a γ_R -function on G . A function $\lambda = (V_0, V_1, V_2)$ is an *outer-connected fair Roman dominating function* (OcFRDF) on G provided that for every vertex in V_0 is adjacent to some vertex in V_2 and for every $u, v \in V_0$, $|N_G(u) \cap V_2| = |N_G(v) \cap V_2| \geq 1$, i.e., V_0 is a fair vertex set, and $\langle V_0 \rangle$ is a connected subgraph of G . The *weight* of OcFRDF λ denoted by $\tilde{\omega}_G^{cFR}(\lambda)$ is the sum $\tilde{\omega}_G^{cFR}(\lambda) = \sum_{x \in V(G)} \lambda(x) = |V_1| + 2|V_2|$. The smallest weight of an OcFRDF on G denoted by $\tilde{\gamma}_{cFR}(G)$ is called *outer-connected fair Roman domination number* and defined as $\tilde{\gamma}_{cFR}(G) = \min\{\tilde{\omega}_G^{cFR}(\lambda) : \lambda \text{ is an OcFRDF on } G\}$. Every OcFRDF λ on graph G with $\tilde{\omega}_G^{cFR}(\lambda) = \tilde{\gamma}_{cFR}(G)$ is called a $\tilde{\gamma}_{cFR}$ -function on G . In this paper, the authors introduce a new restricted variant of a Roman domination in graphs called outer-connected fair Roman domination. Some important theoretical properties of an outer-connected fair Roman dominating function in various classes of connected graphs, bounds of outer-connected fair

Roman domination number, the realization problem, and some characterizations were also discussed.

2. MAIN RESULTS

This section presents some results of an outer-connected fair Roman dominating function on any connected graph.

Proposition 2.1. *Let G be a connected graph of order n where $n \in \mathbb{N}$ and let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on G . Then $|V_0| = |V_2| \geq 0$ if and only if $\tilde{\gamma}_{cFR}(G) = n$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on a connected graph G . Assume that $|V_0| = |V_2| \geq 0$. Then we have $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| = |V_0| + |V_1| + |V_2| = |V(G)| = n$. Conversely, suppose $\tilde{\gamma}_{cFR}(G) = n$. Seeking a contradiction. Assume for a moment that $|V_0| \neq |V_2|$. Then either $|V_0| < |V_2|$ or $|V_0| > |V_2|$. If $|V_0| < |V_2|$, then $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| > |V_0| + |V_1| + |V_2| = |V(G)| = n$. A contradiction. Now, if $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| < |V_0| + |V_1| + |V_2| = |V(G)| = n$. This is again a contradiction to our assumption. Consequently, we conclude that $|V_0| = |V_2| \geq 0$. This completes the proof. $\square$

As illustration of Proposition 2.1., we let $G = P_n = [v_1, v_2, \dots, v_n]$ where $n \geq 5$ and let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on G . Then we set $V_0 = \{v_2, v_3\}$, $V_1 = \{v_5, v_6, \dots, v_n\}$, and $V_2 = \{v_1, v_4\}$. In this case, $|V_0| = |V_2|$. Note that $|N_G(v_2) \cap V_2| = |N_G(v_3) \cap V_2| = 1$ and $\langle V_0 \rangle$ is a connected subgraph of G . By construction, we conclude that $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| = |\{v_5, v_6, \dots, v_n\}| + 2|\{v_1, v_4\}| = (n - 4) + 2(2) = n$. The same argument can be applied if $G = C_n$ where $n \geq 5$. Hence, the same findings we get if we invoke Proposition 2.1 for graph $G = P_n$ or $G = C_n$ where $n \geq 5$. The following result below is a consequence of Proposition 2.1 and the definition of outer-connected fair Roman dominating function on any connected graph.

Proposition 2.2. *Let G be a connected graph with $|V(G)| = n$ where $n \in \mathbb{N}$ and let $\lambda = (V_0, V_1, V_2)$ be $\tilde{\gamma}_{cFR}$ -function on G . Then $\tilde{\gamma}_{cFR}(G) < n$ if and only if $1 \leq |V_2| < |V_0|$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on a connected graph G of order n . Suppose $\tilde{\gamma}_{cFR}(G) < n$. Then $V_0 \neq \emptyset$, and so, $|V_2| \geq 1$. By Proposition 2.1., $|V_0| \neq |V_2|$. If $|V_2| > |V_0|$, then $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| > |V_1| + |V_2| + |V_0| = |V(G)| = n$, a contradiction since λ is a $\tilde{\gamma}_{cFR}$ -function on G . Thus, we can conclude that $1 \leq |V_2| < |V_0|$. As for the converse, suppose $1 \leq |V_2| < |V_0|$. Then it follows that $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| < |V_1| + |V_2| + |V_0| = |V(G)| = n$. This completes the proof. $\square$

The following propositions are vital properties of OcFRDF on a connected graph G .

