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$F-$INDEX AND HYPER-ZAGREB INDEX OF GENERALIZED MIDDLE GRAPHS

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Abstract. Graph invariants are numerical quantities connected with molecular graphs, and topological indices are one of them. Molecular descriptors are what they're called in mathematical chemistry. Quantitative structure-property relationship (QSPR) and quantitative structure-activity relationship (QSAR) studies rely primarily on topological indices in mathematical chemistry. The F-index, or forgotten topological index, for a molecular graph G is equal to the sum of cubes of degrees of all vertices of the graph, while the hyper-Zagreb index is equal to the square of sum of degrees of all adjacent vertices of the graph. We obtain F-index, hyper-Zagreb index, and coindex values for generalised middle graphs G^{xy} and their complements of $\overline{G^{xy}}$ in this study.

1. Introduction

In this paper, we are concerned with simple graphs, having no directed or weighted edges and no loops. Let G be a finite graph with n vertices and m edges is called (n, m) graph. We denote vertex set and edge set of graph G as $V(G)$ and $E(G)$, respectively. The *neighbourhood* of a vertex $u \in V(G)$ is defined as the set $N_G(u)$ consisting of all vertices v which are adjacent to u in G. The *degree* of a vertex $u \in V(G)$, denoted by $d_G(u)$ and is equal to $|N_G(u)|$. If u and v are two adjacent vertices of G, then the edge connecting them will be denoted by $d_G(e)$, and is defined by $d_G(e) = d_G(u)+d_G(v)-2$ with $e = uv$, and $e \sim f(e \nsim f)$ means that the edges e and f are adjacent(not adjacent). The *complement of a graph* G is denoted by $\overline{G}$ and is defined as the graph whose vertex set is $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G. Obviously, $\overline{G}$ has n vertices and $\binom{n}{2} - m$ edges. The line graph $L(G)$ of a graph G is the graph with vertex set $E(G)$ and two vertices are adjacent in $L(G)$ if and only if the corresponding edges in G are adjacent. The line graph $L(G)$ has order $n_L = m$ and size $m_L = -m + \frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$. The complement of line graph is jump graph, which is denoted by $J(G)$. The jump graph has order m and size

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$m_J = \binom{m+1}{2} - \frac{1}{2}\sum_{i=1}^{n} d_G(v_i)^2$. The subdivision graph $S(G)$ of a graph G whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is an edge of G incident with it. The partial complement of subdivision graph $S^*(G)$ of a graph G whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is an edge of G nonincident with it. For undefined graph theoretic terminologies and notations refer [23].

For a molecular graph G, *first Zagreb index* was defined by Gutman and Trinajstić [21] in 1972 as

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2.$$

The second Zagreb index was defined in [20] as

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) \cdot d_G(v).$$

The first Zagreb index [29] can also be expressed as

$$M_1(G) = \sum_{uv \in E(G)} \Big(d_G(u) + d_G(v) \Big).$$

The first and second Zagreb coindices [2] were defined respectively as

$$\overline{M_1}(G) = \sum_{uv \notin E(G)} \Big(d_G(u) + d_G(v) \Big),$$

and

$$\overline{M_2}(G) = \sum_{uv \notin E(G)} \Big(d_G(u) \cdot d_G(v) \Big).$$

For basic properties of Zagreb indices refer [18, 21] and for Zagreb indices of graph operation refer [2, 13, 25, 34]. Another degree based graph invariant called *forgotten topological index* or F- index was put forward by Furtula and Gutman [16] is defined as

$$F(G) = \sum_{v \in V(G)} d_G(v)^3 = \sum_{uv \in E(G)} \Big(d_G(u)^2 + d_G(v)^2 \Big).$$

Its coindex [15] is given by

$$\overline{F}(G) = \sum_{uv \notin E(G)} \Big(d_G(u)^2 + d_G(v)^2 \Big).$$

See [16] for basic properties and [3, 15] for F-index of graph operations.

The *hyper Zagreb index* was introduced by Shirdel et al., in [31] which is defined as

$$HM(G) = \sum_{uv \in E(G)} \Big(d_G(u) + d_G(v) \Big)^2$$

and hyper Zagreb coindex was introduced by Veylanki et al., in [33] as

$$\overline{HM}(G) = \sum_{uv \notin E(G)} \Big(d_G(u) + d_G(v)\Big)^2.$$

For basic properties of hyper Zagreb index and coindex refer [17] and for graph operations refer [5, 7, 31, 33].

