

STUDY OF A HYPERBOLIC SINE MODIFICATION OF THE HILBERT INTEGRAL INEQUALITY

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ABSTRACT. In this article, we study a new modification of the classical Hilbert integral inequality based on the hyperbolic sine function. The resulting upper bound involves weighted integral norms of the main functions, with hyperbolic tangent weight functions. Notably, the constant factor in this inequality remains the classical value π . From our main result, we derive two further propositions. The first proposition is the series analogue of the main integral inequality. The second proposition yields a new one-function integral inequality.

1. INTRODUCTION

Integral inequalities are essential tools in mathematical analysis. They have a wide range of applications, including in harmonic analysis, partial differential equations and approximation theory. More precisely, they provide lower or upper bounds that simplify the evaluation of complicated integrals. One of the most significant results in this field is the Hilbert integral inequality: a fundamental inequality with profound links to Fourier analysis, operator theory, and Hardy space theory.

In its standard form, the Hilbert integral inequality provides an upper bound for a bilinear integral operator with a singular kernel function defined by $k(x, y) = 1/(x + y)$. Specifically, let $f, g : [0, +\infty) \rightarrow [0, +\infty)$ be two functions. Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dx dy \leq \pi \left(\int_0^{+\infty} f^2(x) dx \right)^{1/2} \left(\int_0^{+\infty} g^2(x) dx \right)^{1/2}, \quad (1.1)$$

provided that the two integrals on the right-hand side converge. In this context, the constant factor π is known to be optimal. Further details can be found in the books [4, 10, 11] and the survey [1]. Various facets of the Hilbert integral inequality are covered in the articles [2, 3, 5, 6, 7, 8, 9].

For the purposes of this article, it is useful to recall a bounded-domain version of the Hilbert integral inequality, established in [8, Theorem 1]. Specifically, let

Date: Received: Sep 22, 2025; Revised: Oct 19, 2025; Accepted: Nov 3, 2025.

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2010 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Hilbert integral inequality, hyperbolic sine function, hyperbolic tangent function, change of variable techniques.

$\alpha > 0$ and $f, g : [0, +\infty) \rightarrow [0, +\infty)$ be two functions. Then we have

$$\begin{aligned} & \int_0^\alpha \int_0^\alpha \frac{f(x)g(y)}{x+y} dx dy \\ & \leq \pi \left(\int_0^\alpha \left(1 - \frac{1}{2} \sqrt{\frac{x}{\alpha}} \right) f^2(x) dx \right)^{1/2} \left(\int_0^\alpha \left(1 - \frac{1}{2} \sqrt{\frac{x}{\alpha}} \right) g^2(x) dx \right)^{1/2}, \end{aligned} \quad (1.2)$$

provided that the two integrals on the right-hand side converge. In this bounded-domain formulation, the constant factor π remains best possible. The inequality also highlights how the truncation at the upper limit α modifies the classical bound, naturally leading to the introduction of weight functions in the integral norms. The series analogue of the Hilbert integral inequality also needs to be presented. Let $(a_m)_{m \in \mathbb{N} \setminus \{0\}}$ and $(b_n)_{n \in \mathbb{N} \setminus \{0\}}$ be two sequences of non-negative real numbers. Then we have

$$\sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{a_m b_n}{m+n} \leq \pi \left(\sum_{m=1}^{+\infty} a_m^2 \right)^{1/2} \left(\sum_{n=1}^{+\infty} b_n^2 \right)^{1/2},$$

provided that the two series on the right-hand side converge. This result is among the most classical series inequalities in mathematical analysis.

In this article, we take an innovative approach to this vast topic by examining a hyperbolic sine modification of the Hilbert integral inequality based on the kernel function $k(x, y) = 1/\sinh(x+y)$, recalling that, for any $a \in [0, +\infty)$,

$$\sinh a = \frac{e^a - e^{-a}}{2}.$$

Specifically, let $f, g : [0, +\infty) \rightarrow [0, +\infty)$ be two functions. Then our main result is centered on the following double integral:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{\sinh(x+y)} dx dy.$$

Compared to the classical Hilbert integral inequality, the denominator $x+y$ is replaced by $\sinh(x+y)$. This modification naturally introduces new analytical challenges and structural differences: while $1/(x+y)$ is homogeneous of degree -1 , the hyperbolic sine kernel decays exponentially as $x+y \rightarrow +\infty$, leading to stronger convergence properties. At the same time, the singular behavior near the origin remains comparable to that of the classical case. These features make the study of this inequality both a natural extension of and an intriguing departure from the classical Hilbert integral inequality. The resulting upper bound is expressed in terms of weighted integral norms of f and g , with hyperbolic tangent weight functions.