Proposition 2.3. *Let G be a connected graph. If $\lambda = (V_0, V_1, V_2)$ is a $\tilde{\gamma}_{cFR}$ -function on G , then $V_1 \cup V_2$ is an outer-connected fair dominating set on G .*

Proof. Assume that $\lambda = (V_0, V_1, V_2)$ is a $\tilde{\gamma}_{cFR}$ -function on a connected graph G . Then it implies that λ is an OcFRDF on G . Hence, for every $x \in V_0$, there exists $y \in V_2$ such that $xy \in E(G)$ and for every $u, v \in V_0$, $|N_G(u) \cap V_2| = |N_G(v) \cap V_2| \geq 1$ and the subgraph $\langle V_0 \rangle$ is connected. Therefore, we conclude that $V_1 \cup V_2$ is an outer-connected fair dominating set on G . This completes the proof. $\square$

Proposition 2.4. *Let G be a connected graph and let $\lambda = (V_0, V_1, V_2)$ be an OcFRDF on G such that $V_1 = \emptyset$. Then λ is $\tilde{\gamma}_{cFR}$ -function on G if and only if V_2 is a $\tilde{\gamma}_{cF}$ -set on G .*

Proof. Let $\lambda = (V_0, V_1 = \emptyset, V_2)$ be an OcFRDF on a connected graph G .

($\Rightarrow$) Suppose λ is $\tilde{\gamma}_{cFR}$ -function on G . Assume for a moment that V_2 is not a $\tilde{\gamma}_{cF}$ -set on G . Then it implies that there exists $\tilde{\gamma}_{cF}$ -set V'_2 on G such that $V(G) \setminus V'_2$ is a fair vertex set on G and the subgraph $\langle V(G) \setminus V'_2 \rangle$ is connected. So, we set $W_0 = V(G) \setminus V'_2$, $W_1 = \emptyset$, and $W_2 = V'_2$. Then it suffices to say that $f = (W_0, W_1, W_2)$ is an OConRDF on G . Now, $\tilde{\omega}_G^{cFR}(f) = |W_1| + 2|W_2| = 2|V'_2| < 2|V_2| = |V_1| + 2|V_2| = \tilde{\omega}_G^{cFR}(\lambda) = \tilde{\gamma}_{cFR}(G)$. Hence, $\tilde{\omega}_G^{cFR}(f) < \tilde{\gamma}_{cFR}(G)$. This is a contradiction. Therefore, V_2 is a $\tilde{\gamma}_{cF}$ -set on G .

($\Leftarrow$) Suppose V_2 is a $\tilde{\gamma}_{cF}$ -set on G . Then $V_0 = V(G) \setminus V_2$ is a fair vertex set on G and the subgraph $\langle V_0 \rangle$ is connected. Assume for a moment that $\lambda = (V_0, V_1, V_2)$ is an OcFRDF on G but not a $\tilde{\gamma}_{cFR}$ -function. Then it means that there exists $\tilde{\gamma}_{cFR}$ -function $g = (U_0, U_1 = \emptyset, U_2)$ on G . Observe that $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(g) = |U_1| + 2|U_2| = 2|U_2| < 2|V_2| = |V_1| + 2|V_2| = \tilde{\omega}_G^{cFR}(\lambda)$. Thus, we obtain $|U_2| < |V_2|$. This is a contradiction to our assumption that V_2 is a $\tilde{\gamma}_{cF}$ -set on G . Therefore, we can conclude that $\lambda = (V_0, V_1, V_2)$ is a $\tilde{\gamma}_{cFR}$ -function on G . This completes the proof. $\square$

Proposition 2.5. *Let G be a connected graph and let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on G . Then $V_0 = \emptyset$ if and only if $V_2 = \emptyset$. Hence, $\tilde{\gamma}_{cFR}(G) = n$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on a connected graph G .

($\Rightarrow$) Suppose $V_0 = \emptyset$. Assume for a moment that $V_2 \neq \emptyset$. Then let $x \in V_2$. Set $U_0 = \emptyset$, $U_1 = V_1 \cup \{x\}$, and $U_2 = V_2 \setminus \{x\}$. It implies that $f = (U_0, U_1, U_2)$ is an OcFRDF on G . Observe that $\tilde{\omega}_G^{cFR}(f) = |U_1| + 2|U_2| = (|V_1| + 1) + 2(|V_2| - 1) = |V_1| + 2|V_2| - 1 < \tilde{\omega}_G^{cFR}(\lambda) = \tilde{\gamma}_{cFR}(G)$. Hence, we get $\tilde{\omega}_G^{cFR}(f) < \tilde{\gamma}_{cFR}(G)$, a contradiction. Therefore, we conclude that $V_2 = \emptyset$.

($\Leftarrow$) Conversely, suppose $V_2 = \emptyset$. Since λ is a $\tilde{\gamma}_{cFR}$ -function on G , it clearly follows that $V_0 = \emptyset$ and thus, $|V_2| = |V_0| = 0$. By Proposition 2.1., we conclude that $\tilde{\gamma}_{cFR}(G) = n$.