The Zagreb indices interms of edge-degrees reformulated by Milićević et al. in 2004 [28] . The first and second reformulated Zagreb indices are defined respectively as

$$EM_1(G) = \sum_{e \in E(G)} d_G(e)^2.$$

and

$$EM_2(G) = \sum_{e \sim f} d_G(e) d_G(f).$$

The forgotten topological indices reformulated interms of edge-degrees. The reformulated F-index [26] is defined by,

$$EF(G) = \sum_{e \in E(G)} d_G(e)^3.$$

For topological indices' chemical applications refer to [19, 32].

2. Generalized middle graphs G^{xy}

Akiyama et al. [1] defined the middle graph $M(G)$ of G is the graph whose vertex set is $V(G) \cup E(G)$ in which two vertices u and v are adjacent if and only if at least one of u and v is an edge of G, and they are adjacent or incident in G. Note that middle graph identical to the semi-total line graph introduced by Sampathkumar et al. [30]. Inspired by the definition of middle graph of a graph, Basavanagoud et al. [4] introduced some new transformation graphs, which generalize the concept of middle graph.

Definition 2.1. Let $G = (V, E)$ be a graph and x, y be two variables taking values $+$ or $-$. The generalized middle graph G^{xy}[4] is the graph having $V(G) \cup E(G)$ as the vertex set and $\alpha, \beta \in V(G) \cup E(G)$, α and β are adjacent in G^{xy} if and only if one of the following holds:

(i) $\alpha, \beta \in E(G)$, α, β are adjacent in G if $x = +$ and α, β are not adjacent in G if $x = -$.
(ii) $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $y = +$ and α, β are not incident in G if $y = -$.

In the view of the above one can obtain four generalized middle graphs, since there are four distinct 2-permutation of $\{+, -\}$. Note that G^{++} is just the middle graph $M(G)$ of G, whereas the other generalized middle graphs are G^{+-}, G^{-+} and G^{--}. We denote the complement of the generalized middle graphs G^{xy} by $\overline{G^{xy}}$. The generalized middle graphs and their complements are shown in Figure 1. The vertex u_i of G^{xy} corresponding to a vertex of G is referred to as a *point vertex*. The vertex e of G^{xy} corresponding to an edge e_i of G is referred to as

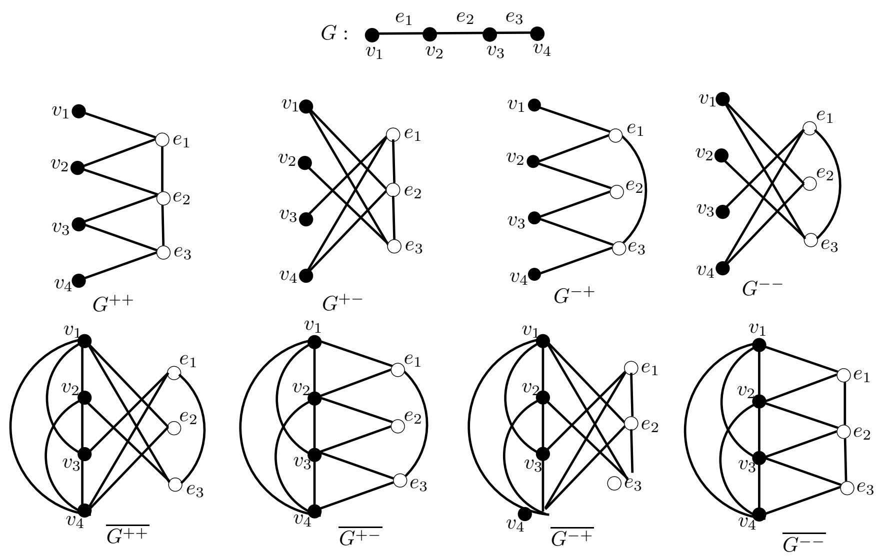

FIGURE 1. Graph G, its generalized transformations G^{xy} and their complements $\overline{G^{xy}}$.

a *line vertex*. Different topological indices of generalized middle graphs have already been studied by many researchers. Basavanagoud et al. in [10] found Zagreb indices of generalized middle graphs. This motivates us to study F-index and hyper-Zagreb index of generalized middle graphs.