Building on this main result, we derive two further propositions. The first proposition offers a valuable upper bound for a double series of the following form:

$$\sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{a_m b_n}{\sinh(m+n)},$$

where $(a_m)_{m \in \mathbb{N} \setminus \{0\}}$ and $(b_n)_{n \in \mathbb{N} \setminus \{0\}}$ are two sequences of non-negative real numbers. The second proposition establishes a one-function version of the main integral inequality.

The rest of the article is organized as follows: In Section 2, we present a first approach and our main theorem. Section 3 is devoted to the two derived propositions. Finally, Section 4 provides some concluding remarks and discusses possible directions for future work.

2. MAIN THEOREM

2.1. First approach. As a first approach to the problem is the proposition below. The proof follows from a hyperbolic sine inequality and the classical Hilbert integral inequality.

Proposition 2.1. *Let $f, g : [0, +\infty) \rightarrow [0, +\infty)$ be two functions. Then we have*

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{\sinh(x+y)} dx dy \leq \pi \left(\int_0^{+\infty} f^2(x) dx \right)^{1/2} \left(\int_0^{+\infty} g^2(x) dx \right)^{1/2},$$

provided that the two integrals on the right-hand side converge.

Proof. For any $a \in [0, +\infty)$, the following inequality holds:

$$\sinh(a) \geq a.$$

This and the Hilbert integral inequality in Equation (1.1) gives

$$\begin{aligned} \int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{\sinh(x+y)} dx dy &\leq \int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dx dy \\ &\leq \pi \left(\int_0^{+\infty} f^2(x) dx \right)^{1/2} \left(\int_0^{+\infty} g^2(x) dx \right)^{1/2}. \end{aligned}$$

This completes the proof. $\square$

Of course, this first approach is not sharp, as it does not fully exploit the specific structure of the hyperbolic sine function. The remainder of the article is devoted to refining this result and obtaining sharper inequalities.

2.2. Statement and proof. The hyperbolic sine modification of the Hilbert integral inequality is stated in the theorem below, followed by its proof.

Theorem 2.2. *Let $f, g : [0, +\infty) \rightarrow [0, +\infty)$ be two functions. Then we have*

$$\begin{aligned} \int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{\sinh(x+y)} dx dy \\ \leq \pi \left(\int_0^{+\infty} \left(1 - \frac{1}{2} \sqrt{\tanh x} \right) f^2(x) dx \right)^{1/2} \left(\int_0^{+\infty} \left(1 - \frac{1}{2} \sqrt{\tanh x} \right) g^2(x) dx \right)^{1/2}, \end{aligned}$$

where we recall that, for any $a \in [0, +\infty)$,

$$\tanh a = \frac{e^a - e^{-a}}{e^a + e^{-a}},$$

provided that the two integrals on the right-hand side converge.

Proof. We set

$$I = \int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{\sinh(x+y)} dx dy$$

and note that, for any $x, y \in [0, +\infty)$,

$$\sinh(x+y) \geq 0.$$

Furthermore, we recall the following classical formulas:

- for any $b, c \in [0, 1)$, we have

$$\tanh^{-1} b + \tanh^{-1} c = \tanh^{-1} \left(\frac{b+c}{1+bc} \right), \quad (2.1)$$

- for any $b \in [0, 1)$, we have

$$\sinh(\tanh^{-1} b) = \frac{b}{\sqrt{1-b^2}}. \quad (2.2)$$

We first perform an integral transformation using the hyperbolic tangent function, making the change of variables

$$u = \tanh x, \quad v = \tanh y,$$

so that

$$x = \tanh^{-1} u, \quad y = \tanh^{-1} v,$$

which maps $x, y \in [0, +\infty)$ to $u, v \in [0, 1)$. The differentials are given by

$$dx = \frac{1}{1-u^2} du, \quad dy = \frac{1}{1-v^2} dv.$$

We now need to transform $\sinh(x+y)$ with the new variables. Using Equations (2.1) and (2.2), and standard developments, we have

$$\begin{aligned} \sinh(x+y) &= \sinh(\tanh^{-1} u + \tanh^{-1} v) \\ &= \sinh \left(\tanh^{-1} \frac{u+v}{1+uv} \right) \\ &= \frac{(u+v)/(1+uv)}{\sqrt{1 - ((u+v)/(1+uv))^2}} \\ &= \frac{(u+v)/(1+uv)}{\sqrt{(1-u^2)(1-v^2)/(1+uv)^2}} \\ &= \frac{u+v}{\sqrt{(1-u^2)(1-v^2)}}. \end{aligned}$$