This completes the proof. $\square$

Proposition 2.6. *Let G be a connected graph and let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on G . If there exists $x \in V(G)$ such that $\deg_{G-V_2}(x) = 0$ and $|V_0| \geq 2$, then $x \in V_1$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on G . Suppose that there exists $x \in V(G)$ such that $\deg_{G-V_2}(x) = 0$ and $|V_0| \geq 2$. In that case, $x \in V(G) \setminus V_2$. Seeking a contradiction. Assume for a moment that $x \notin V_1$. Since $x \in V(G) \setminus V_2$, it follows that $x \in V_0$. Now, since $V_1 \cup V_2$ is an outer-connected fair dominating set

by Proposition 2.3 and $|V_0| \geq 2$, it implies that the subgraph $\langle V_0 \rangle$ is connected and so, $N_G(x) \cap (V_0 \setminus \{x\}) \neq \emptyset$. This follows that $\deg_{G-V_2}(x) \neq 0$, a contradiction to our assumption. Therefore, we conclude that $x \in V_1$. This completes the proof. $\square$

The following corollary is an immediate consequence of Proposition 2.6.

Corollary 2.7. *Let G be a connected graph of order n where $n \in \mathbb{N}$ and let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on G . If there exists $x \in V(G)$ such that for all $y \in V(G) \setminus \{x\}$, $\deg_{G-\{x\}}(y) = 0$, then $\tilde{\gamma}_{cFR}(G) = n$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on G with $|V(G)| = n$ where $n \in \mathbb{N}$. Assume that there exists $x \in V(G)$ such that for all $y \in V(G) \setminus \{x\}$, $\deg_{G-\{x\}}(y) = 0$. Then invoking Proposition 2.6., $y \in V_1$ for all $y \in V(G) \setminus \{x\}$. Since λ is a $\tilde{\gamma}_{cFR}$ -function on G , it also follows that $x \in V_1$. Therefore, we have $|V_0| = |V_2| = 0$ and by Proposition 2.1., we conclude that $\tilde{\gamma}_{cFR}(G) = n$. This completes the proof. $\square$

Remark 2.8. Let $G = S_n$ where $n \geq 3$. Then $\tilde{\gamma}_{cFR}(G) = n$.

Proof. Note that $G = S_n = K_1 + \overline{K_{n-1}}$ where $n \geq 3$. Let $x \in V(K_1)$. Then $G - \{x\} = \overline{K_{n-1}}$. In that case, for all $y \in V(\overline{K_{n-1}})$, $\deg_{G-\{x\}}(y) = \deg_{\overline{K_{n-1}}}(y) = 0$. Invoking Corollary 2.7., we end up with $\tilde{\gamma}_{cFR}(G) = n$. This completes the proof. $\square$

The following main result determines some bounds of the outer-connected fair Roman domination number of a connected graph G .

Theorem 2.9. *Let G be a connected graph with $|V(G)| = n$ where $n \in \mathbb{N}$. Then the following inequality holds:*

$$\max\{\gamma_R(G), \tilde{\gamma}_{cF}(G)\} \leq \tilde{\gamma}_{cFR}(G) \leq \min\{2\tilde{\gamma}_{cF}(G), n\}.$$

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on a connected graph G of order n where $n \in \mathbb{N}$. By Proposition 2.3, it implies that $V_1 \cup V_2$ is an outer-connected fair dominating set on G . Now, $\tilde{\gamma}_{cF}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \tilde{\gamma}_{cFR}(G)$. Then we obtain $\tilde{\gamma}_{cF}(G) \leq \tilde{\gamma}_{cFR}(G)$. Now, since every OcFRDF on graph G is an RDF on G , it simply implies that $\gamma_R(G) \leq \tilde{\gamma}_{cFR}(G)$. Thus, we have $\max\{\tilde{\gamma}_{cF}(G), \gamma_R(G)\} \leq \tilde{\gamma}_{cFR}(G)$. Let $V_0 = \emptyset$. Then by Proposition 2.5, we get $V_2 = \emptyset$. Hence, $\lambda = (\emptyset, V_1 = V(G), \emptyset)$ is an OcFRDF on G . It means that $\tilde{\gamma}_{cFR}(G) \leq \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. Thus, we can say that $\tilde{\gamma}_{cFR}(G) \leq n$ and this inequality is sharp. Let D be a $\tilde{\gamma}_{cF}$ -set on G , that is, $\tilde{\gamma}_{cF}(G) = |D|$. Set $W_0 = V(G) \setminus D$, $W_1 = \emptyset$, and $W_2 = D$. Then it implies that $\lambda^* = (W_0, W_1, W_2)$ is an OcFRDF on G , that is, $W_0 = V(G) \setminus D$ is a fair vertex set on G and the subgraph $\langle W_0 \rangle$ is connected. Thus we have $\tilde{\gamma}_{cFR}(G) \leq \tilde{\omega}_G^{cFR}(\lambda^*) = |W_1| + 2|W_2| = 2|D| = 2\tilde{\gamma}_{cF}(G)$. Accordingly, it implies that $\tilde{\gamma}_{cFR}(G) \leq \min\{n, 2\tilde{\gamma}_{cF}(G)\}$. Therefore, the conclusion follows. $\square$

Theorem 2.10. *Let G be a connected graph. Then $\tilde{\gamma}_{cFR}(G) = 1$ if and only if $G = K_1$.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on G .