3. Preliminaries

The following results are useful for proving our main results.

Proposition 3.1. [4] *Let G be an (n, m)-graph. Then the degrees of point and line vertices in G^{xy} are*

(i) $d_{G^{++}}(u_i) = d_G(u)$ *and* $d_{G^{++}}(e_i) = 2 + d_G(e)$.
(ii) $d_{G^{+-}}(u_i) = m - d_G(u)$ *and* $d_{G^{+-}}(e_i) = n - 2 + d_G(e)$.
(iii) $d_{G^{-+}}(u_i) = d_G(u)$ *and* $d_{G^{-+}}(e_i) = m + 1 - d_G(e)$.
(iv) $d_{G^{--}}(u_i) = m - d_G(u)$ *and* $d_{G^{--}}(e_i) = n + m - 3 - m - d_G(e)$.

Proposition 3.2. [4] *Let G be an (n, m)-graph. Then order of G^{xy} is $n + m$ and*

(i) *size of* $G^{++} = m + \frac{1}{2}M_1(G)$.
(ii) *size of* $G^{+-} = m(n-3) + \frac{1}{2}M_1(G)$.
(iii) *size of* $G^{-+} = \binom{m}{2} + 3m - \frac{1}{2}M_1(G)$.
(iv) *size of* $G^{--} = m(n-1) + \binom{m}{2} - \frac{1}{2}M_1(G)$.

Theorem 3.3. [22] *Let G be a graph with n vertices and m edges. Then*

$$M_1(G^{++}) = M_1(G) + 2M_2(G) + F(G).$$

Theorem 3.4. [4] *Let G be a graph with n vertices and m edges. Then*

(i) $M_1(G^{+-}) = (2n-7)M_1(G) + 2M_2(G) + [nm - 4m + (n-4)^2]m + F(G)$.
(ii) $M_1(G^{-+}) = 2M_2(G) - (2m+5)M_1(G) + m(m+3)^2 + F(G)$.

(iii) $M_1(G^{--}) = (3-2n-2m)M_1(G) + 2M_2(G) + m^2(n-4) + m(n+m-1)^2 + F(G)$.

Theorem 3.5. [4] *Let G be a graph with n vertices and m edges. Then*
$$M_2(G^{++}) = 2EM_1(G) + EM_2(G) + 2[M_1(G) + M_2(G)] + F(G) - 4m.$$

Theorem 3.6. [10] *Let G be a graph with n vertices and m edges. Then*
(i) $M_2(G^{+-}) = m^2[(n-2)(n-6)+4] - m(n-2)^2 + \frac{1}{2}M_1(G)[2(m+1)(n-4) + (n-2)^2] + F(G) + 2M_2(G) + EM_2(G) + (n-2)EM_1(G)$.
(ii) $M_2(G^{-+}) = (m+1)^2\binom{m+1}{2} - (\frac{m^2-5}{2})M_1(G) - (m+1)\overline{EM_1(G)} + \overline{EM_2(G)} - F(G) - 2M_2(G)$.
(iii) $M_2(G^{--}) = (n+m-3)^2 m_J - (n+m-3)\overline{EM_1(G)} + \overline{EM_2(G)} + m^2(n-4)(n+m-3) + 2m^2(n-4) - [m(n-5)-n+1]M_1(G) - F(G) - 2M_2(G)$.

Theorem 3.7. [9] *Let G be a simple graph. Then $HM(G) = F(G) + 2M_2(G)$.*

Theorem 3.8. [15] *Let G be a graph with n vertices and m edges. Then*
(i) $F(\overline{G}) = n(n-1)^3 - 4m(n-1)^2 + 3(n-1)M_1(G) - F(G)$.
(ii) $\overline{F}(G) = (n-1)M_1(G) - F(G)$.
(iii) $\overline{F}(\overline{G}) = 2m(n-1)^2 - 2(n-1)M_1(G) + F(G)$.