Hence, we can express I as

$$\begin{aligned}
I &= \int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{\sinh(x+y)} dx dy \\
&= \int_0^1 \int_0^1 \frac{f(\tanh^{-1}u)g(\tanh^{-1}v)}{(u+v)/(\sqrt{1-u^2}\sqrt{1-v^2})} \times \frac{1}{(1-u^2)(1-v^2)} dudv \\
&= \int_0^1 \int_0^1 \frac{f(\tanh^{-1}u)g(\tanh^{-1}v)}{(u+v)\sqrt{1-u^2}\sqrt{1-v^2}} dudv \\
&= \int_0^1 \int_0^1 \frac{f_*(u)g_*(v)}{u+v} dudv, \tag{2.3}
\end{aligned}$$

where

$$f_*(u) = \frac{f(\tanh^{-1}u)}{\sqrt{1-u^2}}, \quad g_*(v) = \frac{g(\tanh^{-1}v)}{\sqrt{1-v^2}}.$$

Applying Equation (1.2) to the functions f_* and g_* , and $\alpha = 1$, we obtain

$$\begin{aligned}
&\int_0^1 \int_0^1 \frac{f_*(u)g_*(v)}{u+v} dudv \\
&\leq \pi \left(\int_0^1 \left(1 - \frac{1}{2}\sqrt{u} \right) f_*^2(u) du \right)^{1/2} \left(\int_0^1 \left(1 - \frac{1}{2}\sqrt{u} \right) g_*^2(u) du \right)^{1/2}. \tag{2.4}
\end{aligned}$$

Let us examine the first integral. Making the change of variables $u = \tanh x$, we get

$$\begin{aligned}
&\int_0^1 \left(1 - \frac{1}{2}\sqrt{u} \right) f_*^2(u) du = \int_0^1 \left(1 - \frac{1}{2}\sqrt{u} \right) \frac{f^2(\tanh^{-1}u)}{1-u^2} du \\
&= \int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh x} \right) f^2(x) dx. \tag{2.5}
\end{aligned}$$

Similarly but with the change of variables $v = \tanh x$, we obtain

$$\int_0^1 \left(1 - \frac{1}{2}\sqrt{u} \right) g_*^2(u) du = \int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh x} \right) g^2(x) dx. \tag{2.6}$$

It follows from Equations (2.3), (2.4), (2.5) and (2.6) that

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{\sinh(x+y)} dx dy \\
&\leq \pi \left(\int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh x} \right) f^2(x) dx \right)^{1/2} \left(\int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh x} \right) g^2(x) dx \right)^{1/2}.
\end{aligned}$$

This concludes the proof. $\square$

This theorem emphasizes that the constant factor remains π , drawing a clear parallel with the classical Hilbert integral inequality. The upper bound, meanwhile, is formulated in terms of the weighted norms of f and g . The weight functions depend on the hyperbolic tangent function, i.e.,

$$w(x) = 1 - \frac{1}{2}\sqrt{\tanh x}.$$

For any $x \in (0, 1)$, since $\tanh(x) \in (0, 1)$, we have $w(x) \in (1/2, 1)$.

An interesting aspect of Theorem 2.2 is its flexibility: it can be used as a tool to derive further results in integral theory. This versatility is demonstrated in the next section, where we present two related propositions.

3. TWO PROPOSITIONS

3.1. A series analogue. A series analogue of Theorem 2.2 is presented in the proposition below.

Proposition 3.1. *Let $(a_m)_{m \in \mathbb{N} \setminus \{0\}}$ and $(b_n)_{n \in \mathbb{N} \setminus \{0\}}$ be two sequences of non-negative real numbers. Then we have*

$$\begin{aligned} & \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{a_m b_n}{\sinh(m+n)} \\ & \leq \pi \left(\sum_{m=1}^{+\infty} \left(1 - \frac{1}{2} \sqrt{\tanh(m-1)} \right) a_m^2 \right)^{1/2} \left(\sum_{n=1}^{+\infty} \left(1 - \frac{1}{2} \sqrt{\tanh(n-1)} \right) b_n^2 \right)^{1/2}, \end{aligned}$$

provided that the two series on the right-hand side converge.