($\Rightarrow$) Suppose $\tilde{\gamma}_{cFR}(G) = 1$. Assume for a moment that $G \neq K_1$. Then we have $|V(G)| > 1$. If $V_0 \neq \emptyset$, then by Proposition 2.5, $V_2 \neq \emptyset$. This implies that $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| > 1$. This is a contradiction. If $V_0 = \emptyset$, then $V_2 = \emptyset$ by Proposition 2.5. Hence, it means that $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| = |V(G)| > 1$. This is again a contradiction. Therefore, we conclude that $G = K_1$. ($\Leftarrow$) As for the converse, we suppose $G = K_1$. Since $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on G , it is clear that $|V_0| = |V_2| = 0$ and thus, $\tilde{\gamma}_{cFR}(G) = |V_1| = |V(K_1)| = 1$.

This completes the proof. $\square$

The *join* of two graphs G and H denoted by $G + H$ is a graph with vertex set given by $V(G + H) = V(G) \cup V(H)$ and edge set given by $E(G + H) = E(G) \cup E(H) \in \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Theorem 2.11. *Let G be a connected graph with $|V(G)| \geq 2$. Then $\tilde{\gamma}_{cFR}(G) = 2$ if and only if $G = K_1 + H$ where H is any connected graph.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on a connected graph G of order $n \geq 2$.

($\Rightarrow$) Assume that $\tilde{\gamma}_{cFR}(G) = 2$. Then $|V_1| + 2|V_2| = 2$ and it follows that $|V_2| \leq 1$. Suppose that if $|V_2| = 0$, then we obtain $V_0 = \emptyset$ by Proposition 2.5. Thus we get $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| = |V_1| = |V(G)| = 2$. Since G is a connected graph, it implies that $G = K_2 = K_1 + K_1$. Let $H = K_1$. Then H is a trivial connected graph. Now, suppose $|V_2| = 1$. Since $\tilde{\gamma}_{cFR}(G) = 2$, it implies that $V_1 = \emptyset$. Let $u \in V_2 = V(K_1)$. Then $G = K_1 + \langle V(G) \setminus \{u\} \rangle$ and so, $V_0 = \langle V(G) \setminus \{u\} \rangle$. Since λ is a $\tilde{\gamma}_{cFR}$ -function on G , then for every $x, y \in V_0$, $|N_G(x) \cap V_2| = |N_G(y) \cap V_2| = 1$ and the subgraph $\langle V_0 \rangle$ is connected. Hence, we can set $H = \langle V_0 \rangle$. Therefore, we can conclude that $G = K_1 + H$ where H is a connected graph.

($\Leftarrow$) Conversely, we suppose $G = K_1 + H$ where H is any connected graph. Set $V_1 = \emptyset$ and $V_2 = V(K_1)$. Since for every $u, v \in V(H)$, $|N_G(u) \cap V_2| = |N_G(v) \cap V_2| = 1$ and H is connected, it follows that $V_0 = V(H)$. In this case, it is easy to see that V_2 is a $\tilde{\gamma}_{cF}$ -set on G . Therefore, by Proposition 2.4, we end up with $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| = 2|V(K_1)| = 2$.

This completes the proof. $\square$

Corollary 2.12. *If $G \in \{K_n, F_n, W_n\}$ where $n \geq 3$, then $\tilde{\gamma}_{cFR}(G) = 2$.*

Proof. Assume that $G \in \{K_n, F_n, W_n\}$ where $n \geq 3$. Then it follows that $G \in \{K_1 + K_{n-1}, K_1 + P_n, K_1 + C_n\}$ where $n \geq 3$. By Theorem 2.11., the conclusion follows. $\square$

Theorem 2.13. *Let $G \neq K_1 + H$ where H is any connected graph and let $\lambda = (V_0, V_1 = \emptyset, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on G . Then $\tilde{\gamma}_{cFR}(G) = 4$ if and only if there exists a dominating set $\{x, y\}$ such that $V(G) \setminus \{x, y\}$ is a fair vertex set and $\langle V(G) \setminus \{x, y\} \rangle$ is connected.*

Proof. Let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on $G \neq K_1 + H$ where H is any connected graph with $V_1 = \emptyset$.

($\Rightarrow$) Suppose $\tilde{\gamma}_{cFR}(G) = 4$. Since $V_1 = \emptyset$, it follows that $|V_2| = 2$. By Proposition 2.4., it implies that V_2 is a $\tilde{\gamma}_{cF}$ -set on G . So, it follows that V_2 is an outer-connected fair dominating set on G . In that case, V_0 is a fair vertex set and $\langle V_0 \rangle$ is connected. Let $V_2 = \{x, y\}$. Thus, $\{x, y\}$ is a dominating set and $V_0 = V(G) \setminus \{x, y\}$ is a fair vertex set and $\langle V(G) \setminus \{x, y\} \rangle$ is a connected subgraph of G .

($\Leftarrow$) Conversely, suppose there exists a dominating set $\{x, y\}$ such that $V(G) \setminus \{x, y\}$ is a fair vertex set on G and $\langle V(G) \setminus \{x, y\} \rangle$ is a connected graph. Then $F = \{x, y\}$ is an outer-connected fair dominating set on G . Since $\lambda = (V_0, V_1 = \emptyset, V_2)$ is a $\tilde{\gamma}_{cFR}$ -function on G , it follows that V_2 is a $\tilde{\gamma}_{cF}$ -set on G by Proposition 2.4. And so, we have $2 = |F| \geq |V_2|$. Now, if $|V_2| = 1$, then in view of Theorem 2.11, we obtain $G = K_1 + H$ where H is any connected graph, contrary to our assumption. This follows that $|V_2| \geq 2$. Accordingly, $|V_2| = 2$ and we conclude that, $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| = 2|V_2| = 4$.