Theorem 3.9. [17] *Let G be a graph with n vertices and m edges. Then*
(i) $HM(\overline{G}) = 2n(n-1)^3 - 12m(n-1)^2 + 4m^2 + (5n-6)M_1(G) - HM(G)$.
(ii) $\overline{HM}(G) = 4m^2 + (n-2)M_1(G) - HM(G)$.

Theorem 3.10. [24] *Let G be a graph with n vertices and m edges. Then*
$$\overline{HM}(\overline{G}) = 4m(n-2)^2 - 4(n-1)M_1(G) + HM(G)$$
.

4. F-INDEX AND COINDEX OF GENERALIZED MIDDLE GRAPHS AND THEIR COMPLEMENTS

In this section, we obtain F-index and coindex of generalized middle graphs and their complements.

Theorem 4.1. *Let G be an (n,m) graph. Then*
$$F(G^{++}) = F(G) + 8m + EF(G) + 24m_L + 6EM_1(G).$$

Proof. By the definition of F- index and Proposition 3.1, we have
$$\begin{aligned}
F(G^{++}) &= \sum_{u\in V(G^{++})} d^3_{G^{++}}(u) \\
&= \sum_{u\in V(G^{+-})\cap V(G)} d^3_{G^{+-}}(u) + \sum_{e\in V(G^{++})\cap E(G)} d^3_{G^{++}}(e) \\
&= \sum_{u\in V(G)} \big(d_G(u)\big)^3 + \sum_{e\in E(G)} (2+d_G(e))^3 \\
&= F(G) + 8m + EF(G) + 24m_L + 6EM_1(G).
\end{aligned}$$

□

Using Theorems 3.2, 3.3 and 4.1 in Theorem 3.8, we get the following results for F-index of $\overline{G^{++}}$, F-coindex of G^{++} and its complement $\overline{F}(\overline{G^{++}})$, respectively.

Corollary 4.2. *Let G be an (n,m) graph. Then*

$$\begin{aligned} F(\overline{G^{++}}) &= (n+m-1)\Big(3M_1(G) - 6m(n+m-1) + n(n+m-1)^2\Big) - F(G) \\ &\quad +(n+m-3)\Big(3EM_1(G) - 6m_L(n+m-3) + m(n+m-3)^2\Big) - EF(G). \end{aligned}$$

Corollary 4.3. *Let G be an (n,m) graph. Then*

$$\begin{aligned} \overline{F}(G^{++}) &= (n+m-1)\big(M_1(G) + 2M_2(G)\big) + F(G)(n+m-2) - 8m \\ &\quad -EF(G) - 24m_L - 6EM_1(G). \end{aligned}$$

Corollary 4.4. *Let G be an (n,m) graph. Then*

$$\begin{aligned} \overline{F}(\overline{G^{++}}) &= (2m + M_1(G))(n+m-1)^2 - 2(n+m-1)[M_1(G) + 2M_2(G)] \\ &\quad -F(G)[2(n+m)-1] + 8m + EF(G) + 24m_L + 6EM_1(G). \end{aligned}$$

Theorem 4.5. *Let G be an (n,m) graph. Then*

$$\begin{aligned} F(G^{+-}) &= m^3(n-6) - F(G) + 3mM_1(G) + EF(G) \\ &\quad +(n-2)(3EM_1(G) + 6m_L(n-2) + m(n-2)^2). \end{aligned}$$

Proof. By the definition of F- index and Proposition 3.1, we have

$$\begin{aligned} F(G^{+-}) &= \sum_{u\in V(G^{+-})} d^3_{G^{+-}}(u) \\ &= \sum_{u\in V(G^{+-})\cap V(G)} d^3_{G^{+-}}(u) + \sum_{e\in V(G^{+-})\cap E(G)} d^3_{G^{+-}}(e) \\ &= \sum_{u\in V(G)} \big(m - d_G(u)\big)^3 + \sum_{e\in E(G)} (n-2+d_G(e))^3 \\ &= m^3(n-6) - F(G) + 3mM_1(G) + EF(G) \\ &\quad +(n-2)(3EM_1(G) + 6m_L(n-2) + m(n-2)^2). \end{aligned}$$

□

Using Theorems 3.2, 3.4 and 4.5 in Theorem 3.8, we get the following results for F-index of $\overline{G^{+-}}$, F-coindex of G^{+-} and its complement $\overline{F}(\overline{G^{+-}})$, respectively.