Proof. For any $m, n \in \mathbb{N} \setminus \{0\}$, let us set $f_\Delta(x) = a_m$ for $x \in [m-1, m)$ and $g_\Delta(x) = b_n$ for $x \in [n-1, n)$. Since $1/\sinh(x+y)$ is a non-increasing function with respect to x and y , we have

$$\frac{a_m b_n}{\sinh(m+n)} \leq \int_{n-1}^n \int_{m-1}^m \frac{f_\Delta(x) g_\Delta(y)}{\sinh(x+y)} dx dy.$$

This, Theorem 2.2 applied with the functions f_Δ and g_Δ , and the fact that $\tanh(x)$ is non-decreasing give

$$\begin{aligned} & \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{a_m b_n}{\sinh(m+n)} \leq \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \int_{n-1}^n \int_{m-1}^m \frac{f_\Delta(x) g_\Delta(y)}{\sinh(x+y)} dx dy \\ & = \int_0^{+\infty} \int_0^{+\infty} \frac{f_\Delta(x) g_\Delta(y)}{\sinh(x+y)} dx dy \\ & \leq \pi \left(\int_0^{+\infty} \left(1 - \frac{1}{2} \sqrt{\tanh x} \right) f_\Delta^2(x) dx \right)^{1/2} \left(\int_0^{+\infty} \left(1 - \frac{1}{2} \sqrt{\tanh x} \right) g_\Delta^2(x) dx \right)^{1/2} \\ & = \pi \left(\sum_{m=1}^{+\infty} \int_{m-1}^m \left(1 - \frac{1}{2} \sqrt{\tanh x} \right) a_m^2 dx \right)^{1/2} \left(\sum_{n=1}^{+\infty} \int_{n-1}^n \left(1 - \frac{1}{2} \sqrt{\tanh x} \right) b_n^2 dx \right)^{1/2} \\ & \leq \pi \left(\sum_{m=1}^{+\infty} \left(1 - \frac{1}{2} \sqrt{\tanh(m-1)} \right) a_m^2 \right)^{1/2} \left(\sum_{n=1}^{+\infty} \left(1 - \frac{1}{2} \sqrt{\tanh(n-1)} \right) b_n^2 \right)^{1/2}. \end{aligned}$$

This concludes the proof. $\square$

To the best of our knowledge, this is a new result in the literature on series inequalities. It thus completes the collection of manageable series inequality having a constant factor depending on π .

3.2. A one-function integral inequality. The proposition below offers a one-function integral inequality that is closely connected to Theorem 2.2.

Proposition 3.2. *Let $f : [0, +\infty) \rightarrow [0, +\infty)$ be a function. Then we have*

$$\begin{aligned} & \int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh y}\right)^{-1} \left(\int_0^{+\infty} \frac{f(x)}{\sinh(x+y)} dx\right)^2 dy \\ & \leq \pi^2 \int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh x}\right) f^2(x) dx, \end{aligned}$$

provided that the integral on the right-hand side converges.

Proof. We set

$$J = \int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh y}\right)^{-1} \left(\int_0^{+\infty} \frac{f(x)}{\sinh(x+y)} dx\right)^2 dy.$$

Then, by the Fubini-Tonelli integral theorem, we can write

$$\begin{aligned} J &= \int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh y}\right)^{-1} \\ &\quad \times \left(\int_0^{+\infty} \frac{f(x)}{\sinh(x+y)} dx\right) \left(\int_0^{+\infty} \frac{f(x)}{\sinh(x+y)} dx\right) dy \\ &= \int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g_\diamond(y)}{\sinh(x+y)} dx dy, \end{aligned} \tag{3.1}$$

where

$$g_\diamond(y) = \left(1 - \frac{1}{2}\sqrt{\tanh y}\right)^{-1} \int_0^{+\infty} \frac{f(x)}{\sinh(x+y)} dx.$$

Applying Theorem 2.2 to the functions f and $g_\diamond$, we get

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g_\diamond(y)}{\sinh(x+y)} dx dy \\ & \leq \pi \left(\int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh x}\right) f^2(x) dx\right)^{1/2} \\ & \quad \times \left(\int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh x}\right) g_\diamond^2(x) dx\right)^{1/2}. \end{aligned} \tag{3.2}$$