This completes the proof. $\square$

Corollary 2.14. *If $G = K_{m,n}$ where $m, n \geq 2$, then $\tilde{\gamma}_{cFR}(G) = 4$.*

Proof. Assume that $G = K_{m,n} = (U, V, E)$ where U and V are vertex sets, that is, $|U| = m \geq 2$ and $|V| = n \geq 2$, and E is the edge set of G . Let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on G , and let $U = \{u_1, u_2, \dots, u_m\}$ and $V = \{v_1, v_2, \dots, v_n\}$. Since $U \subseteq N_G[v_j]$ for any $j \in \{1, \dots, n\}$ and $V \subseteq N_G[u_i]$ for any $i \in \{1, \dots, m\}$, we can set $D = \{u_1, v_1\}$ as dominating set on G , that is, $V(G) = N_G[D]$. Observe that $|N_G(u_i) \cap D| = |N_G(v_j) \cap D| = 1$ for all $i \in \{2, \dots, m\}$ and $j \in \{2, \dots, n\}$. It follows that $V(G) \setminus D$ is a fair vertex set on G . Additionally, we can easily check that $\langle V(G) \setminus D \rangle$ is a connected subgraph of G . Therefore, invoking Theorem 2.13., we conclude that $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| = 2|D| = 4$. This completes the proof. $\square$

Theorem 2.15. *Let G be a connected graph of order n where $n \in \mathbb{N}$ and let $\lambda = (V_0, V_1 = \emptyset, V_2)$ be an OcFRDF on G . Then λ is a $\tilde{\gamma}_{cFR}$ -function on G and $\tilde{\gamma}_{cFR}(G) = \tilde{\gamma}_{cF}(G) + 1$ if and only if there exists $x \in V(G)$ such that $\deg_G(x) = n - \tilde{\gamma}_{cF}(G)$.*

Proof. Let $\lambda = (V_0, V_1 = \emptyset, V_2)$ be an OcFRDF on connected graph G .

($\Rightarrow$) Suppose λ is a $\tilde{\gamma}_{cFR}$ -function on G and $\tilde{\gamma}_{cFR}(G) = \tilde{\gamma}_{cF}(G) + 1$. This implies that $|V_1| + 2|V_2| = \tilde{\gamma}_{cF}(G) + 1$. Now, since $V_1 = \emptyset$, it means that V_2 is a $\tilde{\gamma}_{cF}$ -set on G by Proposition 2.4. Thus, we obtain $\tilde{\gamma}_{cF}(G) = |V_2|$. Hence, $|V_2| + 1 = 2|V_2|$ and so, $|V_2| = 1$. Let $V_2 = \{x\}$. Since V_2 is a $\tilde{\gamma}_{cF}$ -set on G , it follows that $V(G) = N_G[x]$ and $V(G) \setminus \{x\}$ is a fair vertex set on G , and $\langle V(G) \setminus \{x\} \rangle$ is a connected subgraph. Therefore, we conclude that $\deg_G(x) = N_G(x) = |V(G) \setminus \{x\}| = |V(G)| - |\{x\}| = n - |V_2| = n - \tilde{\gamma}_{cF}(G)$.

($\Leftarrow$) Suppose there exists $x \in V(G)$ such that $\deg_G(x) = n - \tilde{\gamma}_{cF}(G)$. Let $V_2 = \{x\}$. Since $V_1 = \emptyset$, it follows that $|V_2| = \tilde{\gamma}_{cF}(G) = n - \deg_G(x)$, that is, V_2 is a $\tilde{\gamma}_{cF}$ -set on G . This implies that $V(G) = N_G[V_2]$. Since V_2 is an outer-connected fair dominating set on G , it means that $V(G) \setminus V_2$ is a fair vertex set on G and $\langle V(G) \setminus V_2 \rangle$ is connected. Let $V_0 = V(G) \setminus V_2$. Since V_2 is a $\tilde{\gamma}_{cF}$ -set on

G and $V_1 = \emptyset$, by Proposition 2.4., it follows that $\lambda = (V_0, V_1 = \emptyset, V_2)$ is a $\tilde{\gamma}_{cFR}$ -function on G . Therefore, we conclude that $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| = |V_2| + |V_2| = \tilde{\gamma}_{cF}(G) + 1$.

This completes the proof. $\square$

The next result is a realization problem of OcFRDF and RDF on a connected graph G .