Corollary 4.6. *Let G be an (n,m) graph. Then*

$$\begin{aligned} F(\overline{G^{+-}}) &= (n-1)\Big(3M_1(G) + 6m(n-1) + n(n-1)^2\Big) + F(G) \\ &\quad +(m+1)\Big(3EM_1(G) - 6m_L(m+1) + m(m+1)^2\Big) - EF(G). \end{aligned}$$

Corollary 4.7. *Let G be an (n,m) graph. Then*

$$\begin{aligned} \overline{F}(G^{+-}) &= F(G)(n+m) + M_1(G)[-3m + (n+m-1)(2n-7)] \\ &\quad +(n+m-1)(2M_2(G) + [nm - 4m - (n-4)^2]m) \\ &\quad -m^3(n-6) - (n-2)[3EM_1(G) + 6m^2(n-2) + m(n-2)^2]. \end{aligned}$$

Corollary 4.8. *Let G be an (n,m) graph. Then*

$$\begin{aligned}\overline{F}(\overline{G^{+-}}) &= (2m(n-3)+M_1(G))(n+m-1)^2+F(G)[2(n+m)-1] \\ &\quad +M_1(G)[2(n+m-1)(2n-7)-3m]+2(n+m-1)[2m^2 \\ &\quad +(nm-4m+(n-4)^2)m]+m^3(n-6)+EF(G) \\ &\quad +(n-2)[3EM_1(G)+6m_L(n-2)+m(n-2)^2].\end{aligned}$$

Theorem 4.9. *Let G be an (n,m) graph. Then*

$$F(G^{-+}) = F(G)-EF(G)+(m+1)\big(3EM_1(G)-6m_L(m+1)+m(m+1)\big)^2.$$

Proof. By the definition of F- index and Proposition 3.1, we have

$$\begin{aligned}F(G^{-+}) &= \sum_{u\in V(G^{-+})} d^3_{G^{-+}}(u) \\ &= \sum_{u\in V(G^{-+})\cap V(G)} d^3_{G^{-+}}(u)+\sum_{e\in V(G^{-+})\cap E(G)} d^3_{G^{-+}}(e) \\ &= \sum_{u\in V(G)}(d_G(u))^3+\sum_{e\in E(G)}(m+1-d_G(e))^3 \\ &= F(G)-EF(G)+(m+1)\big(3EM_1(G)-6m_L(m+1)+m(m+1)^2\big).\end{aligned}$$

□

Using Theorems 3.2, 3.4 and 4.9 in Theorem 3.8, we get the following results for F-index of $\overline{G^{-+}}$, F-coindex of G^{-+} and its complement $\overline{F}(\overline{G^{-+}})$, respectively.

Corollary 4.10. *Let G be an (n,m) graph. Then*

$$\begin{aligned}F(\overline{G^{-+}}) &= (n+m-1)\Big(3M_1(G)-6m(n+m-1)+n(n+m-1)^2\Big)-F(G) \\ &\quad +(n-2)\Big(3EM_1(G)+6m_L(n-2)+m(n-2)^2\Big)+EF(G).\end{aligned}$$

Corollary 4.11. *Let G be an (n,m) graph. Then*

$$\begin{aligned}\overline{F}(G^{-+}) &= F(G)(n+m-2)-M_1(G)[(n+m-1)(2m+5)] \\ &\quad +(n+m-1)[2M_2(G)+m(m+3)^3]-m(m+3)^3 \\ &\quad +EF(G)+6(m+1)^2m_L-3(m+1)EM_1(G).\end{aligned}$$

Corollary 4.12. *Let G be an (n,m) graph. Then*

$$\begin{aligned}\overline{F}(\overline{G^{-+}}) &= (m^2+5m-M_1(G))(n+m-1)^2 \\ &\quad -2(n+m-1)[2M_2(G)-(2m+5)M_1(G)+m(m+3)^3] \\ &\quad +F(G)[3+2(n+m)]+m(m+1)^3-EF(G) \\ &\quad -6(m+1)^2m_L+3(m+1)EM_1(G).\end{aligned}$$