Let us examine the last integral. Using the definition of $g_\diamond$, we obtain

$$\begin{aligned} & \int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh x}\right) g_\diamond^2(x) dx \\ &= \int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh x}\right) \left(1 - \frac{1}{2}\sqrt{\tanh x}\right)^{-2} \left(\int_0^{+\infty} \frac{f(t)}{\sinh(t+x)} dt\right)^2 dx \\ &= \int_0^{+\infty} \left(1 - \frac{1}{2}\sqrt{\tanh x}\right)^{-1} \left(\int_0^{+\infty} \frac{f(t)}{\sinh(t+x)} dt\right)^2 dx \\ &= J. \end{aligned} \tag{3.3}$$

It follows from Equations (3.1), (3.2) and (3.3) that

$$J \leq \pi \left(\int_0^{+\infty} \left(1 - \frac{1}{2} \sqrt{\tanh x} \right) f^2(x) dx \right)^{1/2} J^{1/2},$$

which implies that

$$J \leq \pi^2 \int_0^{+\infty} \left(1 - \frac{1}{2} \sqrt{\tanh x} \right) f^2(x) dx,$$

so that

$$\begin{aligned} & \int_0^{+\infty} \left(1 - \frac{1}{2} \sqrt{\tanh y} \right)^{-1} \left(\int_0^{+\infty} \frac{f(x)}{\sinh(x+y)} dx \right)^2 dy \\ & \leq \pi^2 \int_0^{+\infty} \left(1 - \frac{1}{2} \sqrt{\tanh x} \right) f^2(x) dx. \end{aligned}$$

This completes the proof. $\square$

Such results can be used in the study of new integral operators based on the hyperbolic sine function.

4. CONCLUSION

In this article, we establish a hyperbolic sine modification of the classical Hilbert integral inequality. The upper bound is defined using weighted integral norms of the main functions with the classical constant π . The derived propositions further demonstrate the versatility of this approach, providing a series analogue and a one-function version of the inequality. Future work may explore other integral inequalities based on hyperbolic functions, multidimensional settings or applications in functional analysis.

REFERENCES

1. Chen, Q., & Yang, B. C. (2015). A survey on the study of Hilbert-type inequalities. *Journal of Inequalities and Applications*, 2015(1), 1-29. <https://doi.org/10.1186/s13660-015-0829-7>
2. Chesneau, C. (2024). Study of two three-parameter non-homogeneous variants of the Hilbert integral inequality. *Lobachevskii Journal of Mathematics*, 45, 4931-4953. <https://doi.org/10.1134/S1995080224605526>
3. Chesneau, C. (2025). Improvement of a known variant of the Hardy-Hilbert integral inequality. *Annals of Mathematics and Computer Science*, 29, 20-27. <https://doi.org/10.56947/amcs.v29.578>
4. Hardy, G. H., Littlewood, J. E., & Polya, G. (1934). *Inequalities*. Cambridge University Press.
5. Huang, Z., & Yang, B. C. (2011). A new Hilbert-type integral inequality with the combination kernel. *International Mathematical Forum*, 6, 1363-1369. <https://www.m-hikari.com/imf-2011/25-28-2011/yangbcIMF25-28-2011.pdf>
6. Li, Y., Wu, J., & He, B. (2006). A new Hilbert-type integral inequality and the equivalent form. *International Journal of Mathematics and Mathematical Sciences*, 8, 1-6. <https://doi.org/10.1155/IJMMS/2006/45378>
7. Sun, B. (2006). Best generalization of a Hilbert-type inequality. *Journal of Inequalities in Pure and Applied Mathematics*, 7, 1-7.

8. Yang, B. C. (1998). On Hilbert's integral inequality. Journal of Mathematical Analysis and Applications, 220, 778-785. <https://doi.org/10.1006/jmaa.1997.5877>
9. Yang, B. C. (2008). A basic Hilbert-type integral inequality with the homogeneous kernel of -1-degree and extensions. Journal of Guangdong Education Institute, 28(3), 1-10.
10. Yang, B. C. (2009). Hilbert-Type Integral Inequalities. Bentham Science Publishers. <https://doi.org/10.2174/97816080505501090101>
11. Yang, B. C. (2009). The Norm of Operator and Hilbert-Type Inequalities. Science Press.

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