Theorem 2.16. *Let p, q and n be positive integers for which $1 \leq p \leq q \leq n$. Then there exists a connected graph G of order n such that $\gamma_R(G) = p$ and $\tilde{\gamma}_{cFR}(G) = q$.*

Proof. Let G be a connected graph with $|V(G)| = n$. Assume that $f = (W_0, W_1, W_2)$ is a γ_R -function on G and $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on G . Then consider the following three cases below:

Case 1. $1 \leq p \leq q = n$

Let $G = K_1$. Then by Theorem 2.10., we obtain $1 = \gamma_R(G) = \tilde{\gamma}_{cFR}(G) = |V(G)| = n$. Let $\gamma_R(G) = p = q = \tilde{\gamma}_{cFR}(G) = 1$. Then $1 = p = q = n$. In addition, consider the graph $G = K_1 + \overline{K_{n-1}}$ where $n \geq 3$. Then we set $W_0 = V(\overline{K_{n-1}})$, $W_1 = \emptyset$, and $W_2 = V(K_1)$. So, we have $\gamma_R(G) = \omega_G^R(f) = |W_1| + 2|W_2| = 2|V(K_1)| = 2$. Note that a trivial graph is a connected graph. Hence, we set $V_0 = V(K_1)$, $V_1 = V(\overline{K_{n-1}} \setminus \{u\})$, and $V_2 = \{u\}$. This implies that $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| = |V(\overline{K_{n-1}} \setminus \{u\})| + 2|\{u\}| = (n-2) + 2 = n$. Let $\gamma_R(G) = 2 = p$ and $\tilde{\gamma}_{cFR}(G) = n = q$. Since $n \geq 3$, we have $1 < p < q = n$. Therefore, Case 1 holds.

Case 2. $1 < p < q < n$

Let k and m be positive integer for which $k \geq 1$ and $m \geq 2$ and let G be a connected graph such that $V(G) = V(\overline{K_k}) \cup V(K_1) \cup V(P_m)$ and $N_G[V(\overline{K_k})] \cap N_G[V(P_m)] = V(K_1)$. It follows that $n = k + m + 1$ and $V(G) = N_G[V(K_1)]$. Thus, we can set $W_0 = V(\overline{K_k}) \cup V(P_m)$, $W_1 = \emptyset$, and $W_2 = V(K_1)$. Hence, we obtain $\gamma_R(G) = \omega_G^R(f) = |W_1| + 2|W_2| = 2|V(K_1)| = 2$. Moreover, let $V_2 = V(K_1)$. Since for every $v \in V(\overline{K_k})$, $\deg_{G-V_2}(v) = 0$, invoking Proposition 2.6, we get $V(\overline{K_k}) \subseteq V_1$. Also, since for every $x, y \in V(P_m)$, $|N_G(x) \cap V_2| = |N_G(y) \cap V_2| = 1$ and $\langle V(P_m) \rangle = P_m$ is a path graph, hence, connected, it implies that $V_1 \subseteq V(\overline{K_k})$ and $V_0 = V(P_m)$. In that case, $V_1 = V(\overline{K_k})$. Thus, $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| = |V(\overline{K_k})| + 2|V(K_1)| = k + 2$. Since $m \geq 2$, it follows that $\tilde{\gamma}_{cFR}(G) = k + 2 < k + m + 1 = n$. Now, let $\gamma_R(G) = 2 = p$ and $\tilde{\gamma}_{cFR}(G) = k + 2 = q$. Therefore, $1 < p < q < n$ and Case 2 holds.

Case 3. $1 < p = q < n$

Consider $G = K_2 + C_{n-2}$ where $n \geq 5$. Since $V(G) = N_G[V(K_2)]$, we can set $W_0 = V(C_{n-2})$, $W_1 = \emptyset$, and $W_2 = V(K_2)$. Thus, we have $\gamma_R(G) = \omega_G^R(f) = |W_1| + 2|W_2| = 2|V(K_2)| = 4$. Moreover, since for every $x, y \in V(C_{n-2})$, $|N_G(x) \cap V(K_2)| = |N_G(y) \cap V(K_2)| = 2$ and $\langle V(C_{n-2}) \rangle = C_{n-2}$ is a cycle graph, and so, connected. Hence, we can set $V_0 = V(C_{n-2})$, $V_1 = \emptyset$, and $V_2 = V(K_2)$. Hence, $\tilde{\gamma}_{cFR}(G) = \tilde{\omega}_G^{cFR}(\lambda) = |V_1| + 2|V_2| = 2|V(K_2)| = 4 < n$. Let $p = \gamma_R(G) = 4 = \tilde{\gamma}_{cFR}(G) = q$. Therefore, $1 < p = q < n$ and Case 3 holds.

This completes the proof. $\square$

The corollary below is a direct consequence of Theorem 2.16.

Corollary 2.17. *Let G be a connected graph. Then the difference $\tilde{\gamma}_{cFR}(G) - \gamma_R(G)$ can be made arbitrarily large.*

Proof. Let k be a positive integer. Then by Theorem 2.16, there exists a connected graph G with $|V(G)| = n$ such that $\tilde{\gamma}_{cFR}(G) = k + 2 < n$ and $\gamma_R(G) = 2$. Hence, it implies that $\tilde{\gamma}_{cFR}(G) - \gamma_R(G) = k$. Therefore, the difference $\tilde{\gamma}_{cFR}(G) - \gamma_R(G)$ can be made arbitrarily large. This completes the proof. $\square$

Remark 2.18. Let G be a connected graph of order n . Then the difference $n - \tilde{\gamma}_{cFR}(G)$ can be made arbitrarily large.

The following result is a characterization of an outer-connected fair Roman dominating function in the join of two connected graphs.