Theorem 4.13. *Let G be an (n,m) graph. Then*

$$\begin{aligned}F(G^{--}) &= m^3(n-6)-F(G)+3mM_1(G)-EF(G) \\ &\quad +(n+m-3)(3EM_1(G)-6m_L(n+m-3) \\ &\quad +m(n+m-3)^2).\end{aligned}$$

Proof. By the definition of F- index and Proposition 3.1, we have

$$\begin{aligned}
F(G^{--}) &= \sum_{u\in V(G^{--})} d^3_{G^{--}}(u) \\
&= \sum_{u\in V(G^{--})\cap V(G)} d^3_{G^{--}}(u) + \sum_{e\in V(G^{--})\cap E(G)} d^3_{G^{--}}(e) \\
&= \sum_{u\in V(G)} (d_G(u))^3 + \sum_{e\in E(G)} (m+1-d_G(e))^3 \\
&= m^3(n-6) - F(G) + 3mM_1(G) - EF(G) \\
&\quad +(n+m-3)(3EM_1(G) - 6m_L(n+m-3) + m(n+m-3)^2).
\end{aligned}$$

□

Using Theorems 3.2, 3.4 and 4.13 in Theorem 3.8, we get the following results for F-index of $\overline{G^{--}}$, F-coindex of G^{--} and its complement $\overline{F}(\overline{G^{--}})$, respectively.

Corollary 4.14. *Let G be an (n,m) graph. Then*

$$\begin{aligned}
F(\overline{G^{--}}) &= (n-1)\Big(3M_1(G) + 6m(n-1) + n(n-1)^2\Big) + F(G) \\
&\quad +6EM_1(G) + 8m + 24m_L + EF(G).
\end{aligned}$$

Corollary 4.15. *Let G be an (n,m) graph. Then*

$$\begin{aligned}
\overline{F}(G^{--}) &= (n+m-1)[(3-2n-2m)M_1(G) + 2M_2(G) \\
&\quad +m^2(n-4) + m(n+m-1)^2 + F(G)] \\
&\quad -[nm^3 - F(G) - 6m^3 + 3mM_1(G) + m(n+m-3)^3 \\
&\quad -EF(G) - 6m_L(n+m-3)^2 + 3EM(G)(n+m-3)].
\end{aligned}$$

Corollary 4.16. *Let G be an (n,m) graph. Then*

$$\begin{aligned}
\overline{F}(\overline{G^{--}}) &= [2m(n-1) + m(m-1) - M_1(G)](n+m-1)^2 \\
&\quad -2(n+m-1)[(3-2n-2m)M_1(G) + 2M_2(G) \\
&\quad +m^2(n-4) + m(n+m-1)^2 + F(G)] + [nm^3 - F(G) \\
&\quad -6m^3 + 3mM_1(G) + m(n+m-3)^3 - EF(G) \\
&\quad -6m_L(n+m-3)^2 + 3EM(G)(n+m-3)].
\end{aligned}$$

5. Hyper-Zagreb index and coindex of generalized middle graphs and their complements

In this section, we obtain hyper-Zagreb index and coindex of generalized middle graphs and their complements.

Theorem 5.1. *Let G be an (n,m) graph. Then*

i. $HM(G^{++}) = 3F(G) + EF(G) + 24m_L + 10EM_1(G) + 2EM_2(G) + 4[M_1(G) + M_2(G)]$.
ii. $HM(G^{+-}) = F(G) + EF(G) + \{3m + [2(m+1)(n-4) + (n-2)^2]\}M_1(G) + 4(n-2)EM_1(G) + 4M_2(G) + 2EM_2(G) + m^3(n-6) + (n-2)[6m_L(n-2) + m(n-2)^2] + 2[m^2[(n-2)(n-6)+4] - m(n-2)^2]$.

iii. $HM(G^{-+}) = 3(m+1)EM_1(G) - F(G) - EF(G) - 6(m+1)^2 m_L + m(m+1)^3 + 2[\frac{m(m+1)^3}{2} - (\frac{m^2-5}{2})M_1(G) - (m+1)\overline{EM_1(G)} + \overline{EM_2(G)} - 2M_2(G)]$.

iv. $HM(G^{--}) = m^3(n-6) - 3F(G) + 3m - [m(n-5) - n + 1]M_1(G) + (n+m-3)[3EM_1(G) - 6m_L(n+m-3) + m(n+m-3)^2] + 2[(n+m-3)^2 m_J - (n+m-3)\overline{EM_1(G)} + \overline{EM_2(G)} + m^2(n-4)(n+m-3) + 2m^2(n-4) - 2M_2(G)]$.

Proof. By using Theorems 3.5, 3.6, 4.1, 4.5, 4.9 and 4.13 in Theorem 3.7, we get the desired result. $\square$

Theorem 5.2. *Let G be an (n, m) graph. Then*

i. $HM(\overline{G^{++}}) = M_1(G)[5n + 3m - 10] + M_2(G)[4n - 10] + F(G)[2n - 2] + EM_1(G)[3n + 3n - 13] - 2EM_2(G) - EF(G) + (n + m - 1)^2[n(n + m - 1) - 6m] + (n + m - 3)^2[m(n + m - 3) - 6m_L] - 6m(n - 1)^2 + 4m^2 - 8m$.

ii. $HM(\overline{G^{+-}}) = F(G) + EF(G) + \{3m + [2(m+1)(n-4) + (n-2)^2]\}M_1(G) + 4(n-2)EM_1(G) + 4M_2(G) + 2EM_2(G) + m^3(n-6) + (n-2)[6m_L(n-2) + m(n-2)^2] + 2[m^2[(n-2)(n-6) + 4] - m(n-2)^2]$.

iii. $HM(\overline{G^{-+}}) = 3(m+1)EM_1(G) - F(G) - EF(G) - 6(m+1)^2 m_L + m(m+1)^3 + 2[\frac{m(m+1)^3}{2} - (\frac{m^2-5}{2})M_1(G) - (m+1)\overline{EM_1(G)} + \overline{EM_2(G)} - 2M_2(G)]$.

iv. $HM(\overline{G^{--}}) = m^3(n-6) - 3F(G) + 3m - [m(n-5) - n + 1]M_1(G) + (n+m-3)[3EM_1(G) - 6m_L(n+m-3) + m(n+m-3)^2] + 2[(n+m-3)^2 m_J - (n+m-3)\overline{EM_1(G)} + \overline{EM_2(G)} + m^2(n-4)(n+m-3) + 2m^2(n-4) - 2M_2(G)]$.

Proof. By using Theorems 3.5, 3.6, 4.1, 4.5, 4.9 and 4.13 in Theorem 3.7, we get the desired result. $\square$

Theorem 5.3. *Let G be an (n, m) graph. Then*

i. $\overline{HM}(G^{++}) = M_1(G)\Big[n + 5m - 6 + M_1(G)\Big] + 2M_2(G)(n+m) + F(G)\Big[n + m - 5\Big] + EF(G) + 10EM_1(G) + 2EM_2(G) + 24m_L + 4m^2$.

ii. $\overline{HM}(G^{+-}) = M_1(G)\Big[M_1(G) + 4m(n-3) + (n+m-2)(2n-7) - 3m - 2(m+1)(n-4) - (n-2)^2\Big] + 2M_2(G)\big[n+m\big] + F(G)(n+m-3) - EM_1(G)(4n-8) - 2EM_2(G) - EF(G) + 4m^2(n-3)^2 + m(n+m-2)\Big[nm - 4m + (n-4)^2\Big] - m^3(n-6) - (n-2)\big[6m_L(n-2) + m(n-2)^2\big] - 2\Big[m^2[(n-2)(n-6)+4] - m(n-2)^2\Big]$.

iii. $\overline{HM}(G^{-+}) = M_1(G)\Big[M_1(G) - 12m - 2m(m-1) + (m^2-5) - (n+m-2)(2m+5)\Big] + 2M_2(G)(n+m)F(G)(n+m-1) + EF(G) - 3(m+1)EM_1(G) - 2\Big[\overline{EM_2(G)} - (m+1)\overline{EM_1(G)}\Big] + m^2(m-1)^2 + 36m^2 + 12m^2(m-1) + 6m_L(m+1)^2 - 2m(m+1)^3 + m(n+m-2)(m+3)^2$.