Theorem 2.19. *Let G and H be connected graphs. Then $\lambda = (V_0, V_1, V_2)$ is an OcFRDF on $G + H$ if and only if one of the following holds:*

- (i.) $V_1 \cup V_2 = V_1^G \cup V_2^G$; or
- (ii.) $V_1 \cup V_2 = V_1^H \cup V_2^H$; or
- (iii.) $V_1 \cup V_2 = (V_1^G \cup V_2^G) \cup (V_1^H \cup V_2^H)$.

where $\lambda|_G = (V_0^G, V_1^G, V_2^G)$ is an OcFRDF on G such that for all $x \in V_0^G$, $V_2^G \subseteq N_G(x)$ and $\lambda|_H = (V_0^H, V_1^H, V_2^H)$ is an OcFRDF on H such that for all $y \in V_0^H$, $V_2^H \subseteq N_H(y)$, for which $V_i^G = V_i \cap V(G)$ and $V_i^H = V_i \cap V(H)$ for each $i \in \{0, 1, 2\}$.

Proof. Let $\lambda = (V_0, V_1, V_2)$ be an RDF on $G + H$ where G and H are connected graphs.

($\Rightarrow$) Assume that λ is an OcFRDF on $G + H$. Set $F_G = (V_1 \cup V_2) \cap V(G)$ and $F_H = (V_1 \cup V_2) \cap V(H)$. Let $x \in V_0$. Then $x \in V(G) \setminus F_G$ or $x \in V(H) \setminus F_H$. Suppose $x \in V(G) \setminus F_G$. Since λ is an OcFRDF on $G + H$, there exists $y \in V_2$ such that $xy \in E(G + H)$, and $V(G) \setminus F_G$ is a fair vertex set on G and $\langle V(G) \setminus F_G \rangle$ is a connected subgraph of G . This follows that for all $v \in V(G) \setminus F_G$, $V_2^G \subseteq N_G(v)$. In that case, either $y \in F_G$ or $y \in F_H$. So, if $y \in F_G$, then it means that F_G is an outer-connected fair dominating set on G . Hence, we can let $V_0^G = V(G) \setminus F_G$, $V_1^G = F_G \setminus V_2$ and $V_2^G = F_G \setminus V_1$. This follows that for each $i \in \{0, 1, 2\}$, $V_i^G = V_i \cap V(G)$. Thus, $\lambda|_G = (V_0^G, V_1^G, V_2^G)$ is an OcFRDF on G . Observe $V(G + H) = N_{G+H}[V_1^G \cup V_2^G]$ and $V_0^G \cup V(H)$ is a fair vertex set on $G + H$ and $\langle V_0^G \cup V(H) \rangle$ is a connected subgraph of $G + H$. This follows that $V_1 \cup V_2 = V_1^G \cup V_2^G$ which indicates that (i.) holds. Meanwhile, we can set $V_0^H = V(H) \setminus F_H$, $V_1^H = F_H \setminus V_2$ and $V_2^H = F_H \setminus V_1$. By similar argument, it implies that for each $i \in \{0, 1, 2\}$, $V_i^H = V_i \cap V(H)$ and it follows that $\lambda|_H = (V_0^H, V_1^H, V_2^H)$ is an OcFRDF on H . And as we can also observe, $V(G + H) = N_{G+H}[V_1^H \cup V_2^H]$ and $V_0^H \cup V(G)$ is a fair vertex set on $G + H$ and $\langle V_0^H \cup V(G) \rangle$ is a connected subgraph of $G + H$. This means that $V_1 \cup V_2 = V_1^H \cup V_2^H$ and consequently, (ii.) holds. On the other hand, if $u \in F_H$, then $V_0^G \cup V_0^H$ is a fair vertex set on $G + H$ and $\langle V_0^G \cup V_0^H \rangle$ is a connected subgraph of $G + H$. Hence, we obtain $V_1 \cup V_2 = (V_1^G \cup V_2^G) \cup (V_1^H \cup V_2^H)$. Therefore, we can conclude that (iii.) holds. ($\Leftarrow$) Assume that either (i.) or (ii.) or (iii.) hold. Since $\lambda|_G = (V_0^G, V_1^G, V_2^G)$ is an OcFRDF on G such that for all $x \in V_0^G$, $V_2^G \subseteq N_G(x)$ and $\lambda|_H = (V_0^H, V_1^H, V_2^H)$

is an OcFRDF on H such that for all $y \in V_0^H, V_2^H \subseteq N_G(y)$, for which $V_i^G = V_i \cap V(G)$ and $V_i^H = V_i \cap V(H)$ for each $i \in \{0, 1, 2\}$, we can therefore conclude that $\lambda = (V_0, V_1, V_2)$ is an OcFRDF on $G + H$. This completes the proof. $\square$

The following corollary is a direct consequence of Theorem 2.19.