iv. $\overline{HM}(G^{--}) = M_1(G)\Big[M_1(G) - 4m(n-1) - 2m(m-1)$
$+ (n+m-2)(3-2n-2m) + m + mn - 5m - n + 1\Big]$
$+ M_2(G)[2n+2m-8] + F(G)(n+m-5) + 3(n+m-3)EM_1(G) - 2\overline{EM_1(G)} + 4m^2(n-1)^2 + m^2(m-1)^2 + 4m^2(n-1)(m-1)$
$+ m^2(n-4)(n+m-2) + m(n+m-2)(n+m-1)^2 - m^3(n-6)$
$- 3m + (n+m-3)\Big[m(n+m-3)^2 - 6m_L(n+m-3)\Big]$
$+ 2\Big[(n+m-3)^2 m_J + m^2(n-4)(n+m-3) + 2m^2(n-4)\Big].$

Proof. By using Theorems 3.2, 3.3, 3.4, 3.7 and 5.3 in Theorem 3.9, we get the desired result. □

Theorem 5.4. *Let G be an (n,m) graph. Then*

i. $\overline{HM}(\overline{G^{++}}) = M_1(G)\Big[10 + 2(n^2+m^2) - 8(n+m) + 4nm\Big]$
$+ M_2(G)\Big[12 - 8(n+m)\Big] + F(G)\Big[7 - 4(n+m)\Big] + EF(G) + 10EM_1(G) + 2EM_2(G) + 24m_L + 4m(n+m-1)^2.$

ii. $\overline{HM}(\overline{G^{+-}}) = M_1(G)\Big[2(n+m-1)^2 - 4(n+m-1)(2n-7) + 3m$
$+ 2(m+1)(n-4) + (n-2)^2\Big] + M_2(G)\Big[12 - 8(n+m)\Big] + F(G)\Big[5 - 4(n+m)\Big]$
$+ 4(n-2)EM_1(G) + 2EM_2(G) + EF(G) + (n+m-1)^2\Big[4m(n-2)\Big]$
$- 4m(n+m-1)\Big[nm - 4m + (n-4)^2\Big] + (n-2)\Big[6m_L(n-2) + m(n-2)^2\Big]$
$+ 2\Big[m^2\big((n-2)(n-6)+4\big) - m(n-2)^2\Big].$

iii. $\overline{HM}(\overline{G^{-+}}) = M_1(G)\Big[2(n+m-1)^2 + 4(n+m-1)(2m+5) - (m^2-5)\Big]$
$- M_2(G)\Big[-4 + 8(n+m)\Big] - F(G)\Big[-3 + 4(n+m)\Big] + 3(m+1)EM_1(G)$
$- EF(G) + 2\Big[\overline{EM_2(G)} - (m+1)\overline{EM_1(G)}\Big] + (n+m-1)^2[2m^2+10m]$
$- 4m(n+m-1)(m+3)^2 - 6m_L(m+1)^2 + 2m(m+1)^3.$

iv. $\overline{HM}(\overline{G^{--}}) = M_1(G)\Big[-4(n+m-1)(3-2n-2m) - 2(n+m-1)^2$
$- [m(n-5-n+1)]\Big] + M_2(G)\Big[4 - 8(n+m)\Big] + F(G)\Big[1 - 4(n+m)\Big]$
$+ 4(n+m-1)^2\Big[m(n-1) + \frac{m(m-1)}{2}\Big] - 4(n+m-1)\Big[m^2(n-4) + m(n+m-1)^2$
$+ m^3(n-6) + 3m + (n+m-3)\Big[3EM_1(G) - 6m_L(n+m-3) + m(n+m-3)^2\Big]$
$+ 2\Big[m_J(n+m-3)^2 - (n+m-3)\overline{EM_1(G)} + \overline{EM_2(G)} + m^2(n-4)(n+m-3)$
$+ 2m^2(n-4)\Big].$

Proof. By using Theorems 3.2, 3.3, 3.4, 3.7 and 5.3 in Theorem 3.10, we get the desired result. □

6. Conclusion

In this paper, we have obtained for F- index, hyper-Zagreb index and their coindices of generalized middle graphs and their complements.

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