Corollary 2.20. *Let $\lambda = (V_0, V_1, V_2)$ be an RDF on $G + H$ where G and H are connected graphs. Let $\lambda|_G = (V_0^G, V_1^G, V_2^G)$ be an RDF on G such that for all $x \in V_0^G, V_2^G \subseteq N_G(x)$ and $\lambda|_H = (V_0^H, V_1^H, V_2^H)$ is an RDF on H such that for all $y \in V_0^H, V_2^H \subseteq N_G(y)$. Then $\tilde{\gamma}_{cFR}(G + H) = \min\{\tilde{\gamma}_{cFR}(G), \tilde{\gamma}_{cFR}(H)\}$.*

Proof. Let G and H be connected graphs and let $\lambda = (V_0, V_1, V_2)$ be an RDF on $G + H$. Assume that $\lambda|_G = (V_0^G, V_1^G, V_2^G)$ be an RDF on G such that for all $x \in V_0^G, V_2^G \subseteq N_G(x)$ and $\lambda|_H = (V_0^H, V_1^H, V_2^H)$ is an RDF on H such that for all $y \in V_0^H, V_2^H \subseteq N_G(y)$. Suppose $\lambda|_G = (V_0^G, V_1^G, V_2^G)$ is a $\tilde{\gamma}_{cFR}$ -function on G such that for all $x \in V_0^G, V_2^G \subseteq N_G(x)$. Then $V(G + H) \setminus (V_1^G \cup V_2^G)$ is a fair vertex set and the subgraph $\langle V(G + H) \setminus (V_1^G \cup V_2^G) \rangle$ is connected. Since $V(G + H) = N_{G+H}[V_1^G \cup V_2^G]$, we set $V_j = V_j^G$ for each $j \in \{0, 1, 2\}$. In view of Theorem 2.19, it implies $\lambda = (V_0, V_1, V_2)$ is an OcFRDF on $G + H$. Thus, we have $\tilde{\gamma}_{cFR}(G + H) \leq \tilde{\omega}_{G+H}^{cFR}(\lambda) = |V_1| + 2|V_2| = |V_1^G| + 2|V_2^G| = \tilde{\omega}_G^{cFR}(\lambda|_G) = \tilde{\gamma}_{cFR}(G)$. On the other hand, let $\lambda|_H = (V_0^H, V_1^H, V_2^H)$ be a $\tilde{\gamma}_{cFR}$ -function on H such that for all $y \in V_0^H, V_2^H \subseteq N_G(y)$. Then this implies that $V(G + H) \setminus (V_1^H \cup V_2^H)$ is a fair vertex set and the subgraph $\langle V(G + H) \setminus (V_1^H \cup V_2^H) \rangle$ is connected. Invoking Theorem 2.19 and by similar argument, it follows that $\tilde{\gamma}_{cFR}(G + H) \leq \tilde{\omega}_H^{cFR}(\lambda|_H) = \tilde{\gamma}_{cFR}(H)$. Accordingly, we get $\tilde{\gamma}_{cFR}(G + H) \leq \min\{\tilde{\gamma}_{cFR}(G), \tilde{\gamma}_{cFR}(H)\}$. Define $l = \min\{\tilde{\gamma}_{cFR}(G), \tilde{\gamma}_{cFR}(H)\}$. Let $\lambda = (V_0, V_1, V_2)$ be a $\tilde{\gamma}_{cFR}$ -function on $G + H$. Using the idea of Theorem 2.19, we can have $\lambda|_G = (V_0 \cap V(G), V_1 \cap V(G), V_2 \cap V(G))$ is an OcFRDF on G such that for all $x \in V_0 \cap V(G), V_2 \cap V(G) \subseteq N_G(x)$ and $\lambda|_H = (V_0 \cap V(H), V_1 \cap V(H), V_2 \cap V(H))$ is OcFRDF on H such that for all $y \in V_0 \cap V(H), V_2 \cap V(H) \subseteq N_G(y)$. Now, since $V(G + H) = N_{G+H}[(V_1 \cup V_2) \cap V(G)]$ and $V(G + H) \setminus ((V_1 \cup V_2) \cap V(G))$ is a fair vertex set and the subgraph $\langle V(G + H) \setminus ((V_1 \cup V_2) \cap V(G)) \rangle$ is connected, it implies that $\tilde{\gamma}_{cFR}(G + H) = \tilde{\omega}_{G+H}^{cFR}(\lambda) = |V_1| + 2|V_2| = |V_1 \cap V(G)| + 2|V_2 \cap V(G)| = \tilde{\omega}_G^{cFR}(\lambda|_G) \geq \tilde{\gamma}_{cFR}(G) \geq l$. In a similar manner, we obtain $\tilde{\gamma}_{cFR}(G + H) \geq \tilde{\gamma}_{cFR}(H) \geq l$. Accordingly, we get $\tilde{\gamma}_{cFR}(G + H) \geq \min\{\tilde{\gamma}_{cFR}(G), \tilde{\gamma}_{cFR}(H)\}$. Therefore, we conclude that $\tilde{\gamma}_{cFR}(G + H) = \min\{\tilde{\gamma}_{cFR}(G), \tilde{\gamma}_{cFR}(H)\}$. This completes the proof. $\square$

Remark 2.21. Let G and H be complete graphs. Then $\tilde{\gamma}_{cFR}(G + H) = 2$.

3. CONCLUSION

This paper has introduced a new restricted parameter of domination in graphs involving the Roman dominating function, which is called the outer-connected fair Roman dominating function. Important combinatorial properties of outer-connected fair Roman dominating functions in some classes of graphs were discussed, including some bounds on outer-connected fair Roman domination number, characterization, and the realization problem. This paper suggests future

research to characterize the outer-connected fair Roman dominating function under some binary operations, including vertex corona, edge corona, Cartesian, and lexicographic products, to strengthen the current results.

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